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On Connections between the Cauchy Index, the Sylvester Matrix, Continued Fraction Expansions, and Circuit Synthesis

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Abstract— **A fundamental result in circuit synthesis states that the McMillan degree of a passive circuit's impedance is less than or equal to the number of reactive elements in the circuit. More recently, Hughes and Smith [1] connected the individual numbers of inductors and capacitors in a circuit to a generalisation of the Cauchy index for the circuit's impedance, which was named the extended Cauchy index. There is a close connection between the Cauchy index of a real-rational function and many classical algebraic results relating to pairs of polynomial functions [2]. Using this connection, it is possible to derive algebraic constraints on circuit impedance functions relating to the precise numbers of inductors and capacitors in that circuit. In this paper, we first present these algebraic constraints. We will then show a relationship between the extended Cauchy index and properties of continued fraction expansions of real-rational functions, which we use to provide insight into circuit synthesis procedures.**

Key words. Circuit synthesis, mechanical control, passivity, realisation, inerter

AMS subject classification. 70Q05, 93C05, 94C99

I. Introduction

In this paper, we consider electric circuits possessing one or more ports and comprising an interconnection of inductors, capacitors, resistors, and transformers (RLCT circuits). In [3], the synthesis problem of realising a given symmetric bounded-real function as the scattering matrix of a circuit was considered. There, the technique of reactance extraction was first introduced. This technique allows the circuit synthesis problem to be viewed as the problem of constructing a state-space realisation with a particular structure for a given symmetric bounded-real function. By considering circuit realisations which correspond to the system matrices for such a state-space realisation, [3] showed how any symmetric bounded-real function may be realised as the scattering matrix of a circuit containing resistors, inductors, capacitors, and transformers. Moreover, it was shown how the numbers of capacitors and inductors in the circuit are related to the internal signature of this state-space realisation.

The requirement for transformers in the scheme of [3] poses issues with practical realisability, and research has continued into circuit synthesis when transformers are excluded, with a particular focus on one-port circuits (RLC synthesis). Further motivation for this research stems from the recent invention of a new mechanical component, called the inerter [4]. There is a direct analogy between electric circuits containing resistors, inductors and capacitors, and mechanical networks containing dampers, springs and inerters. Consequently, the RLC synthesis problem has immediate relevance to passive mechanical control, with applications including automobile suspension [5], railway suspension [6], motorcycle steering compensators [7], vibration absorption [4], and building suspension [8]. Despite ongoing efforts, circuit realisation procedures as simple as [3] have remained elusive for the RLC synthesis problem.

For RLC synthesis, the focus is typically on the realisation of a given positive-real function as the impedance of the circuit. Recent papers [9–13] have sought a description of those impedance functions which can be realised by circuits which have limitations on the numbers of elements of the various types (resistors, capacitors, and inductors). In [3], the numbers of capacitors and inductors in the circuit were related to a property of a state-space realisation of the scattering matrix for the circuit. Given the focus on the impedance function in RLC synthesis, it is desirable to have a description of the associated constraints which are imposed on the impedance of a circuit by restrictions on the numbers of inductors and capacitors it contains. Moreover, rather than expressing these constraints as a property of a state-space realisation, it is preferable to express these constraints directly in terms of the parameters in the impedance function. Such constraints were described in [1], where they were expressed in terms of an extended Cauchy index for the circuit impedance, as well as a Sylvester, Bezoutian, and Hankel matrix associated with the impedance. In this paper, we show an additional relationship between the extended Cauchy index and the properties of continued fraction expansions of real-rational functions. This relationship has implications for circuit synthesis. In particular, we will exploit this relationship to derive algebraic expressions for element impedances in certain one-port circuits in terms of the overall circuit impedance.

The paper is structured as follows. We first describe the reactance extraction technique in Section II. In Section III, we then provide the definition of the extended Cauchy index, and describe the connection between the numbers of inductors and capacitors in a one-port circuit and the extended Cauchy index of that circuit's impedance. We then describe the associated algebraic constraints on one-port circuit impedances in Section IV. In Section V, we describe the relationship between the extended Cauchy index and the properties of continued fraction expansions of real-rational function, and the implications for circuit synthesis. We finally describe how the results generalise to multi-port circuits

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in Section VI, before offering some concluding remarks in Section VII.

II. Reactance Extraction

In a seminal paper in circuit synthesis [3], Youla and Tissi introduced the concept of reactance extraction, a technique which brought circuit synthesis into the domain of conventional linear systems theory. In that paper, the question of the realisation of a given symmetric *bounded-real* function as the *scattering* matrix of a circuit was considered. The scattering matrix of a circuit is the mapping in the Laplace domain from the *incident excitation* $\mathbf{v}(s) + \Lambda\mathbf{i}(s)$ to the *reflected response* $\mathbf{v}(s) - \Lambda\mathbf{i}(s)$, where $\mathbf{v}(s)$ and $\mathbf{i}(s)$ are vectors of port voltages and currents respectively, and Λ is a diagonal matrix of positive (otherwise arbitrary) port-normalisation constants. As was already well known at the time, the scattering matrix of a circuit is necessarily symmetric and bounded-real. Through use of the reactance extraction technique, together with the transformation of the Laplace domain variable:

$$\phi(s) := \frac{s+\alpha}{s-\alpha}, \qquad \phi^{-1}(s) = \frac{\alpha(s+1)}{s-1}, \qquad \alpha > 0,$$

Youla and Tissi showed how the problem of synthesising a symmetric bounded-real matrix $S(s)$ as the scattering matrix of a circuit may be posed as the realisation problem of finding a symmetric matrix

$$S_a = \begin{bmatrix} S_{11} & S_{12} \\ S_{12}^T & S_{22} \end{bmatrix}, \tag{1}$$

with $I - S_a^T S_a$ positive semi-definite, such that

$$S(\phi^{-1}(s)) = S_{11} + S_{12}(sI - \Sigma S_{22})^{-1}\Sigma S_{12}^T. \tag{2}$$

The matrix Σ in the above equation is a signature matrix, i.e. a diagonal matrix whose diagonal entries are either $+1$ or -1. The difference between the number of $+1$ and -1 entries in this matrix is known as the *internal signature* of the realisation. As recognised in [3], this internal signature is an invariant of internally symmetric realisations of a proper symmetric real-rational function. In other words, any two realisations of a given proper symmetric real-rational function with the form of (2), for which S_a in (1) is symmetric, have the same internal signature. By using this invariance property, Youla and Tissi showed that the number of inductors (resp. capacitors) in any circuit whose scattering matrix is equal to $S(s)$ must be greater than or equal to the number of $+1$ entries (resp. -1 entries) in the matrix Σ in an internally symmetric realisation of $S(\phi^{-1}(s))$.

By presenting a factorisation of the matrix S_a in (1) which corresponds to the scattering matrix of a circuit containing only resistors and transformers, Youla and Tissi then showed how any bounded-real matrix may be realised as the scattering matrix of the circuit shown in Fig. 1. Here, N_r is the circuit containing resistors and transformers whose scattering matrix is S_a, and the values of the inductances and capacitances are related to the port-normalisation constants associated with this scattering matrix.

The presence of multi-port transformers in the proposed realisation poses a problem for physical realisability, since the behaviour of such idealised multi-port transformers differs considerably from physical devices. Hence, research has continued into the RLC synthesis problem, for which realisation procedures as simple as [3] have proved elusive. Research in this area has been further motivated by the invention of a new mechanical component, called the inerter, which has established a direct analogy between electric circuits containing resistors, inductors and capacitors, and passive mechanical networks containing dampers, springs, and inerters.

In the RLC synthesis problem, it is more usual to consider the impedance (or admittance) of a circuit in preference to the scattering matrix. This is particularly true of mechanical networks, whose impedance relates the force applied across the network to the relative velocity of the two driving-point terminals. It is therefore instructive to obtain conditions on the realisable impedance functions of one-port circuits which relate to the numbers of inductors and capacitors (or, equivalently, springs and inerters) they contain. Moreover, it is desirable to have an external description of these constraints in preference to a description involving properties of an internal realisation as provided in [3]. Such descriptions were recently obtained in [1], and will be outlined briefly in the next two sections. These results enable us to establish certain properties of continued fraction expansions of real-rational functions which relate to the extended Cauchy index, which was not done in [1]. We derive these properties in Section V, and offer associated insights into RLC circuit synthesis procedures.

III. The Extended Cauchy Index

A unifying notion introduced in [1] is that of the *extended Cauchy index*, which we describe in this section. This is a generalisation of the Cauchy index for a real-rational function which accounts for cases where there are poles at infinity. In contrast to the scattering matrix of a circuit, a

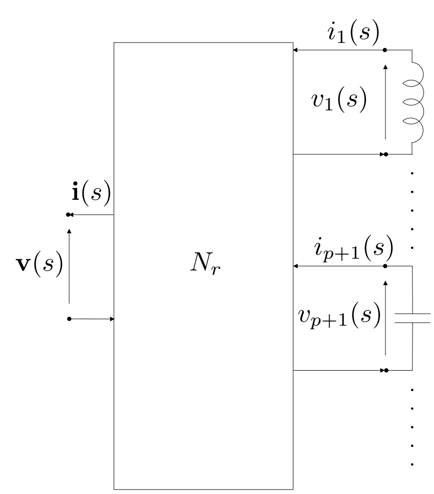

Fig. 1. Circuit with reactive elements extracted.

circuit's impedance need not be proper, and so the impedance may have a pole at infinity. Using the extended Cauchy index, together with the McMillan degree, it is possible to give a neat description of the constraints imposed on a one-port circuit's impedance function by the precise numbers of inductors and capacitors in the circuit. For a scalar real-rational function, [1] defined the extended Cauchy index as follows:

Definition 1 (extended Cauchy index): For a real-rational function $F(s)$, we define the *extended Cauchy index*, denoted by $\gamma(F(s))$, to be the difference between (i) the number of jumps of $F(s)$ from $-\infty$ to $+\infty$, and (ii) the number of jumps from $+\infty$ to $-\infty$ as s is increased in $\mathbb{R}$ from a point a through $+\infty$ and then from $-\infty$ to a again, for any $a \in \mathbb{R}$ which is not a pole of $F(s)$.

The McMillan degree of a real-rational function $F(s)$, which we denote by $\delta(F(s))$, is the number of poles of $F(s)$, counted according to their multiplicities, and including poles at ∞. Equivalently, when $F(s)$ is proper, it is the number of state variables in a minimal realisation of $F(s)$.

The following lemma is straightforward to verify.

Lemma 2 ([1], Lemma 6): Let $F(s)$, $F_1(s)$ and $F_2(s)$ be real-rational functions. Then

1) $\gamma(F(s)) = -\gamma(1/F(s))$.
2) If $F(s) = F_1(s) + F_2(s)$ and $\delta(F(s)) = \delta(F_1(s)) + \delta(F_2(s))$ then $\gamma(F(s)) = \gamma(F_1(s)) + \gamma(F_2(s))$.

By considering the reactance extraction scheme in [14, Chapter 4], Hughes and Smith then showed the following Theorem:

Theorem 3 ([1], Theorem 10): If $Z(s)$ is the impedance of a one-port circuit containing exactly p inductors and q capacitors, then

$$q \geq \frac{1}{2}\left(\delta\left(Z(s)\right) + \gamma\left(Z(s)\right)\right),$$
$$\text{and} \quad p \geq \frac{1}{2}\left(\delta\left(Z(s)\right) - \gamma\left(Z(s)\right)\right).$$

The above theorem provides lower bounds on the individual numbers of inductors and capacitors (or, equivalently, springs and inerters) necessary to realise a given impedance function. In the next section, we describe equivalent algebraic conditions in terms of the parameters in the impedance function.

IV. Algebraic Conditions

For a proper real-rational function, there is a close connection between the Hankel matrix and the Cauchy index between $-\infty$ and $+\infty$ [15, Theorem 9, p. 210]. Similarly, in [1], a connection was established between the extended Cauchy index and the Sylvester and Bezoutian matrices. This connection holds for all real-rational functions, including those which are not proper. The Sylvester and Bezoutian matrices feature in classical algebraic results relating to pairs of polynomial functions. In particular, they allow one to determine the number of roots common to two polynomials. Any real-rational function, expressed as a ratio of two polynomials, therefore has a natural association with a Sylvester and a Bezoutian matrix.

We consider a real-rational function $Z(s)$. For some integer n, $Z(s)$ may be written in the form

$$Z(s) = \frac{a(s)}{b(s)} = \frac{a_n s^n + a_{n-1}s^{n-1} + \ldots + a_0}{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}, \tag{3}$$

where at least one of a_n or b_n is strictly positive (n must be greater than or equal to $\delta(Z(s))$, with equality if and only if $a(s)$ and $b(s)$ are coprime). Associated with this representation of $Z(s)$ are the matrices

$$\mathcal{S}_j := \overbrace{\begin{bmatrix} b_n & b_{n-1} & b_{n-2} & \cdots \\ a_n & a_{n-1} & a_{n-2} & \cdots \\ 0 & b_n & b_{n-1} & \cdots \\ 0 & a_n & a_{n-1} & \cdots \\ 0 & 0 & b_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{j \text{ columns}} \Bigg\} j \text{ rows} \tag{4}$$

for $j = 1, 2, \ldots$. The determinant of the matrix $\mathcal{S}_{2n}$ is then the Sylvester resultant for the polynomials $a(s)$ and $b(s)$.

The two polynomials $a(s)$ and $b(s)$ may also be associated with a *Bezoutian* matrix. This is the symmetric matrix $\mathcal{B}(b, a)$ whose elements $\mathcal{B}_{ij}$ satisfy

$$a(w)b(z) - b(w)a(z) =: \sum_{i=1}^{n}\sum_{j=1}^{n} \mathcal{B}_{ij} z^{i-1}(z-w)w^{j-1}.$$

Theorem 3 may be shown to have an algebraic interpretation in terms of the Sylvester and Bezoutian matrices, which we present in Theorem 4. In that theorem, and hereafter, we denote the determinant of a square matrix M by $|M|$, and we denote the number of positive (resp. negative) eigenvalues of a symmetric matrix M by $\pi(M)$ (resp. $\nu(M)$). Moreover, for a sequence $x_1, \ldots, x_r$, we define $\mathbf{P}(x_1, \ldots, x_r)$ to be the number of permanences of sign, and $\mathbf{V}(x_1, \ldots, x_r)$ to be the number of variations of sign.

Theorem 4 ([1], Section 6 and Theorem 10): Let $Z(s)$ in (3) be the impedance of a one-port circuit containing exactly p inductors and q capacitors, let $\delta(Z(s)) = r$, and let $\mathcal{S}_j$ be as in (4) for $j = 1, 2, \ldots$. Then $|\mathcal{S}_{2r}| \neq 0$, $|\mathcal{S}_{2k}| = 0$ for $k > r$, and

$$q \geq \mathbf{P}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2r}|) = \pi\left(\mathcal{B}(b, a)\right),$$
$$\text{and} \quad p \geq \mathbf{V}(1, |\mathcal{S}_2|, |\mathcal{S}_4|, \ldots, |\mathcal{S}_{2r}|) = \nu\left(\mathcal{B}(b, a)\right).$$

In any sub-sequence of zero values ($|\mathcal{S}_{2k}| \neq 0$, $|\mathcal{S}_{2(k+1)}| = |\mathcal{S}_{2(k+2)}| = \ldots = 0$) signs are assigned to the zero values as follows: $\text{sign}\left(|\mathcal{S}_{2(k+j)}|\right) = (-1)^{\frac{j(j-1)}{2}}\text{sign}\left(|\mathcal{S}_{2k}|\right)$.

Finally, if $Z(s)$ is proper, then let the Laurent series of $Z(s)$ about the point at ∞ be:

$$Z(s) = h_{-1} + \frac{h_0}{s} + \frac{h_1}{s^2} + \frac{h_2}{s^3} + \ldots. \tag{5}$$

We define Hankel matrices for $Z(s)$ as:

$$\mathcal{H}_k := \begin{bmatrix} h_0 & h_1 & \cdots & h_{k-1} \\ h_1 & h_2 & \cdots & h_k \\ \vdots & \vdots & \ddots & \vdots \\ h_{k-1} & h_k & \cdots & h_{2k-2} \end{bmatrix}, \tag{6}$$

for $k = 1, 2, \ldots$. Since $Z(s)$ in (5) is proper, then $b_n \neq 0$, and

$$|\mathcal{S}_{2k}| = b_n^{2k} |\mathcal{H}_k|, \tag{7}$$

by [1, equation (21)]. Hence, if $Z(s)$ is proper, then equivalent conditions to those of Theorem 4 can be obtained in terms of the Hankel matrices $\mathcal{H}_k$ in (6) by using the relationship (7).

Theorem 4 provides algebraic constraints on those impedance functions which may be realised by circuits containing limited numbers of inductors and capacitors (or, equivalently, springs and inerters). The constraints are expressed in terms of the parameters in the impedance function. In the special case of a biquadratic function (a function with McMillan degree equal to two), Theorem 4 shows that the Sylvester resultant is positive if the circuit contains two reactive elements which are of the same kind, and negative if the circuit contains two reactive elements of opposite kind. This was hypothesised by Foster in [16] without proof, as noted by Kalman [17].

In addition to the connections with the Hankel, Sylvester, and Bezoutian matrices, the extended Cauchy index can also be related to the properties of continued fraction expansions of real-rational functions. This relationship, and its implications for circuit synthesis, will be discussed in the next section. In Section V, we then briefly describe the generalisation of the results presented thus far to multi-port circuits.

V. Continued Fraction Expansions

As will be shown in this section, the extended Cauchy index is related to the properties of continued fraction expansions of real-rational functions. Such continued fractions appear prevalently in circuit synthesis, since the impedance of a circuit which comprises alternating series and parallel connected elements is a continued fraction. As will be shown in this section, for a positive-real function $F(s)$ which satisfies $\delta(F(s)) = \gamma(F(s))$, a continued fraction expansion leads to the *Cauer forms*, which were first introduced in [18]. In this section, we also exploit the relationship between the extended Cauchy index and the Sylvester matrix to give explicit algebraic expressions for the element impedances in the circuits of Cauer, in terms of the overall circuit impedance. These algebraic expressions are summarised in Theorem 8, where an explicit realisation is provided for any impedance function which can be realised by a one-port circuit which contains resistors, capacitors and transformers only. The proof of that theorem requires Lemmas 5 to 7, which describe the properties of the impedance functions of one-port circuits containing only resistors, capacitors and transformers.

The extended Cauchy index of a real-rational function $F(s)$ cannot exceed the McMillan degree. Hence, from Theorem 3, if a circuit has impedance $Z(s)$ and contains no inductors, then $\delta(Z(s)) = \gamma(Z(s))$. Since, in addition, the impedance of a RLCT circuit is necessarily positive-real, then the following lemma must hold:

Lemma 5: Let $Z(s)$ be the impedance of a one-port circuit containing resistors, capacitors and transformers only. Then $Z(s)$ is real-rational, analytic in the open right-half plane, $Z(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$, and $\delta(Z(s)) = \gamma(Z(s))$.

Any given real-rational function $F(s)$ possesses a partial fraction expansion

$$F(s) = F_1(s) + F_2(s) + \sum_{i=1}^{k} \frac{A_1^{(i)}}{s - \alpha_i} + \ldots + \frac{A_{n_i}^{(i)}}{(s - \alpha_i)^{n_i}},$$

where $F_1(s)$ is a polynomial in s, $F_2(s)$ is a strictly proper real-rational function with no real poles, n_i is a strictly positive integer and $A_{n_i}^{(i)} \neq 0$, and all the α_i are distinct ($i = 1, 2, \ldots, k$). From the properties of the McMillan degree (see, e.g. [3] or [19]) it may be verified that

$$\delta(F(s)) = \deg(F_1(s)) + \delta(F_2(s)) + \sum_{i=1}^{k} n_i. \tag{8}$$

Furthermore, whenever $F(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$, we have

$$\gamma(F(s)) = -\deg(F_1(s)) \bmod 2 + \sum_{i=1}^{k} (n_i \bmod 2) \operatorname{sign}\left(A_{n_i}^{(i)}\right). \tag{9}$$

From equations (8) and (9) we see that $\gamma(F(s)) = \delta(F(s))$ implies $\deg(F_1(s)) = \delta(F_2(s)) = 0$. Furthermore, we find that $n_i = 1$ and $A_{n_i}^{(i)} > 0$ for $i = 1, 2, \ldots, k$.

From Lemma 5 and the preceding argument, it follows that if $Z(s)$ is the impedance of a one-port circuit containing resistors, capacitors, and transformers only, then $Z(s)$ has the partial fraction expansion

$$Z(s) = Z_0 + \sum_{i=1}^{r} \frac{Z_i}{s - \alpha_i}, \tag{10}$$

with $Z_0 \geq 0$, and both $Z_i > 0$ and $\alpha_i \leq 0$ for $i = 1, 2, \ldots r$ with $\alpha_i < \alpha_j$ for $i < j$. Here $\delta(Z(s)) = \gamma(Z(s)) = r$. It is then clear that $Z(s)$ is the impedance of the *Foster form* in Fig. 2. In that figure, the dotted resistor without a continuous line through it is to be replaced with an open circuit whenever $\alpha_r = 0$, and the dotted resistor with a continuous line through it is to be replaced with a short circuit whenever $Z_0 = 0$.

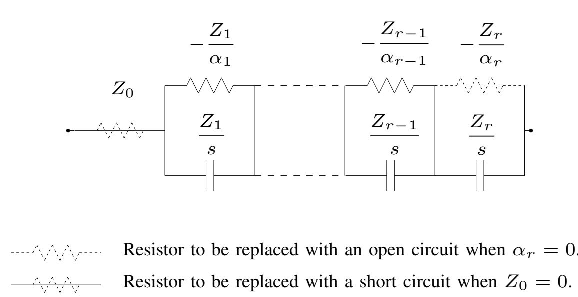

Fig. 2. Foster's form for the realisation of the function $Z(s)$ in (10).

Furthermore, from the partial fraction decomposition (10), we obtain

$$\frac{dZ}{ds} = -\sum_{i=1}^{r} \frac{Z_i}{(s-\alpha_i)^2},$$

and we see that $Z(s)$ is a monotonically decreasing function of s for $s \in \mathbb{R}$ except at the poles α_i. We thus obtain the interlacing property summarised in the next lemma.

Lemma 6: Let $F(s)$ be a real-rational function which is analytic in the open right-half plane and satisfies $F(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$. Then $\gamma(F(s)) = \delta(F(s)) = r$ if and only if the poles of $F(s)$ (denoted α_i for $i = 1, 2, \ldots, r$), and zeros of $F(s)$ (denoted α'_i for $i = 1, 2, \ldots, r$), satisfy the interlacing property:

$$-\infty \leq \alpha'_1 < \alpha_1 < \alpha'_2 < \alpha_2 < \ldots < \alpha'_{r-1} < \alpha_{r-1} < \alpha'_r < \alpha_r \leq 0.$$

We will now use the connection between the extended Cauchy index and the Sylvester matrix to show the following lemma:

Lemma 7: Let $Z(s)$ in (3) be analytic in the open right-half plane, and let $Z(s)$ satisfy $Z(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$ and $\gamma(Z(s)) = \delta(Z(s)) = r$. Further let $\mathcal{S}_j$ be as in (4) ($j = 1, 2, \ldots$). Then $|\mathcal{S}_j| > 0$ for $j = 1, 2, \ldots, 2r$, and $|\mathcal{S}_{2r+1}| \geq 0$ with $|\mathcal{S}_{2r+1}| = 0$ if and only if $b_0 = 0$.

Proof: Since $Z(s)$ does not have a pole at $s = \infty$ by Lemma 6, then $Z(s)$ is proper, and hence $b_n > 0$. Furthermore, since $Z(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$ then the leading coefficient of $a(s)$ is also positive. Moreover, since $Z(s)$ is proper, then the extended Cauchy index for $Z(s)$ is equal to the Cauchy index between $-\infty$ and $+\infty$. From [15, Theorem 9, p. 210], it follows that the Hankel matrix $\mathcal{H}_k$ in (6) is positive definite for $k = 1, 2, \ldots, r$, where $h_0, h_1, \ldots$ are the parameters in the Laurent series for $Z(s)$ about ∞ in (5). Hence, for $k = 1, 2, \ldots, r$, we require $|\mathcal{H}_k| > 0$, which implies $|\mathcal{S}_{2k}| > 0$ by equation (7). Moreover, since $\delta(Z(s)) = r$, then $|\mathcal{H}_{r+1}| = 0$, and accordingly $|\mathcal{S}_{2(r+1)}| = 0$, by [15, Theorem 8, p. 207].

Now, note that

$$\frac{1}{sZ(s)} = \frac{b_n s^n + b_{n-1}s^{n-1} + \ldots + b_0}{a_n s^{n+1} + a_{n-1}s^n + \ldots + a_0 s},$$

so, in a similar fashion to the matrices $\mathcal{S}_j$ associated with the real-rational function $Z(s)$ in (4), and providing $a_n \neq 0$, we associate the matrices

$$\hat{\mathcal{S}}_j := \left.\begin{bmatrix} \overbrace{\begin{matrix} a_n & a_{n-1} & a_{n-2} & \cdots \end{matrix}}^{j \text{ columns}} \\ \begin{matrix} 0 & b_n & b_{n-1} & \cdots \\ 0 & a_n & a_{n-1} & \cdots \\ 0 & 0 & b_n & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \end{bmatrix}\right\} j \text{ rows} \qquad (11)$$

with the real-rational function $1/(sZ(s))$ ($j = 1, 2, \ldots$). In the case $a_n = 0$, we instead let

$$\hat{\mathcal{S}}_j := \left.\begin{bmatrix} \overbrace{\begin{matrix} a_{n-1} & a_{n-2} & a_{n-3} & \cdots \end{matrix}}^{j \text{ columns}} \\ \begin{matrix} b_n & b_{n-1} & b_{n-2} & \cdots \\ 0 & a_{n-1} & a_{n-2} & \cdots \\ 0 & b_n & b_{n-1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix} \end{bmatrix}\right\} j \text{ rows}. \qquad (12)$$

From the interlacing property, and since $Z(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$, it is clear that $W(s) := 1/sZ(s)$ is analytic in the open right-half plane, that $W(s_0) \geq 0$ for all $s_0 \in \mathbb{R}, s_0 > 0$, and that

$$\delta\left(\frac{1}{sZ(s)}\right) = \gamma\left(\frac{1}{sZ(s)}\right) = \tilde{r},$$

where $\tilde{r} = r + 1 - \epsilon_p - \epsilon_z$ with $\epsilon_p = 1$ if $Z(s)$ has a pole at $s = 0$ and 0 otherwise, and $\epsilon_z = 1$ if $Z(s)$ has a zero at $s = -\infty$ and 0 otherwise. Hence, by a similar argument to the preceding paragraph, we find $|\hat{\mathcal{S}}_{2k}| > 0$ for $k = 1, 2, \ldots, \tilde{r}$, and $|\hat{\mathcal{S}}_{2(\tilde{r}+1)}| = 0$.

There are now two cases to consider: (i) $\epsilon_z = 0$, and (ii) $\epsilon_z = 1$.

In case (i), $Z(s)$ does not have a zero at $-\infty$, and hence $a_n > 0$. Furthermore, $\tilde{r} = r + 1 - \epsilon_p$, and from equation (11) we find

$$\hat{\mathcal{S}}_{2(\tilde{r}+1)} = \begin{bmatrix} a_n & a_{n-1} \quad a_{n-2} \quad \cdots \\ 0 & \\ 0 & \mathcal{S}_{2\tilde{r}+1} \\ \vdots & \end{bmatrix}.$$

Since $|\hat{\mathcal{S}}_{2k}| > 0$ for $k = 1, 2, \ldots, r + 1 - \epsilon_p$ and $|\mathcal{S}_{2k}| > 0$ for $k = 1, 2, \ldots, r$, and $a_n > 0$, then $|\mathcal{S}_j| > 0$ for $j \leq \max\{2r, 2(r - \epsilon_p) + 1\}$. Moreover, since $|\hat{\mathcal{S}}_{2(r+2-\epsilon_p)}| = 0$, then $|\mathcal{S}_{2r+1}| = 0$ if and only if $\epsilon_p = 1$.

In case (ii), $Z(s)$ has a zero at $s = -\infty$ so $a_n = 0$. Also, $\tilde{r} = r - \epsilon_p$, and from equation (12) we have

$$\mathcal{S}_{2(r+1)} = \begin{bmatrix} b_n & b_{n-1} \quad b_{n-2} \quad \cdots \\ 0 & \\ 0 & \hat{\mathcal{S}}_{2r+1} \\ \vdots & \end{bmatrix},$$

which again implies that $|\mathcal{S}_j| > 0$ for $j \leq \max\{2r, 2(r - \epsilon_p) + 1\}$, and $|\mathcal{S}_{2r+1}| = 0$ if and only if $\epsilon_p = 1$.

Since $\epsilon_p = 1$ if and only if $b_0 = 0$, we conclude that $|\mathcal{S}_j| > 0$ for $j = 1, 2, \ldots, 2r$, $|\mathcal{S}_{2r+1}| \geq 0$, and $|\mathcal{S}_{2r+1}| = 0$ if and only if $b_0 = 0$. ■

Finally in this section, we will use a continued fraction expansion of $Z(s)$ to show the following theorem:

Theorem 8: Let $Z(s)$ be the impedance of a one-port circuit containing resistors, capacitors, and transformers only. Then $Z(s)$ is realised by the circuit in Fig. 3.

Proof: Let $Z(s)$ be as in (3), let $\mathcal{S}_j$ be as in (4) for $j = 1, 2, \ldots$, and let $r = \delta(Z(s))$. Since $Z(s)$ cannot have a pole at $s = \infty$ by Lemmas 5 and 6, then $Z(s)$ is proper, so without loss of generality we let the leading coefficient of $b(s)$ be strictly positive. From Lemma 5, $\gamma(Z(s)) = \delta(Z(s))$. To prove the present theorem, we will show that $Z(s)$ has the continued fraction expansion

$$Z(s) = u_r + \cfrac{1}{v_r s + \cfrac{1}{u_{r-1} + \cfrac{1}{v_{r-1}s + \ldots + \cfrac{1}{u_1 + \cfrac{1}{v_1 s + t}}}}}, \quad (13)$$

where

$$u_r = \frac{a_n}{|\mathcal{S}_1|}, \quad (14)$$

$$v_r = \frac{|\mathcal{S}_1|^2}{|\mathcal{S}_2|}, \quad (15)$$

and

$$u_k = \frac{|\mathcal{S}_{2(r-k)}|^2}{|\mathcal{S}_{2(r-k)-1}||\mathcal{S}_{2(r-k)+1}|}, \quad (16)$$

$$v_k = \frac{|\mathcal{S}_{2(r-k)+1}|^2}{|\mathcal{S}_{2(r-k)}||\mathcal{S}_{2(r-k+1)}|}, \quad (17)$$

for $k = 1, 2, \ldots, r-1$. Furthermore,

$$t = \frac{|\mathcal{S}_{2r-1}||\mathcal{S}_{2r+1}|}{|\mathcal{S}_{2r}|^2}, \quad (18)$$

or, equivalently,

$$t = \frac{|\mathcal{S}_{2n-1}|b_0}{|\mathcal{S}_{2n}|} \quad (19)$$

when $r = n$. By Lemma 7, it follows that $u_k, v_k > 0$ ($k = 1, 2, \ldots, r-1$), $v_r > 0$, $u_r \geq 0$ with $u_r = 0$ if and only if $a_n = 0$, and $t \geq 0$ with $t = 0$ if and only if $b_0 = 0$. It is then clear that $Z(s)$ is realised by the circuit in Fig. 3.

To show that $Z(s)$ has a continued fraction expansion of the form of (13), suppose $Z_k(s)$ is a real-rational function with $\delta(Z_k(s)) = \gamma(Z_k(s)) = k$. Then $U_k(s) = 1/(Z_k(s) - \lim_{s\to\infty} Z_k(s))$ satisfies $\delta(U_k(s)) = -\gamma(U_k(s)) = k$ by Lemma 2, and has a pole at $s = \infty$ which must be simple as a consequence of Lemma 6. Now, consider $V_{k-1}(s) = U_k(s) - s\lim_{s\to\infty}(U_k(s)/s)$. Since $|\gamma(F(s))| \leq \delta(F(s))$ for any real-rational function $F(s)$, and both $\delta(V_{k-1}(s)) = k-1$ and $\gamma(V_{k-1}(s)) \leq -(k-1)$ by Lemma 2, then $\delta(V_{k-1}(s)) = -\gamma(V_{k-1}(s)) = k-1$. Hence, $Z_{k-1}(s) = 1/V_{k-1}(s)$ satisfies

$$Z_{k-1}(s) = \frac{1}{\frac{1}{Z_k(s)-u_k} - v_k s}, \quad (20)$$

with

$$u_k = \lim_{s\to\infty} Z_k(s),$$
$$v_k = \lim_{s\to\infty}\left(\frac{1}{(Z_k(s)-u_k)s}\right),$$
$$\text{and} \quad \delta(Z_{k-1}(s)) = \gamma(Z_{k-1}(s)) = k-1.$$

Now, let $Z(s) = Z_r(s)$. Then $Z_r(s)$ satisfies $\delta(Z_r(s)) = \gamma(Z_r(s)) = r$ by Lemma 5, and from the preceding argument we obtain functions $Z_j(s)$ with $\delta(Z_j(s)) = \gamma(Z_j(s)) = j$ ($j = 0, 1, \ldots, r-1$). Moreover, from equation (20), we obtain

$$Z_k(s) = u_k + \frac{1}{v_k s + \frac{1}{Z_{k-1}(s)}},$$

and therefore $Z(s)$ has a continued fraction expansion of the form (13).

It remains to show that the parameters u_j, v_j ($j = 1, 2, \ldots, r$), and t, in the continued fraction expansion (13) are given by the equations (14) to (19). To see this, let $p(s)$ be the (monic) greatest common divisor of $a(s)$ and $b(s)$ in (3), let $m = n - r$ be the degree of $p(s)$, and write $Z_k(s) = a_k(s)/b_k(s)$ with

$$a_k(s) = a_{k,m+k}s^{m+k} + a_{k,m+k-1}s^{m+k-1} + \ldots + a_{k,0},$$
$$\text{and} \quad b_k(s) = b_{k,m+k}s^{m+k} + b_{k,m+k-1}s^{m+k-1} + \ldots + b_{k,0},$$

for $k = 0, 1, \ldots, r$, so $p(s)$ divides both $a_r(s)$ and $b_r(s)$. Then

$$Z_{k-1}(s) = \frac{a_k(s) - u_k b_k(s)}{(1 + u_k v_k s)b_k(s) - v_k s a_k(s)},$$

so let

$$a_{k-1}(s) = a_k(s) - u_k b_k(s), \quad (21)$$
$$\text{and} \quad b_{k-1}(s) = (1 + u_k v_k s)b_k(s) - v_k s a_k(s). \quad (22)$$

Then, by induction, $p(s)$ divides both $a_j(s)$ and $b_j(s)$ ($j = 0, 1, \ldots, r$). Moreover, we find

$$u_k = \lim_{s\to\infty}\frac{a_k(s)}{b_k(s)} = \frac{a_{k,m+k}}{b_{k,m+k}}, \quad (23)$$

$$\text{and} \quad v_k = \lim_{s\to\infty}\frac{b_k(s)}{s(a_k(s) - u_k b_k(s))} = \lim_{s\to\infty}\frac{b_k(s)}{s a_{k-1}(s)} = \frac{b_{k,m+k}}{a_{k-1,m+k-1}}, \quad (24)$$

for $k = 1, 2 \ldots r$.

Now, let $\mathcal{S}^{(k)}$ be the $(2k+1) \times (2k+1)$ matrix,

$$\overbrace{\begin{bmatrix} b_{k,m+k} & b_{k,m+k-1} & b_{k,m+k-2} & \cdots \\ a_{k,m+k} & a_{k,m+k-1} & a_{k,m+k-2} & \cdots \\ 0 & b_{k,m+k} & b_{k,m+k-1} & \cdots \\ 0 & a_{k,m+k} & a_{k,m+k-1} & \cdots \\ 0 & 0 & b_{k,m+k} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{2k+1 \text{ columns}} \left.\vphantom{\begin{matrix}1\\1\\1\\1\\1\\1\end{matrix}}\right\} 2k+1 \text{ rows}.$$

From equations (21) and (22), we obtain

$$a_k(s) = (1 + u_k v_k s)a_{k-1}(s) + u_k b_{k-1}(s),$$
$$\text{and} \quad b_k(s) = v_k s a_{k-1}(s) + b_{k-1}(s),$$

which implies

$$\mathcal{S}^{(k)} = \hat{L}^{(k)} \begin{bmatrix} b_{k,m+k} & b_{k,m+k-1} & b_{k,m+k-2} & b_{k,m+k-3} & \cdots \\ 0 & a_{k-1,m+k-1} & a_{k-1,m+k-2} & a_{k-1,m+k-3} & \cdots \\ 0 & 0 & & & \\ 0 & 0 & & \mathcal{S}^{(k-1)} & \\ \vdots & \vdots & & & \end{bmatrix},$$

where $\hat{L}^{(k)}$ is the $(2k+1) \times (2k+1)$ lower triangular matrix

$$\left.\overbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ u_k & 1 & 0 & 0 & 0 & 0 & \cdots \\ 0 & v_k & 1 & 0 & 0 & 0 & \cdots \\ 0 & u_k v_k & u_k & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & v_k & 1 & 0 & \cdots \\ 0 & 0 & 0 & u_k v_k & u_k & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}}^{2k+1 \text{ columns}}\right\} 2k+1 \text{ rows}.$$

Proceeding by induction we find

$$\mathcal{S}^{(r)} = LU, \tag{25}$$

where U is the $(2r+1) \times (2r+1)$ upper triangular matrix

$$\begin{bmatrix} b_{r,m+r} & b_{r,m+r-1} & \cdots & b_{r,m-r+2} & b_{r,m-r+1} & b_{r,m-r} \\ 0 & a_{r-1,m+r-1} & \cdots & a_{r-1,m-r+2} & a_{r-1,m-r+1} & a_{r-1,m-r} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & b_{1,m+1} & b_{1,m} & b_{1,m-1} \\ 0 & 0 & \cdots & 0 & a_{0,m} & a_{0,m-1} \\ 0 & 0 & \cdots & 0 & 0 & b_{0,m} \end{bmatrix},$$

in which $b_{k,j}, a_{k,j} = 0$ for $j < 0$. Moreover, L is the product of lower triangular matrices whose diagonal entries are all one so L is itself lower triangular with ones on the diagonal. Then, from the Binet Cauchy Theorem [20, Chapter I, Section 2] and equations (23) and (24), we obtain the relationships (14), (15), (16), and (17). This may be seen by multiplying the numerator and denominator of u_k in (23) by $a_{k,m+k} \times (b_{r,m+r} \times \Pi_{i=1}^{r-k-1} b_{k+i,m+k+i} a_{k+i,m+k+i})^2$, and by multiplying the numerator and denominator of v_k in (24) by $b_{k,m+k} \times (\Pi_{i=1}^{r-k} b_{k+i,m+k+i} a_{k+i-1,m+k+i-1})^2$, and then factoring the resulting expressions into products of the leading principal minors of the matrix U. Moreover, since $\delta(Z_0(s)) = 0$, then $t = 1/Z_0(s) = b_0(s)/a_0(s) = b_{0,m}/a_{0,m}$, and equation (18) must hold. Finally, if $r = n$, then $m = 0$, and since $b_{k,j} = a_{k,j} = 0$ for $j < 0$ then the first $2r$ entries in the final column of U are zero. By equating the entry in the bottom right-hand corner of the matrix equation (25), we obtain the relationship (19), which must hold whenever $r = n$. ■

The circuit in Fig. 3 is referred to as Cauer's form, and Theorem 8 expresses the impedances of the elements in this circuit in terms of the overall circuit impedance. A dual result may be obtained for one-port circuits which contain no capacitors, for which the impedance $Z(s)$ necessarily satisfies $\delta(Z(s)) = -\gamma(Z(s))$.

VI. Multi-port Circuits

Many of the results stated in Sections III and IV generalise in a natural way to the multi-port case. In this case, we consider the matrix extended Cauchy index, which was defined in [1] as follows:

Definition 9 (matrix extended Cauchy index): For a real-rational symmetric matrix $F(s)$ we define the extended matrix Cauchy index, denoted by $\gamma(F(s))$, to be the difference between (i) the number of jumps in the eigenvalues of $F(s)$ from $-\infty$ to $+\infty$, and (ii) number of jumps in the eigenvalues of $F(s)$ from $+\infty$ to $-\infty$ as s is increased in $\mathbb{R}$ from a point a through $+\infty$ and then from $-\infty$ to a again, for any $a \in \mathbb{R}$ which is not a pole of $F(s)$.

Again, we denote the McMillan degree of a real-rational matrix $F(s)$ as $\delta(F(s))$. As in the scalar case, this is equal to the number of poles of $F(s)$, including poles at ∞, and is equal to the number of states in a minimal realisation of $F(s)$ when $F(s)$ is proper.

The notion of a matrix Bezoutian was defined in [21]. For a symmetric real-rational matrix $F(s)$ with a left matrix factorisation $F(s) = B^{-1}(s)A(s)$ ($A(s)$ and $B(s)$ are polynomial matrices which need not be left coprime), the matrix Bezoutian $\mathcal{B}(B, A)$ is defined as the symmetric matrix with block entries $\mathcal{B}_{ij}$ satisfying

$$B(z)A^T(w) - A(z)B^T(w) =: \sum_{i=1}^{n} \sum_{j=1}^{n} \mathcal{B}_{ij} z^{i-1}(z-w)w^{j-1}.$$

Here, n is the maximum of the degree of the entries in $A(s)$ and $B(s)$.

In the case of multi-port circuits, an impedance (admittance) description may not exist. Nonetheless, it is shown in [14] that any circuit possesses a hybrid matrix, corresponding to the mapping (in the Laplace domain) from the current through m_1 of the ports and the voltage across the remaining m_2 ports, to the corresponding voltages and currents for these ports. In other words, there exists a matrix $M(s)$ corresponding to some compatible partitioning of the port voltages and currents into vectors $\begin{bmatrix} \mathbf{v}_\alpha(s) & \mathbf{v}_\beta(s) \end{bmatrix}^T$ and $\begin{bmatrix} \mathbf{i}_\alpha(s) & \mathbf{i}_\beta(s) \end{bmatrix}^T$ respectively, such that

$$\begin{bmatrix} \mathbf{v}_\alpha(s) \\ \mathbf{i}_\beta(s) \end{bmatrix} = M(s) \begin{bmatrix} \mathbf{i}_\alpha(s) \\ \mathbf{v}_\beta(s) \end{bmatrix}. \tag{26}$$

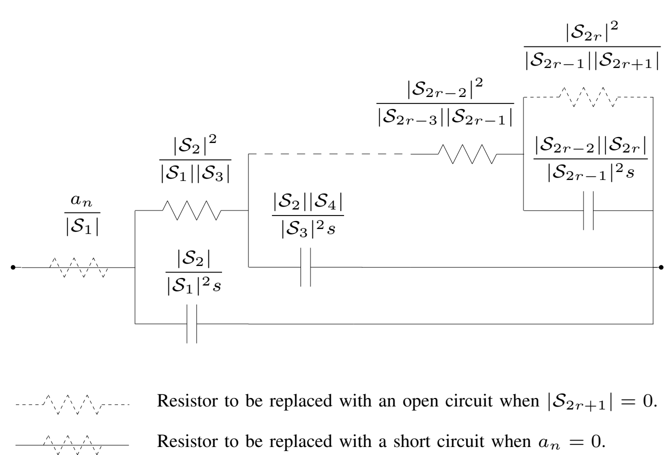

Fig. 3. Circuit realisation for the function $Z(s)$ in (3), in which the leading coefficient of $b(s)$ is strictly positive and $r = \delta(Z(s)) = \gamma(Z(s))$. Here, $\mathcal{S}_j$ is as in (4) $(j = 1, 2, \ldots)$.

By considering the reactance extraction scheme in [3], together with properties of the extended Cauchy index presented in [1], the following theorem may be shown:

Theorem 10 ([1], Theorem 15): Let $M(s)$ be the hybrid matrix of an m-port circuit containing exactly p inductors and q capacitors, with current excitation at the first m_1 ports and voltage excitation at the remaining m_2 ports as in (26), and let

$$\Sigma_e = \begin{bmatrix} I_{m_1} & 0_{m_1 \times m_2} \\ 0_{m_2 \times m_1} & -I_{m_2} \end{bmatrix}.$$

Then $M(s)\Sigma_e$ is symmetric. Further, let $M(s)\Sigma_e$ have a left matrix factorisation $M(s)\Sigma_e = B^{-1}(s)A(s)$. Then

$$q \geq \frac{1}{2}\left(\delta\left(M(s)\Sigma_e\right) + \gamma\left(M(s)\Sigma_e\right)\right) = \pi\left(\mathcal{B}\left(B, A\right)\right),$$
$$p \geq \frac{1}{2}\left(\delta\left(M(s)\Sigma_e\right) - \gamma\left(M(s)\Sigma_e\right)\right) = \nu\left(\mathcal{B}\left(B, A\right)\right).$$

VII. Conclusions

In this paper, lower bounds on the numbers of inductors and capacitors (or springs and inerters) required to realise a given impedance function were presented. These lower bounds were presented in terms of an extended Cauchy index, a Sylvester matrix, and a Bezoutian matrix for the circuit impedance. A relationship between the extended Cauchy index and the properties of continued fraction expansions of real-rational functions was shown, with implications for circuit synthesis. In particular, explicit expressions for the element impedances in the realisation of Cauer were obtained in terms of the overall circuit impedance.

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