OPTIMAL ACTUATOR/SENSOR LOCATION

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Some Examples
Actuator/Sensor Location
Actuator/Sensor Location

Where to place the control hardware?
Transient response with single actuator on a simply supported beam

The system impulse response

Damped Amplitude (m)

Time (Sec.)

Actuator at 0.5L and 0.55L
Actuator location for noise control

- **Optimization of control actuators and error sensors provides a method for realizing adaptive structures for active structural acoustic control, rivalling in importance the performance increases gained when acoustic control is achieved with microphone error sensors and multiple control actuators.** [Clark & Fuller, 1992]

- Better noise reduction with optimally placed static output feedback than with full state feedback placed elsewhere. (43dB vs 25dB) [Fahroo & Demetriou 2000]
Actuator/sensor location

- controlled performance depends on actuator location
- often hardware can’t be easily moved once placed
- need to determine locations before implementation
- trial-and-error process impractical with multiple actuators/sensors
- locations of control hardware as well as the feedback gain are controller design parameters.
State-space Formulation

PDE

\[ \dot{z}(t) = \frac{\partial^2 z(x, t)}{\partial x^2} + b(x)u(t), \quad 0 < x < 1, \]
\[ z(0, t) = 0, \quad z(1, t) = 0. \]
State-space Formulation

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State-space description

\[ \dot{z}(t) = Az(t) + Bu(t) \]

- state-space \( \mathcal{H} = \mathcal{L}_2(0, 1) \)
- \( A = \frac{d^2}{dx^2} ; \quad \text{dom}(A) = \{ z(x) \in \mathcal{H}^2(0, 1) ; z(0) = z(1) = 0 \} \)
- \( B = b(x) \)
Strongly Continuous Semigroups

\[ \dot{z}(t) = Az(t), \quad z(0) = z_0 \]

**Definition: Strongly continuous semigroup** \( S(t) \) on Hilbert space \( \mathcal{H} \)

- \( S(0) = I \),
- \( S(t)S(s) = S(t + s) \),
- \( \lim_{t \downarrow 0} S(t)z = z \), for all \( z \in \mathcal{H} \).

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{H} \) means that for all \( z \in D(A) \),
  \[ Az = \lim_{t \downarrow 0} \frac{S(t)z - z}{t}, \]
  \[ z(t) = S(t)z_0. \]

- Generalization of matrix exponential
General State-space Description

\[ \dot{z}(t) = Az(t) + Bu(t), \quad z(0) = z_0 \]
\[ y(t) = Cz(t). \]

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{H} \)
- \( B \in \mathcal{L}(U, \mathcal{H}), \ C \in \mathcal{L}(\mathcal{H}, \mathcal{Y}) \)
- Input/output map: \( y = C \int_0^t S(t - \tau) Bu(\tau) d\tau \)
Modeling of Actuator/Sensors

Control of Acoustic Noise (v1)

\[
\frac{\partial p(x, t)}{\partial t} = -\frac{\partial v(x, t)}{\partial x},
\]

\[
\frac{\partial v(x, t)}{\partial t} = -\frac{\partial p(x, t)}{\partial x},
\]

\[v(0, t) = u(t),\]

\[p(L, t) = 0.\]
Modeling of Actuator/Sensors

Control of Acoustic Noise (v2)

\[
\begin{align*}
\frac{\partial p(x, t)}{\partial t} &= -\frac{\partial v(x, t)}{\partial x}, \\
\frac{\partial v(x, t)}{\partial t} &= -\frac{\partial p(x, t)}{\partial x}, \\
A\dot{c}(t) &= \pi a^2 v(0, t), \\
m\ddot{c}(t) + d\dot{c}(t) + kc(t) &= Bu(t) - Ap(0, t).
\end{align*}
\]

Better model includes loudspeaker dynamics
Modeling of Actuator/Sensors

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\]

Better model includes loudspeaker dynamics

Unbounded control/observation operators typically become bounded if actuator dynamics are included.
Optimal Actuator Location Problem

\[ \dot{z}(t) = Az(t) + B(r)u(t), \quad z(0) = z_0 \]

- \( A \) generates a strongly continuous semigroup \( S(t) \) on \( \mathcal{Z} \)
- \( B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{Z}) \)
- \( M \) actuators with locations in some closed and bounded set \( \Omega \subset \mathbb{R}^N \)
- location \( r \) is a vector of length \( M \), \( r \in \Omega^M \)

Optimal Actuator Location Problem

\[
\min_{r \in \Omega^M} \hat{\mu} [r]
\]

for some measure of performance \( \hat{\mu} \).
## Typical Performance Objectives

- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
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- controllability
- linear-quadratic cost (response to initial condition)
- $H_2$-cost (known disturbance)
- $H_\infty$-cost (unknown disturbance)
Controllability

\((A, B(r))\) controllable \(\iff\) solution \(L_c(r)\) to

\[
AL_c(r) + L_c(r)A^* + B(r)B(r)^* = 0
\]

is positive definite.

- Minimum energy to steer \(z(0) = 0 \rightarrow z_f\) is \(z_f^*L_c(r)^{-1}z_f\).
- The most energy required over all targets \(z_f\) is

\[
\sup_{\|z_f\| \leq 1} \|z_f^*L_c(r)^{-1}z_f\| = \lambda_{\text{max}}L_c(r)^{-1}
\]

Maximize \(\hat{\mu}(r) = \lambda_{\text{min}}(L_c(r))\)
Controllability

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\]

Maximize \(\hat{\mu}(r) = \lambda_{min}(L_c(r))\)

Most common criterion in engineering literature
Illustrative Example

**Simply Supported Beam**

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^4 w}{\partial x^4} + c_v \frac{\partial w}{\partial t} + c_d \frac{\partial^5 w}{\partial t \partial x^4} = b_r(x)u(t), \quad 0 < x < 1,
\]

\[
b_r(x) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2}
\end{cases}
\]
Controllability of Beam

![Graph showing controllability of beam as a function of actuator location. The x-axis represents actuator location from 0 to 1, and the y-axis represents controllability from 0 to 0.18. Peaks indicate areas of high controllability.]

- [Graph showing controllability of beam as a function of actuator location. The x-axis represents actuator location from 0 to 1, and the y-axis represents controllability from 0 to 0.18. Peaks indicate areas of high controllability.]
Controllability of Beam

5 modes (--) and 10 modes (-----)
Optimal Controllability vs Approximation Order
Approximate and Exact Controllability

\((A, B(r))\) controllable \(\Leftrightarrow\) solution \(L_c(r)\) to

\[
AL_c(r) + L_c(r)A^* + B(r)B(r)^* = 0.
\]

is positive definite.

- **Exact controllability**: Can steer to any point in \(Z\). Equivalent to \(L_c^{-1}\) bounded.
- **Approximate controllability**: Can steer arbitrarily close to any point. Equivalent to \(L_c^{-1}\) exists; i.e.

\[
L_c = \begin{bmatrix}
1 & 0 & \ldots \\
0 & \frac{1}{2} & \ldots \\
0 & 0 & \frac{1}{3} & \ldots \\
0 & 0 & 0 & \frac{1}{4} & \ldots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{bmatrix}.
\]
Comparision of optimal controllability and LQ-optimal actuator locations
Design Objectives

- controllability
- linear-quadratic cost
- $H_2$-cost
- $H_\infty$-cost
Design Objectives

- controllability
- linear-quadratic cost
- $H_2$-cost
- $H_\infty$-cost

locate actuators as part of controller design
Linear Quadratic (LQ) Control

\[
\inf_{u \in L_2(0, \infty; \mathcal{U})} \int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), u(t) \rangle \, dt
\]

\[
J(u, z_0)
\]

Theorem

If \((A, B(r))\) is stabilizable then there exists a unique \(\Pi \geq 0\) such that for all \(z \in D(A)\),

\[
(\Pi A + A^* \Pi + C^* C - \Pi B(r)B(r)^* \Pi)z = 0,
\]

- Optimal cost \(\inf_{u \in L_2(0, \infty; \mathcal{U})} J(u, z_0) = \langle z_0, \Pi(r)z_0 \rangle\)
- Optimal control \(u(t) = -K(r)z(t)\) where \(K = B^*(r)\Pi(r)\)
LQ-optimal actuator location

Find controller $u$, locations $r$ to

$$\int_0^\infty \langle Cz(t), Cz(t) \rangle + \langle u(t), Ru(t) \rangle dt$$

$$J_r(u,z_0)$$

$$\dot{z}(t) = Az(t) + B(r)u(t), \quad z(0) = z_0.$$ 

For each $r$, $z_0$ optimal cost is $\langle \Pi(r)z_0, z_0 \rangle$ where $\Pi(r)$ solves ARE.
Treatment of Initial Condition in LQ-optimal actuator location

\[ \langle \Pi(r)z_0, z_0 \rangle \]

- For a particular initial condition \( z_0 \),

\[ \hat{\mu}(r) = \langle \Pi(r)z_0, z_0 \rangle \]

- Minimize response to the worst \( z_0 \):

\[ \hat{\mu}(r) = \max_{\|z_0\|=1} \langle \Pi(r)z_0, z_0 \rangle = \| \Pi(r) \| \]

- \( z(0) \) random variable with variance \( V \):

\[ \hat{\mu}(r) = \text{trace} V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} = \| V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \|_1 \]
Worst Initial Condition

**Theorem 1 (Continuity of Performance)**

Let $B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $r \in \Omega^m$, be a family of compact input operators such that for any $r_0 \in \Omega^m$, $u \in \mathcal{U}$,

$$\lim_{r \to r_0} B(r)u = B(r_0)u.$$

Assume that $(A, B(r))$ are all stabilizable and that $(A, C)$ is detectable where $C$ is a compact operator. Then the corresponding Riccati operators $\Pi(r)$ are continuous functions of $r$ in the operator norm:

$$\lim_{r \to r_0} \| \Pi(r) - \Pi(r_0) \| = 0.$$
Proof of Theorem (outline)

\[ \Pi(r) = \int_{0}^{\infty} S_{Kr}(t)^* \left[ C^* C + \Pi(r)B(r)R^{-1}B(r)^*\Pi(r) \right] S_{Kr}(t)dt \]

1. Since \( B(r) \) is continuous, semigroups generated by \( A - B(r)K(r_0) \) bounded by \( M_1 e^{-\alpha_1 t} \), and so
\[
\lim_{r \to r_0} \| \Pi(r)z - \Pi(r_0)z \| = 0, \quad \forall z \in \mathcal{H}.
\]

2. \( S_{Kr}(t) \) converges strongly to \( S_{K0}(t) \), uniformly on bounded intervals of time.

3. Since \( B(r)K(r) \) converges uniformly to \( B(r_0)K(r_0) \), \( S_{Kr}^*(t) \) also converge strongly, uniformly on bounded intervals of time to \( S_{Kr}^*(t) \). Compactness of \( B \Rightarrow \) uniform convergence of \( B(r)^*\Pi(r) \) to \( B(r_0)^*\Pi(r_0) \).

Compactness of $\Omega$ then implies

**Corollary 2**

*With the assumptions of the previous theorem, there exists an optimal actuator location $\hat{r}$ such that*

$$\|\Pi(\hat{r})\| = \inf_{r \in \Omega^m} \|\Pi(r)\| = \hat{\mu}.$$
Example: Optimal actuator location for beam

\[
\frac{\partial^2 w}{\partial t^2} + c_d \frac{\partial w}{\partial t} + \frac{\partial^4 w}{\partial x^4} = b(x)u(t), \quad t \geq 0, \; 0 < x < 1,
\]

\[w(0, t) = 0, \; w_{xx}(0, t) = 0, \; w(1, t) = 0, \; w_{xx}(1, t) = 0.\]

\[b(r) = \begin{cases} 
1/\delta, & |x - r| < \frac{\delta}{2} \\
0, & |x - r| \geq \frac{\delta}{2}
\end{cases}.\]

- reduce state uniformly: \( C = I \)
- eigenfunction approximations
Optimal performance ($||\Pi_n||$), $C = I$

\[
\min || P_N (r) || : C=I
\]

\begin{figure}
\centering
\includegraphics[width=\textwidth]{chart.png}
\end{figure}
Optimal actuator location \((||\Pi_n||)\), \(C = I\)
Calculation of Linear Quadratic Regulator

Operator ARE

\[ A^* \Pi + \Pi A - \Pi BB^* \Pi + C^* C = 0 \]

- Need to approximate solution
- Approximate \( A, B, C \) by \( A_n, B_n, C_n \)
- Solve finite-dimensional ARE for \( \Pi_n \)
- Approximation \( K_n = R^{-1} B_n^* \Pi_n \) used to control original system
Standard Approximation Assumptions

Assume that for each $z \in \mathcal{H}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$,

(A1i) $\|S_n(t)P_nz - S(t)z\| \to 0$ (uniformly on $[0, T]$)

(A2i) $\|B_nu - Bu\| \to 0$, $\|C_nP_nz - Cz\| \to 0$,
Standard Approximation Assumptions for Controller Design

Assume that for each \( z \in \mathcal{H} \), \( u \in \mathcal{U} \), \( y \in \mathcal{Y} \),

(A1i) \( \| S_n(t)P_nz - S(t)z \| \to 0 \) (uniformly on \([0, T]\))

(A1ii) \( \| S_n^*(t)P_nz - S^*(t)z \| \to 0 \) (same)

(A2i) \( \| B_nu - Bu \| \to 0 \), \( \| C_nP_nz - Cz \| \to 0 \),

(A2ii) \( \| B_n^*z - B^*z \| \to 0 \), \( \| C_n^*y - C^*y \| \to 0 \)
Assume that for each $z \in \mathcal{H}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$,

(A1i) $\|S_n(t)P_n z - S(t)z\| \to 0$ (uniformly on $[0, T]$)

(A1ii) $\|S_n^*(t)P_n z - S^*(t)z\| \to 0$ (same)

(A2i) $\|B_n u - Bu\| \to 0$, $\|C_n P_n z - Cz\| \to 0$

(A2ii) $\|B_n^* z - B^* z\| \to 0$, $\|C_n^* y - C^* y\| \to 0$

(A3i) $(A_n, B_n)$ is uniformly exponentially stabilizable:

$\exists K_n \in \mathcal{L}(\mathcal{H}_n, \mathcal{U})$, $\|K_n\| \leq M$,

$\|e^{(A_n-B_nK_n)t}P_n z\| \leq M_1 e^{-\omega_1 t} |z|$

(A3ii) $(A_n, C_n)$ is uniformly exponentially detectable:

$\exists F_n \in \mathcal{L}((\mathcal{Y}, \mathcal{H}_n)$, $\|F_n\| \leq M$,

$\|e^{(A_n-F_nC_n)t}P_n \| \leq M_2 e^{-\omega_2 t}$
Theorem 3 (Convergence of $\Pi_n$)

(Banks, Kunisch, Burns, Ito, 1980’s) If the standard assumptions for controller design hold, then for all $z \in \mathcal{H}$,

$$\|\Pi_n P_n z - \Pi z\| \to 0.$$
Convergence of $LQ$–Optimal Actuator Location

**Theorem 4**

*In addition to the standard assumptions on approximations for controller design, if*$

- $C$ is a compact operator

*then*$

$$\|\Pi(r) - \Pi_n(r)\| \to 0$$

*and*$

$$\hat{\mu} = \lim_{n \to \infty} \hat{\mu}_n,$$

*and there exists a subsequence $\{\hat{r}_m\}$ of $\{\hat{r}_n\}$ such that*$

$$\hat{\mu} = \lim_{m \to \infty} \|\Pi(\hat{r}_m)\|.$$
Optimal performance $\|\Pi_n\|$, state weight is $C = [I \ 0]$
Optimal actuator location, state weight is \( C = [I \ 0] \)
Solution to ARE for analytic semigroups

**Theorem 5 (Lasiecka & Triggiani, 2000)**

If $A$ generates an analytic semigroup, then $\Pi$ is a compact operator and $\|\Pi_n(r) - \Pi(r)\| \to 0$ if standard approximation assumptions satisfied.

- Applies to unbounded control operators $B$ and observation $C$
- Convergence of optimal actuator locations, even for non-compact weight on state
- Examples include diffusion, structures with Kelvin-Voigt damping
Control of Cantilevered Beam

- 70 × 7 × .85cm beam
- 2 actuators, each 7 × 7cm
- optimal locations: 1 and 2
- 4 patches attached; only 2 activated at a time
- 2 laser sensors
Tip Displacement of Cantilevered Beam
Optimal location for 10 actuators on a cantilevered plate

Genetic algorithm

Gradient algorithm
Comparision to Genetic Algorithm

<table>
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<tr>
<th></th>
<th>Objective Value</th>
<th>Elapsed time (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Method</td>
<td>71.98</td>
<td>4.78e2</td>
</tr>
<tr>
<td>GA</td>
<td>72.17</td>
<td>4.14e4</td>
</tr>
</tbody>
</table>

Optimal location of 10 actuators, pinned beam

<table>
<thead>
<tr>
<th></th>
<th>Objective Value</th>
<th>Elapsed time (sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current Method</td>
<td>1.584</td>
<td>4.920e2</td>
</tr>
<tr>
<td>Genetic algorithm</td>
<td>1.748</td>
<td>4.443e4</td>
</tr>
</tbody>
</table>

Optimal location of 10 actuators, cantilevered plate
Random Initial Condition

- If \( z(0) \) is random, with zero mean and variance \( V \) then expected cost is

\[
\text{trace} \left( V \frac{1}{2} \Pi(r) V^\frac{1}{2} \right),
\]

or since \( \Pi \) is self-adjoint and non-negative,

\[
\| \left( V^\frac{1}{2} \Pi(r) V^\frac{1}{2} \right) \|_1
\]

where \( \| \cdot \|_1 \) indicates the nuclear norm.

- Let \( V = I \) to simplify.

**Performance**

\[
\hat{\mu}(r) = \| \Pi(r) \|_1.
\]

(If \( U \) and \( Y \) are both finite-dimensional, then \( \Pi(r) \) is nuclear.)
Random Initial Condition

- If $z(0)$ is random, with zero mean and variance $V$ then expected cost is

$$\text{trace} \left( \frac{1}{2} V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right),$$

or since $\Pi$ is self-adjoint and non-negative,

$$\| \left( V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right) \|_1$$

where $\| \cdot \|_1$ indicates the nuclear norm.

- Let $V = I$ to simplify.

Performance

$$\hat{\mu}(r) = \| \Pi(r) \|_1.$$
Random Initial Condition

- If $z(0)$ is random, with zero mean and variance $V$ then expected cost is

$$\text{trace} \left( V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right),$$

or since $\Pi$ is self-adjoint and non-negative,

$$\| \left( V^{\frac{1}{2}} \Pi(r) V^{\frac{1}{2}} \right) \|_1$$

where $\| \cdot \|_1$ indicates the nuclear norm.

- Let $V = I$ to simplify.

Performance

$$\hat{\mu}(r) = \| \Pi(r) \|_1.$$
Optimal trace $\Pi_n(r)$ for viscously damped beam, State weight $C = I$
Continuity of Performance for Trace Norm

**Theorem 6**

Let $B(r) \in \mathcal{L}(\mathcal{U}, \mathcal{H})$, $r \in \Omega^m$, be a family of input operators such that for any $r_0 \in \Omega^m$, $u \in \mathcal{U}$,

$$\lim_{r \to r_0} B(r)u = B(r_0)u.$$ 

Assume that $(A, B(r))$ are all stabilizable and that $(A, C)$ is detectable where $C \in \mathcal{L}(\mathcal{H}, \mathcal{Y})$. If $\mathcal{U}$ and $\mathcal{Y}$ are finite-dimensional, then

$$\lim_{r \to r_0} \|\Pi(r) - \Pi(r_0)\|_1 = 0.$$
Proof of Theorem (outline)

Defining $C_r(t) = \begin{bmatrix} C \\ R^{1/2}K_r \end{bmatrix} S_r(t)$, $\Pi(r) = \int_0^\infty C_r(t)^*C_r(t)dt$.

1. As for norm convergence, the semigroups $S_r(t)$ generated by $A - B(r)K(r)$ are uniformly exponentially bounded, $S_r(t)$ converges strongly to $S_0(t)$, uniformly on bounded intervals of time, and similarly for $S_r^*(t)$.

2. Defining $M = \dim(\mathcal{Y} + \mathcal{U})$, $c_{ri}(t) = S_r(t)^* \begin{bmatrix} C^* & K_r^*R^{1/2} \end{bmatrix} e_i$,

$$\|C_0 - C_r\|^2_{HS} = \sum_{i=1}^{M} \int_0^\infty \|c_{0i}(t) - c_{ri}(t)\|^2_{\mathcal{H}} dt.$$  

Since $\|c_{0i}(t) - c_{ri}(t)\|_{\mathcal{H}} \to 0$ uniformly on bounded intervals of time, $\|S_r(t)\| \leq Me^{-\alpha t}$, $\|C_0 - C_r\|_{HS} \to 0$.

3. $\Pi(r) = C_r^*C_r \Rightarrow \Pi(r)$ converges to $\Pi(r_0)$ in nuclear norm.
Optimal Damping

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0, \\
w(0, t) = 0, w(1, t) = 0.
\]
Optimal Damping

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + a(x) \frac{\partial w}{\partial t} = 0,
\]

\[w(0, t) = 0, w(1, t) = 0.\]

- need to minimize a weakly lower semi-continuous objective function over weakly compact set
- non-convergence of approximating solutions
- decay rate yields upper bound on energy of system
- considering any \(a(\cdot) \in L_2(0, 1)\), arbitrarily large decay rate can be achieved. [Cox&Zuazua, 1994], [Castro&Cox, 2001]
Localized Damping

\[ \frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + 2k \chi_\omega(x) \frac{\partial w}{\partial t} = 0, \]

\[ w(0, t) = 0, w(1, t) = 0. \]

Choose \( \omega \subset [0, 1] \) total length \( \ell \) to maximize decay rate of system.
Localized Damping

\[
\frac{\partial^2 w}{\partial t^2} + \frac{\partial^2 w}{\partial x^2} + 2k \chi_\omega(x) \frac{\partial w}{\partial t} = 0,
\]

\[w(0, t) = 0, w(1, t) = 0.\]

Choose \(\omega \subset [0, 1]\) total length \(\ell\) to maximize decay rate of system

- spectral abscissa of generator \(A\) is good measure of decay rate for reasonably small \(k\).
- if \# intervals not constrained, no optimum (unless \(\ell = 0, 1, \frac{1}{2}\)) \cite{HebrardHenrot03,PrivatTrelatZuazua13,PrivatTrelatZuazua15}
- maximizing observability & controllability yields similar results
- optimum for \(N\) modes is at node of \(N + 1\)st and is bad choice
  Spillover! \cite{HebrardHenrot05}
Reducing Response to Disturbances

\[ \dot{z}(t) = Az(t) + B_1d(t) + B(r)u(t), \quad z(0) = 0 \]

- \( B_1 \in \mathcal{L}(\mathcal{V}, \mathcal{Z}) \)
- full information: input to the controller is \([z(t) \ v(t)]\)
- known (white noise) or unknown disturbance \(d\)

**Cost with known disturbance**

\[ \mu(r) = \sqrt{\int_{0}^{\infty} \|y_1(t)\|^2 dt} = \|y\|_{H_2}, \quad y_1 = Cz(t) + E_{12}u(t) \]

- If \( E_{12}^*C = 0 \) cost is identical to LQ cost with \( R = E_{12}^*E_{12} \)
- Typical: \( C = \begin{bmatrix} C_1 \\ 0 \end{bmatrix}, \quad E_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix} \)
Theorem 7 (\(H_2\)-optimal control)

If \(B_1\) is a Hilbert-Schmidt operator, the \(H_2\)-optimal control is the state feedback

\[
u(t) = -B^*(r)\Pi(r)z(t)
\]

where \(\Pi(r)\) solves an ARE and optimal cost

\[
\hat{\mu}(r) = \sqrt{Tr(B_1^*\Pi(r)B_1)}.
\]

If there is a single disturbance, so \(B_1d\) can be written \(b_1d\), \(b_1 \in \mathbb{Z}\),

\[
\hat{\mu}(r) = \sqrt{\langle b_1, \Pi(r)b_1 \rangle}.
\]

\(H_2\)-optimal actuator location

\[
\min_{r \in \Omega^M} Tr(B_1^*\Pi(r)B_1).
\]
Theorem 8 (Well-posedness of $H_2$-optimal actuator location)

Assume for any $r \in \Omega^M$,

$$\lim_{r \to r_0} \| B(r) - B(r_0) \| = 0.$$

and the standard approximation assumptions are satisfied for each $(A_n, [B_{1,n} B_n(r)], C_n)$, and that $B_1$ is Hilbert-Schmidt. Then the approximating optimal costs converge to the exact optimal cost, that is

$$\inf_{r \in \Omega^M} Tr(B_1^* \Pi(r) B_1) = \lim_{n \to \infty} \inf_{r \in \Omega^M} Tr(B_{1,n}^* \Pi_n(r) B_{1,n}),$$

There is a subsequence of approximating actuator locations $\hat{r}_n$ so

$$\hat{\mu} = \lim_{n \to \infty} Tr(B_{1,n}^* \Pi(\hat{r}_n) B_{1,n}).$$
$H_2$-cost for beam: various weights

Local disturbance at $x = 0.3$
$H_2$-cost for beam: various weights

Local disturbance at $x = 0.3$
Cost of Control Signal at Different Actuator Locations

$L_2$-norm

$L_{\infty}$-norm

Optimal location improves performance without increasing cost
Optimal actuator location for various disturbances on cantilevered beam

\[
C = C_{\text{tip}}
\]

<table>
<thead>
<tr>
<th>( \max b_1(\xi)/L )</th>
<th>( \hat{\xi}/L )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
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<td>0.50</td>
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</tr>
<tr>
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<td>0.4475</td>
</tr>
<tr>
<td>1</td>
<td>0.9950</td>
</tr>
</tbody>
</table>
Diffusion

\[ \Omega \]

\[ \kappa(x_1, x_2) \]
Disturbance location and optimal actuator locations, cost $C = I$
Minimization over all spatial disturbances $b_1(x)$

Minimize the $H_2$-cost over all disturbance shapes:

$$\inf_{r \in \Omega^M} \sup_{b_1 \in \mathbb{Z}} \langle b_1, \Pi(r)b_1 \rangle = \inf_{r \in \Omega^M} \| \Pi(r) \|.$$ 

Worst disturbance shape is eigenfunction of $\lambda_{max} \Pi(r)$.

Worst disturbance for $r = .509L, r = .995L$ on cantilevered beam.
Sensor Location

\[
\dot{z}(t) = Az(t) + Bu(t) + B_1 d(t) \\
y(t) = Cz(t) + E_{12}d(t) \quad \Leftarrow \text{Cost} \\
y_{\text{sen}}(t) = C_2(s)z(t) + E_{21}d(t)
\]

- \( M \) sensors with locations in some compact set \( \Omega \subset R^n \).

- Estimation is dual to control

- Research in 1970’s on finite-time optimal estimation (Bensoussan, Curtain, Ichikawa ...)

- LQ output feedback: control and estimation entirely decoupled.

- \( H_2 \) output feedback: decoupled calculation, but coupled cost:

\[
\min \| G_{d,y} \|_2^2 = \text{trace}(B_1 \Pi B_1) + \text{trace}(B \Pi X B)
\]

- \( H_\infty \) output feedback: both calculation and cost are coupled.
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$H_2$-Output Feedback - Simply Supported Beam

Disturbance and Actuator Shape

$H_2$ Output Feedback Cost vs Sensor Position

Sensor Position on Beam

$H_2$ Output Feedback Cost

Sensor Position on Beam
$H_2$-Output Feedback - Simply Supported Beam

Disturbance and Actuator Shape

$H_2$ Output Feedback Cost vs Sensor Position

Sensor Position on Beam
$H_2$-Output Feedback - Simply Supported Beam

**Disturbance and Actuator Shape**

**$H_2$ Output Feedback Cost vs Sensor Position**

Optimal sensor location not affected by actuator location
$H_2$-Output Feedback - Simply Supported Beam (larger control cost)

Disturbance and Actuator Shape

Control operator $B$ scaled by 1000
$H_2$-Output Feedback - Simply Supported Beam (larger control cost)

**Disturbance and Actuator Shape**

**$H_2$ Output Feedback Cost vs Sensor Position**

Control operator $B$ scaled by 1000
$H_2$-Output Feedback - Simply Supported Beam (larger control cost)

Disturbance and Actuator Shape

$H_2$ Output Feedback Cost vs Sensor Position

Control operator $B$ scaled by 1000
$H_2$-Output Feedback - Simply Supported Beam

Comparision of Optimal Actuator/Sensor Pairs
Summary

- Locate actuators as part of controller design
- Compactness is critical for well-posedness of problem and for validity of calculations
- Numerics an issue: even if controllers designed with fixed actuator location converge, actuator locations may not
- Problem needs to be properly formulated
- Optimal location depends on cost criterion
- Optimal location not always intuitive
Some Open Problems

- Efficient procedure for calculating $\mathcal{H}_\infty$-optimal locations
- Incorporating statistical information about hardware, model
- Robustness
- Optimal shape of actuator/sensor
- Nonlinearities
REFERENCES


