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Robust Synchronization by Open-Loop Control

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***Abstract*— In this paper we consider networks of identical linear time-invariant single-input-single-output systems. We consider the situation in which the couplings between the individual systems depend on a parameter varying over compact real interval. We examine a family of circularly interconnected harmonic oscillators. Also, we consider a family of systems that are arranged in a row. We show that in both cases there exists a broadcasted parameter-independent open-loop control input steering all individual system towards a predefined parameter-dependent family of terminal states. The investigation of this problem leads naturally to the analysis of one-parameter dependent linear systems and, in particular, uniform ensemble controllability. To this end, we generalize existing results on uniform ensemble controllability of parameter-dependent single-input linear systems so that the parameter space can be any compact subset of the real line.**

Keywords - *Families of networks of linear systems, ensemble controllability, synchronization, complex approximation.*

MSC 2010 - 30E10, 34D06, 93B05

I. INTRODUCTION

The analysis of networks of systems has reached considerable attraction in recent years. The core of network theory is the interplay between graph theory and continuous or discrete dynamic equations. The vertices in the graph represent the various systems whose evolution is governed by differential or difference equations. The set of edges in the directed or undirected graphs render the interconnection structure among the single systems. As a consequence, the characteristics of the behavior of the network's state depends on one hand on the network topology, which is naturally given in terms of a graph. On other hand the behavior of the network is determined by dynamic equations describing the evolution of the vertices.

A topic which has been studied intensively is controllability of the overall network, cf. [3], [9], [10], [11], [15], [16]. Another objective which has widely been studied is the synchronization problem, cf. [6], [12]. That is, given a set of vertices with maybe identical systems the task is to derive conditions on the information exchange so that the interconnected system converge towards a common trajectory. In [14] the robust synchronization problem was studied, where the dynamics in the vertices is subject to additive perturbations which are bounded by a priori given tolerance. In this work the authors use output feedback and the information exchange is described by the Laplace matrix in order to achieve synchronization, where the Laplace matrix corresponds to an undirected graph or a directed graph containing a spanning tree. Moreover, in the context of synchronization of (non-)linear systems the pinning control technique was also used. The essence of pinning control is the usage of feedback control to a certain subset of vertices and the feedback law contains the difference to the given synchronizing trajectory, cf. [2].

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In this paper, we also consider the robust synchronization problem, but we focus on the situation where the interconnection structure is fixed but the coupling strength is uncertain and varies over compact interval. The networks under consideration consist of N identical single-input-single-output (SISO) systems $\Sigma = (A, b, c)$ defined by

$$\dot{x}(t) = A\,x(t) + b\,v(t)$$
$$y(t) = c\,x(t),$$

where $A \in \mathbb{R}^{n\times n}, b \in \mathbb{R}^n$ and $c \in \mathbb{R}^{1\times n}$. Throughout the paper we consider systems Σ that are controllable and observable. The systems are connected by directed links and the interconnection is described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ denotes the set of vertices, i.e. the N SISO systems, and $\mathcal{E}$ describes the set of edges, i.e. the couplings. The network topology is represented by the adjacency matrix which will be denoted by $\mathcal{A} \in \mathbb{R}^{N\times N}$. The links are weighted and uncertain. That is, the topology of the network does not change and the weights are not known precisely but depend on a parameter η varying over a given compact interval $\mathbf{P} \subset \mathbb{R}_+ = (0,\infty)$. The adjacency matrix is given by

$$\mathcal{A}(\eta) = \begin{cases} k_{ij}(\eta) & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{else,} \end{cases}$$

where $k_{ij}\colon \mathbf{P} \to (0,\infty)$ is continuous if $(i,j) \in \mathcal{E}$. Thus,

$$\mathcal{A}_{\mathbf{P}} = \{\mathcal{A}(\eta)\,:\,\eta \in \mathbf{P}\}$$

defines a family of adjacency matrices. Moreover, there is also an external input u which is distributed to the systems Σ via the input-to-state interconnection vector $\mathcal{B} \in \mathbb{R}^N$. The overall network can be described by the following dynamic equation

$$\begin{aligned} \dot{x}(t) &= \big(I \otimes A + \mathcal{A}(\eta) \otimes bc\big)x(t) + \big(\mathcal{B} \otimes b\big)u(t) \\ x(0,\eta) &= \mathbf{1} \otimes x^0, \end{aligned} \tag{1}$$

where $\mathbf{1} = (1 \cdots 1)^\top \in \mathbb{R}^N$ and $x^0 \in \mathbb{R}^n$ denotes the initial state of the system Σ. The triple $(\Sigma, \mathcal{A}_{\mathbf{P}}, \mathcal{B})$ defines a family of networks. We emphasize that the interconnection of the individual system in the dynamics (1) is given in terms of the adjacency matrices and not in terms of the Laplace matrices. The setting is illustrated in Figure 1.

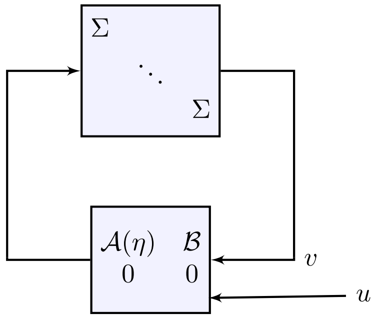

Fig. 1. Block diagram of the network.

We note that the structure of (1) is a special case of the general setting described in [3]. Throughout the paper we denote by $x(T, \mathcal{A}(\eta), \mathbf{1} \otimes x^0)$ the solution to (1) at time $T > 0$ starting in $\mathbf{1} \otimes x^0$ under the interconnection $\mathcal{A}(\eta)$. The problem we are addressing in the sequel is the following: Given a family of desired terminal state $x^* \in C(\mathbf{P}, \mathbb{R}^n)$ we aim to steer the whole family of networks $(\Sigma, \mathcal{A}_{\mathbf{P}}, \mathcal{B})$ form $\mathbf{1} \otimes x^0$ to an ε-neighborhood of the desired network state $\mathbf{1} \otimes x^* \in C(\mathbf{P}, \mathbb{R}^{nN})$ by applying an open-loop control input function $u\colon [0,T] \to \mathbb{R}$ independently of the coupling strength $\eta \in \mathbf{P}$. This is formalized in the following definition.

Definition 1: The family of networks $(\Sigma, \mathcal{A}_{\mathbf{P}}, \mathcal{B})$ is called robustly synchronizable from $\mathbf{1} \otimes x^0 \in \mathbb{R}^{nN}$ to $\mathbf{1} \otimes x^* \in C(\mathbf{P}, \mathbb{R}^{nN})$ if for every $\varepsilon > 0$ there is a $T > 0$ and an input-function $u\colon [0,T] \to \mathbb{R}$ such that

$$\sup_{\eta \in \mathbf{P}} \| x(T, \mathcal{A}(\eta), \mathbf{1} \otimes x^0) - (\mathbf{1} \otimes x^*(\eta)) \| < \varepsilon.$$

We emphasize that the input u is broadcasted to the systems Σ within the network according to the input-to-state vector $\mathcal{B}$. This phenomenon may be interpreted as that u serves as an universal input for a class of networks that steers the initial state $\mathbf{1} \otimes x^0$ to a desired synchronized state $\mathbf{1} \otimes x^*$ in finite time T uniformly for all $\eta \in \mathbf{P}$ and interconnection matrices $\mathcal{A}(\eta)$. It seems surprising that a single input function exists that robustly synchronizes states for the whole family of networks. The analysis of this problem is based on the concept of uniform ensemble controllability, which is the content of the subsequent section.

II. ENSEMBLE CONTROLLABILITY

In this section we consider parameter-dependent linear time-invariant systems of the form

$$\begin{aligned} \tfrac{\partial}{\partial t} x(t,\theta) &= A(\theta)x(t,\theta) + b(\theta)u(t) \\ x(0,\theta) &= x^0(\theta), \end{aligned} \tag{2}$$

where $A(\theta) \in \mathbb{R}^{n \times n}$, $b(\theta) \in \mathbb{R}^n$ and $x^0(\theta)$ are assumed to depend continuously on a single parameter $\theta \in \mathbf{P} \subset \mathbb{R}$. Let $x(T,\theta)$ denote the solution to (2) at time $T > 0$. In the following, let $x^*\colon \mathbf{P} \to \mathbb{R}^n$ denote a given family of desired terminal states. For uniform ensemble controllability it is natural to assume that $x^*(\theta)$ depends continuously on the parameter. Thus, we assume in the following that x^* is continuous. Our aim is to investigate the existence of an open-loop input function $u\colon [0,T] \to \mathbb{R}$ that steers the initial states $x(0,\theta) = x^0(\theta)$ in time T into an ε-neighborhood of the desired terminal state $x^*(\theta)$, simultaneously for all parameters $\theta \in \mathbf{P}$. This is in principle what we call uniform ensemble controllability. The precise definition is as follows.

Definition 2: System (2) is called *uniformly ensemble controllable* if for any $x^* \in C(\mathbf{P}, \mathbb{R}^n)$ and for any $\varepsilon > 0$ there exists a $T > 0$ and an input function $u\colon [0,T] \to \mathbb{R}$ such that

$$\sup_{\theta \in \mathbf{P}} \| x(T,\theta) - x^*(\theta) \| < \varepsilon. \tag{3}$$

If $\mathbf{P} \subset \mathbb{R}$ is compact and connected sufficient conditions for uniform ensemble controllability have been shown in [5]. As this result holds also for discrete-time systems, it is assumed that the individual systems are zero, i.e. $x(0,\eta) = 0$ for all $\eta \in \mathbf{P}$. In the following result we will show that this result can be generalized to disconnected compact sets $\mathbf{P} \subset \mathbb{R}$. In addition, we show that in continuous time the initial conditions are also allowed to depend continuously on the parameter. Although the verification is close to the one in [13] we will proof the result as in Example 1 we will partly make use of it.

Theorem 1: Let $\mathbf{P} \subset \mathbb{R}$ be compact. A continuous family $(A(\theta), b(\theta))$ of linear systems (2) is uniformly ensemble controllable in arbitrary time $T > 0$ provided the following conditions are satisfied:

(i) $(A(\theta), b(\theta))$ is controllable for all $\theta \in \mathbf{P}$.

(ii) For any pair of distinct parameters $\theta, \theta' \in \mathbf{P}, \theta \neq \theta'$, the spectra of $A(\theta)$ and $A(\theta')$ are disjoint:

$$\sigma(A(\theta)) \cap \sigma(A(\theta')) = \emptyset.$$

(iii) For each $\theta \in \mathbf{P}$, the eigenvalues of $A(\theta)$ have algebraic multiplicity one.

Proof: Let $\mathbf{P}$ be disconnected compact subset in $\mathbb{R}$ and $\varepsilon > 0$ and $T > 0$ be fixed. According to the assumptions on controllability and assuming in addition that $A(\theta)$ has distinct eigenvalues we consider w.l.o.g. a family of single-input systems of the form

$$A(\theta) = \begin{bmatrix} a_1(\theta) & & \\ & \ddots & \\ & & a_n(\theta) \end{bmatrix} \quad \text{and} \quad b(\theta) = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}.$$

Then, the corresponding dynamic equations are

$$\begin{aligned} \tfrac{\partial}{\partial t} x_1(t,\theta) &= a_1(\theta)x_1(t,\theta) + u(t) \\ &\;\vdots \\ \tfrac{\partial}{\partial t} x_n(t,\theta) &= a_n(\theta)x_n(t,\theta) + u(t), \end{aligned} \tag{4}$$

with continuous and injective functions $a_1, \cdots, a_n\colon \mathbf{P} \to \mathbb{C}$, cf. Chapter II, § 5, 3 in [7]. The solution to (4) with input function u at time $T>0$ is given by

$$x(T,\theta)=\begin{bmatrix}\mathrm{e}^{a_1(\theta)T}\,x_1^0(\theta)+\int_0^T \mathrm{e}^{sa_1(\theta)}\,u(T-s)\,\mathrm{d}s\\ \vdots\\ \mathrm{e}^{a_n(\theta)T}\,x_n^0(\theta)+\int_0^T \mathrm{e}^{sa_n(\theta)}\,u(T-s)\,\mathrm{d}s\end{bmatrix}. \quad (5)$$

Note that, since we applied a coordinate transformation to diagonalize the system matrix $A(\theta)$, the initial states $x^0(\theta)$ as well as the terminal states $x^*(\theta)$ have to be transformed accordingly and might become complex in general. For simplicity, we do not introduce a new notation for the transformed system. Moreover, we consider piecewise constant input functions. To this end, we divide $[0,T]$ into $K\in\mathbb{N}$ intervals of length $\tau>0$ so that the mappings $\theta\mapsto \mathrm{e}^{\tau\, a_j(\theta)}$ are injective for all $j=1,...,n$. That is, we have

$$[0,T]=\bigcup_{k=0}^{K} I_k, \quad (6)$$

where $I_k=[k\tau,(k+1)\tau)$ for $k=0,...,K-1$ and $I_K=[(K-1)\tau,T]$. The piecewise constant input function $u\colon[0,T]\to\mathbb{R}$ is then given by

$$u|_{I_k}(t):=u_k \quad (7)$$

for some real coefficients $u_0,...,u_K$. Also, let $\mathbb{1}_{I_k}$ denote the indicator function defined by

$$s\mapsto \mathbb{1}_{I_k}(s)=\begin{cases}1 & \text{if } s\in I_k\\ 0 & \text{else}\end{cases}.$$

The input function u is then of the form

$$u(T-s)=\sum_{k=0}^{K}u_k\,\mathbb{1}_{I_k}(T-s)=\sum_{k=0}^{K}u_{K-k}\,\mathbb{1}_{I_k}(s).$$

If $a_j(\theta)\neq 0$ the jth component of the solution to (5) at time $T>0$ is

$$\begin{aligned}x_j(T,\theta)&=\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)+\int_0^T \mathrm{e}^{a_j(\theta)s}\,u(T-s)\,\mathrm{d}s\\ &=\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)+\sum_{k=0}^{K}\int_{k\tau}^{(k+1)\tau}\mathrm{e}^{a_j(\theta)s}\,u_{K-k}\mathbb{1}_{I_k}(s)\,\mathrm{d}s\\ &=\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)+\sum_{k=0}^{K}\frac{u_{K-k}}{a_j(\theta)}\,\mathrm{e}^{k\tau a_j(\theta)}\left(\mathrm{e}^{\tau a_j(\theta)}-1\right)\\ &=\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)+\left(\frac{\mathrm{e}^{\tau a_j(\theta)}-1}{\tau a_j(\theta)}\right)\sum_{k=0}^{K}\tau u_{K-k}\,\mathrm{e}^{k\tau a_j(\theta)}\end{aligned}$$

If $a_j(\theta)=0$, we have

$$\begin{aligned}x_j(T,\theta)&=x_j^0(\theta)+\sum_{k=0}^{K}\int_0^{\tau}u_{K-k}\mathbb{1}_{I_k}(k\tau+s)\,\mathrm{d}s\\ &=x_j^0(\theta)+\sum_{k=0}^{K}\tau\,u_{K-k}.\end{aligned}$$

For any $\varepsilon>0$ there is a $\tau^*>0$ so that for any $\tau\in(0,\tau^*)$ we have

$$\left|\frac{\mathrm{e}^{\tau a_j(\theta)}-1}{\tau a_j(\theta)}-1\right|<\tfrac{\varepsilon}{4} \quad (8)$$

for any $a_j(\theta)\neq 0$, $j=1,...,n$. Let $K^*:=\left\lfloor\frac{T}{\tau^*}\right\rfloor$, where $\lfloor x\rfloor=\max\{z\in\mathbb{Z}:z\le x\}$. Furthermore, it holds

$$\lim_{\tau\to 0}\frac{\mathrm{e}^{\tau z}-1}{\tau z}=1 \qquad \text{for all } z\in\mathbb{C}\setminus\{0\}.$$

We emphasize that

$$\sum_{k=0}^{K}u_{K-k}\,\mathrm{e}^{k\tau a_j(\theta)}$$

defines a polynomial whose coefficients are exactly the values of the input function u defined in (7). That is, for $z\in\mathbb{C}$ we define

$$p(z):=\sum_{k=0}^{K}u_{K-k}\,z^k. \quad (9)$$

In terms of polynomial p the jth component of the solution to (4) reads as

$$x_j(T,\theta)=\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)+\tau\left(\frac{\mathrm{e}^{\tau a_j(\theta)}-1}{\tau a_j(\theta)}\right)p\left(\mathrm{e}^{\tau a_j(\theta)}\right)$$

if $a_j(\theta)\neq 0$ and

$$x_j(T,\theta)=x_j^0(\theta)+\tau\,p(1)$$

if $a_j(\theta)=0$. By (8), it holds

$$\left|\tau p(\mathrm{e}^{\tau a_j(\theta)})-\left(x_j(T,\theta)-\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)\right)\right|<\tfrac{\varepsilon}{4} \quad (10)$$

for all $\tau\in(0,\tau^*)$. The significance of (10) is that the verification of the assertion can be traced back to the problem of approximating a continuous function uniformly by a polynomial.

For $j\in\{1,...,n\}$ let $\Omega_j:=\{\mathrm{e}^{\tau a_j(\theta)}:\theta\in\mathbf{P}\}\subset\mathbb{C}$ and $\Omega=\bigcup_{j=1}^{n}\Omega_j$. Note that, by the assumptions of Theorem 1 and by the choice of τ, the set Ω is compact with empty interior and $\mathbb{C}\setminus\Omega$ is connected. Moreover, let the modified family of terminal states $\widetilde{x}^*\in C(\mathbf{P},\mathbb{C}^n)$ be defined by

$$\widetilde{x}_j^*(\theta):=x_j^*(\theta)-\mathrm{e}^{a_j(\theta)T}\,x_j^0(\theta)$$

and consider the continuous function $f\colon\Omega\to\mathbb{C}$ defined by

$$f|_{\Omega_j}(z)=\widetilde{x}_j^*\left(a_j^{-1}\left(\tfrac{\ln z}{\tau}\right)\right).$$

By Mergelyan's theorem (cf. [4] Theorem 1 in Chapter II 2) there exists a polynomial p with degree $M\in\mathbb{N}$ so that

$$|f(z)-p(z)|<\tfrac{\varepsilon}{4} \quad (11)$$

for all $z\in\Omega$. Note that $p(z)=c_0+c_1z+\cdots+c_Mz^M$ is a polynomial with complex coefficients in general. In order to obtain a polynomial which has real coefficients, we consider the polynomial

$$q(z)=\frac{p(z)+\overline{p}(z)}{2}=\mathrm{Re}(c_0)+\mathrm{Re}(c_1)z+\cdots+\mathrm{Re}(c_M)z^M.$$

If $M \geq K^*$ we choose the coefficients in (9) as

$$u_0 = \mathrm{Re}(c_M) \quad u_1 = \mathrm{Re}(c_{M-1}) \quad \cdots \quad u_M = \mathrm{Re}(c_0).$$

If $M < K^*$ we pick $u_0 = \cdots = u_{K^*-M-1} = 0$ and

$$u_{K^*-M} = \mathrm{Re}(c_M) \qquad \cdots \qquad u_K^* = \mathrm{Re}(c_0).$$

Then, in either case (10) and (11) imply that for any $\theta \in \mathbf{P}$ and for any $j \in \{1, ..., n\}$ it holds

$$\left|x_j^*(\theta) - x_j(T,\theta)\right| \leq \left|\widetilde{x}_j^*(\theta) - \tau q(\mathrm{e}^{\tau\lambda_j(\theta)})\right| + \left|\tau q(\mathrm{e}^{\tau\lambda_j(\theta)}) - \left(x_j(T,\theta) - \mathrm{e}^{a_j(\theta)T}\, x_j^0(\theta)\right)\right| < \varepsilon.$$

Consequently, given $x^*(\theta)$, $T > 0$ and $\varepsilon > 0$ there exist real values $u_0, ..., u_{\max\{M^*,K^*\}}$ defining a piecewise constant input function u so that

$$\sup_{\theta \in \mathbf{P}} \|x^*(\theta) - x(T,\theta)\| < \varepsilon.$$

This shows the assertion. ■

We note that in [5] it is shown that under the assumption of Theorem 1 uniform ensemble controllability holds also for discrete-time systems. However, in discrete-time the family of initial states has to be zero for the all parameters, i.e. $x(0,\theta) = 0$. To make this intuitively clear, notice that in continuous-time by using piecewise constant control inputs, fixing the terminal time $T > 0$ and subdividing the interval $[0,T]$ with step size $\tau > 0$ any approximation error $\varepsilon > 0$ can be achieved by reducing the step size τ in the subdivision the interval $[0,T]$.

III. RINGS OF HARMONIC OSCILLATORS

In this section we consider N identical single-input-single-output (SISO) harmonic oscillators $\Sigma_{ho} = (A,b,c)$ defined by

$$A := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{12}$$

The harmonic oscillators Σ_{ho} are connected and the coupling strength η is assumed to vary over a compact interval $\mathbf{P} := [\eta^-, \eta^+]$. The oscillators are arranged so that they form a ring. The network topology can be described a directed graph Γ with N nodes and the weighted adjacency matrix $\mathcal{A}(\eta)$ is given by

$$\mathcal{A}_{\text{ring}}(\eta) := \begin{bmatrix} 0 & \eta & & \eta \\ \eta & \ddots & \ddots & \\ & \ddots & \ddots & \eta \\ \eta & & \eta & 0 \end{bmatrix}, \quad \eta \in \mathbf{P}. \tag{13}$$

The network is depicted in Figure 2. In the following we aim to investigate the question whether it is possible to steer the oscillators from a given initial state $x^0 \in \mathbb{R}^2$ to any state $\bar{x}^* \in \mathbb{R}^2$ robustly using an external open-loop control input u which is *independent* of the coupling strength $\eta \in \mathbf{P}$. In addition, this goal should be achieved by applying the external input to a single harmonic oscillator.

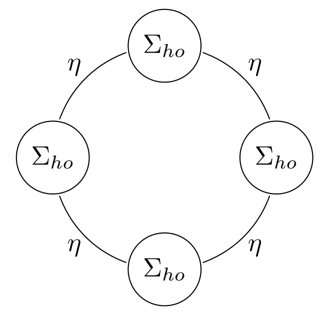

Fig. 2. Ring of 4 identical harmonic oscillators.

Without loss of generality, let the numbering of the harmonic oscillators be such that the external input is applied to the first harmonic oscillator. Thus, the input-to-state interconnection vector is $\mathcal{B} = e_1 = (1, 0, ..., 0)^\top$. The dynamics of the overall network of systems is of the form

$$\begin{aligned} \tfrac{\partial}{\partial t} x(t,\eta) &= \big(I \otimes A + \mathcal{A}_{\text{ring}}(\eta) \otimes bc\big)x(t) + \big(e_1 \otimes b\big)u(t) \\ x(0,\eta) &= \mathbf{1} \otimes x^0, \end{aligned}$$

where $x^0 \in \mathbb{R}^2$ denotes the initial state of the harmonic oscillators. In addition, we denote by $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ the desired terminal states of the harmonic oscillators.

Corollary 1: For any $x^0 \in \mathbb{R}^2$ and for any $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ the family of rings of harmonic oscillators $(\Sigma_{ho}, \mathcal{A}_{\text{ring}}(\mathbf{P}), e_1)$ with $\mathbf{P} \subset (0,1)$ is robustly synchronizable from $\mathbf{1} \otimes x^0$ to $\mathbf{1} \otimes x^*$.

Proof: First, observe that the adjacency matrix is a circulant matrix defined by the vector $v = (0, \eta, 0, ..., \eta)^\top$. From [8] we quote that the spectrum of $\mathcal{A}(\eta) = \mathrm{circ}(0, \eta, 0, ..., \eta)$ is

$$\sigma(\mathcal{A}(\eta)) = \left\{ \eta \left(\mathrm{e}^{2\pi i \frac{l}{N}} + \mathrm{e}^{2\pi i \frac{(N-1)l}{N}} \right),\ l = 0, ..., N-1 \right\}.$$

Using trigonometric identities for $\sin$ and $\cos$ the spectrum of the family of adjacency matrices is

$$\sigma(\mathcal{A}(\eta)) = \{\lambda_l(\eta) := \eta \cos\left(2\pi \tfrac{l}{N}\right),\ l = 0, ..., N-1\}.$$

Let $\epsilon_l := \mathrm{e}^{2\pi i \frac{l}{N}}$ the family $\{\mathcal{A}(\eta) : \eta \in \mathbf{P}\}$ is simultaneously diagonalizable using the matrix

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & \epsilon_1 & \cdots & \epsilon_{N-1} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & \epsilon_1^{N-1} & \cdots & \epsilon_{N-1}^{N-1} \end{bmatrix}. \tag{14}$$

Hence, applying the change of coordinates $S \otimes I$ and since $S\mathbf{1} = Ne_1$, the network dynamics is given by

$$\begin{aligned} \tfrac{\partial}{\partial t} x(t,\eta) &= \mathrm{diag}\left(A + \lambda_l(\eta) bc\right) x(t) + (\mathbf{1} \otimes b)\, u(t) \\ x(0,\eta) &= Ne_1 \otimes x^0. \end{aligned} \tag{15}$$

The corresponding desired terminal states are $Ne_1 \otimes x^*(\eta) \in C(\mathbf{P}, \mathbb{R}^{2N})$. We show that Theorem 1 can be applied to (15). As the spectrum of $\operatorname{diag}(A + \lambda_l(\eta)bc)$ is given by

$$\bigcup_{l=1}^{N} \left\{ w \in \mathbb{R} : w^2 - (\eta \cos(2\pi l/N) - 1) = 0 \right\} \tag{16}$$

we have

$$\sigma(A + \lambda_l(\eta)bc) \cap \sigma(A + \lambda_k(\eta)bc) = \emptyset$$

for all $\eta \in \mathbf{P}$ and for all $l \neq k \in \{1, ..., N\}$. In addition, as for all $\eta \in \mathbf{P}$ and all $l \in \{1, ..., n\}$ the pair $(A + \lambda_l(\eta)bc, b)$ is controllable we conclude that the parallel connection (15) is controllable. Furthermore, by inspection it follows from (16) that the conditions (ii) and (iii) hold. Finally, we apply Theorem 1 to (15) and conclude that for any $\varepsilon > 0$, $T > 0$ and for any continuous family $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ there exists an input function $u\colon [0, T] \to \mathbb{R}$ such that

$$\sup_{\eta \in \mathbf{P}} \| x(T, \mathcal{A}(\eta), \mathbf{1} \otimes x^0) - (\mathbf{1} \otimes x^*(\eta)) \| < \varepsilon.$$

This shows the assertion. ■

Subsequently, we consider the task of determining the control input values of a piecewise constant input function u such that the corresponding state trajectory steers the initial states into an ε-neighborhood of the terminal state x^*.

Example 1: Let $N = 4$, $\mathbf{P} := [\delta, 1 - \delta]$, $\delta \in (0, \frac{1}{2})$ and $x^*(\eta) := \mathbf{1} \in \mathbb{R}^2$. After applying the change of coordinates $S \otimes I$ with S given in (14), the evolution of lth subsystem Σ is governed by a dynamic equation of the form

$$\begin{aligned} \tfrac{\partial}{\partial t} z_l(t, \eta) &= \begin{bmatrix} 0 & -1 + \lambda_{l-1}(\eta) \\ 1 & 0 \end{bmatrix} z_l(t, \eta) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ z_l(0, \eta) &= \mathbb{1}_{(0,\infty)}(\lambda_{l-1}(\eta))\, 4\, x^0, \end{aligned} \tag{17}$$

where $l = 1, 2, 3, 4$ and

$$\lambda_0(\eta) = \eta, \quad \lambda_1(\eta) = 0, \quad \lambda_2(\eta) = -\eta, \quad \lambda_3(\eta) = 0.$$

Further, we define

$$\Lambda := [-(1 - \delta), -\delta] \cup \{0\} \cup [\delta, 1 - \delta]$$

and $\theta^2 := 1 - \lambda$, $\lambda \in \Lambda$. Noticing that Λ is compact, we consider the one-parameter-dependent family of systems

$$\begin{aligned} \tfrac{\partial}{\partial t} z(t, \theta) &= \begin{bmatrix} 0 & -\theta^2 \\ 1 & 0 \end{bmatrix} z(t, \theta) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ z(0, \theta) &= 4\, \mathbb{1}_{[\sqrt{\delta}, \sqrt{1-\delta}]}(\theta)\, x^0, \end{aligned} \tag{18}$$

with $\theta \in \mathbf{P}' = [\sqrt{\delta}, \sqrt{1 - \delta}] \cup \{1\} \cup [\sqrt{1 + \delta}, \sqrt{2 - \delta}]$. To obtain the sequence of control input variables we apply the change of coordinates defined by the matrix

$$\tfrac{1}{\theta} \begin{bmatrix} \theta & i \\ \theta & -i \end{bmatrix}.$$

Then, the system (18) is transformed into

$$\begin{aligned} \tfrac{\partial}{\partial t} z(t, \theta) &= \begin{bmatrix} -i\theta & 0 \\ 0 & i\theta \end{bmatrix} z(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t) \\ z(0, \theta) &= \begin{bmatrix} z^0(\theta) \\ \overline{z^0(\theta)} \end{bmatrix} := \begin{bmatrix} x_1^0 + \frac{i}{\theta} x_2^0 \\ x_1^0 - \frac{i}{\theta} x_2^0 \end{bmatrix}. \end{aligned} \tag{19}$$

The corresponding terminal state is given by

$$z^*(\theta) = \mathbb{1}_{[\sqrt{\delta}, \sqrt{1-\delta}]}(\theta) \begin{bmatrix} x_1^0 + \frac{i}{\theta} x_2^0 \\ x_1^0 - \frac{i}{\theta} x_2^0 \end{bmatrix}.$$

Let $T = 1$ and $\tau = \frac{1}{K}$, $K \in \mathbb{N}$. To obtain the input values we consider the polynomial

$$p(z) = u_{2K-1} + u_{2K-2} z + \cdots + u_0 z^{2K-1}.$$

The coefficients of p can be determined via a interpolation problem. To this end, one can pick Chebyshev sampling points θ_l in the set $\mathbf{P}'$ and determine the coefficients by the following interpolation conditions

$$\begin{aligned} p\left(\mathrm{e}^{i\theta_l/K}\right) &= \left(\frac{i\theta_l}{\mathrm{e}^{i\theta_l/K} - 1} \right) \left(z^*(\theta_l) - \mathrm{e}^{i\theta_l} z^0(\theta_l) \right) \\ p\left(\mathrm{e}^{-i\theta_l/K}\right) &= \left(\frac{-i\theta_l}{\mathrm{e}^{-i\theta_l/K} - 1} \right) \left(\overline{z^*(\theta_l)} - \mathrm{e}^{-i\theta_l}\, \overline{z^0(\theta_l)} \right) \end{aligned}$$

for $l = 1, ..., K$. Note that this defines a Lagrange interpolation problem, which has a unique solution p. In addition, the coefficients of p are real. For a simulation of this approach for the case of harmonic oscillators we refer to [13].

IV. SERIES CONNECTIONS OF SISO SYSTEMS

In this section we consider a family of networks of a two-dimensional single-input-output system $\Sigma = (A, b, c)$ defined by

$$A := \begin{bmatrix} 2 & 0 \\ 1 & -2 \end{bmatrix}, \quad b := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad c := \begin{bmatrix} 0 & 1 \end{bmatrix}. \tag{20}$$

The topology of the network is as follows. The systems are arranged in a row and they are connected via directed links. There are two different coupling strength present. Let k_1, k_2 be positive real numbers and let $\eta \in [\eta^-, \eta^+] =: \mathbf{P}$. Then, the coupling strength of a system to its left and right neighboring system is $k_1\eta$ and $k_2\eta$, respectively, see Figure 3.

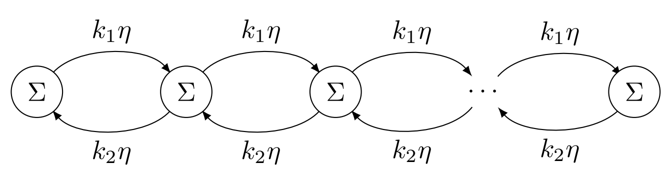

Fig. 3. Series connection of a SISO system Σ.

The adjacency matrix of the graph describing the network is the tridiagonal Toeplitz matrix, cf. [1]

$$\mathcal{A}_{\text{series}}(\eta) = \eta \begin{bmatrix} 0 & k_1 & & & \\ k_2 & 0 & k_1 & & \\ & \ddots & \ddots & \ddots & \\ & & k_2 & 0 & k_1 \\ & & & k_2 & 0 \end{bmatrix}.$$

Denoting the input-to-state interconnection vector by $\mathcal{B}$ the dynamics of the overall networked system is

$$\begin{aligned} \dot{x}(t) &= \big(I \otimes A + \mathcal{A}_{\text{series}}(\eta) \otimes bc \big) x(t) + \big(\mathcal{B} \otimes b \big) u(t) \\ x(0, \eta) &= \mathcal{B} \otimes x^0, \end{aligned} \tag{21}$$

where $x^0 \in \mathbb{R}^2$ is the initial state of the systems Σ. The desired terminal states is denoted by $x^* \in C(\mathbf{P}, \mathbb{R}^2)$.

Corollary 2: There exists an input-to-state interconnection vector $\mathcal{B}$ so that for any $x^0 \in \mathbb{R}^2$ and for any $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ the family of series connected systems $(\Sigma, \mathcal{A}_{\text{series}}(\mathbf{P}), \mathcal{B})$ with $\mathbf{P} \subset (0, 2\sqrt{k_1 k_2}^{-1})$ is robustly synchronizable from $\mathcal{B} \otimes x^0$ to $\mathbf{1} \otimes x^*$.

Proof: Observe that the adjacency matrix is a banded Toeplitz matrix. The spectrum is given by

$$\sigma(\mathcal{A}_{\text{series}}(\eta)) = \{2\eta\sqrt{k_1 k_2}\cos\tfrac{l\pi}{N+1} \,:\, l = 1, ..., N\}.$$

Note that, if N is even the adjacency matrix is invertible as $\lambda_l(\eta) := 2\eta\sqrt{k_1 k_2}\cos\frac{l\pi}{N+1} \neq 0$ for all $l = 1, ..., N$. Regardless of N, there exists a matrix $S \in \mathrm{GL}(N, \mathbb{R})$ so that $S\mathcal{A}(\eta)S^{-1}$ diagonal, cf. [1]. The matrix $S = (s_{uv})$ is given by

$$s_{uv} = \left(\sqrt{\tfrac{k_2}{k_1}}\right)^{v} \sin\tfrac{uv\pi}{N+1} \qquad u, v = 1, ..., N.$$

Defining $\mathcal{B} := S^{-1}\mathbf{1}$ and applying the change of coordinates $S \otimes I$ the dynamic equation describing the lth system is given by

$$\begin{aligned} \tfrac{\partial}{\partial t}x(t,\eta) &= \mathrm{diag}\,(A + \lambda_l(\eta)bc)\,x(t) + (\mathbf{1}\otimes b)\,u(t) \\ x(0,\eta) &= \mathbf{1}\otimes x^0. \end{aligned} \tag{22}$$

We show that Theorem 1 can be applied to (22). As the spectrum of $\mathrm{diag}\left(A + 2\eta\sqrt{k_1 k_2}\cos\frac{l\pi}{N+1}bc\right)$ is given by

$$\begin{aligned} &\bigcup_{l=1}^{N} \sigma\left(A + 2\eta\sqrt{k_1 k_2}\cos\tfrac{l\pi}{N+1}bc\right) = \\ &\bigcup_{l=1}^{N} \left\{w \in \mathbb{R} : w^2 - (2\eta\sqrt{k_1 k_2}\cos\tfrac{l\pi}{N+1} + 4) = 0\right\} \end{aligned} \tag{23}$$

we have that the spectra of $A + 2\eta\sqrt{k_1 k_2}\cos\frac{l\pi}{N+1}bc)$ and $A + 2\eta\sqrt{k_1 k_2}\cos\frac{k\pi}{N+1}bc$ are disjoint for all $\eta \in \mathbf{P}$ and for all $l \neq k \in \{1, ..., N\}$. Moreover, since (A, b) is controllable, the pair $(A + 2\eta\sqrt{k_1 k_2}\cos\frac{l\pi}{N+1}bc, b)$ is controllable for all $\eta \in \mathbf{P}$ and all $l \in \{1, ..., n\}$. Thus, the parallel connection (22) is controllable. Furthermore, (23) implies that the conditions (ii) and (iii) in Theorem 1 hold. Finally, we apply Theorem 1 to (22) and conclude that for any $\varepsilon > 0$, $T > 0$ and for any continuous family $x^* \in C(\mathbf{P}, \mathbb{R}^2)$ there exists an input function $u\colon [0, T] \to \mathbb{R}$ such that

$$\sup_{\eta\in\mathbf{P}} \|x(T, \mathcal{A}(\eta), \mathcal{B}\otimes x^0) - (\mathbf{1}\otimes x^*(\eta))\| < \varepsilon.$$

This shows the assertion. ■

V. CONCLUSIONS AND FUTURE WORKS

In this paper we examined families of networks of linear systems. On one hand the interconnection structure was given by a ring. On other hand, we considered series connections. In both cases the strength of the couplings were not known precisely. The uncertainty was assumed to be within a predefined compact real interval. In the dynamic equation of the overall network the interconnection is represented by the adjacency matrix and not in terms of the Laplace matrix. We investigated the task of robust synchronization by applying an external open-loop control which is broadcasted over the network and is independent of the coupling strength. Our approach was essentially based on uniform ensemble controllability. In the future we will consider heterogenous networks and general topologies. Following this approach, the derivation of criteria that allow for robust synchronization by open-loop control will rest critically on the analysis of the spectrum of the resulting overall system.

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