When Do Gossip Algorithms Converge in Finite Time?

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Abstract—In this paper, we study finite-time convergence of gossip algorithms. We show that there exists a symmetric gossip algorithm that converges in finite time if and only if the number of network nodes is a power of two, while there always exists a globally finite-time convergent gossip algorithm despite the number of nodes if asymmetric gossiping is allowed. For \( n = 2^n \) nodes, we prove that a fastest convergence can be reached in \( mn \) node updates via symmetric gossiping. On the other hand, for \( n = 2^n + r \) nodes with \( 0 \leq r < 2^n \), it requires at least \( mn + 2r \) node updates for achieving a finite-time convergence in cooperation with asymmetric interactions.

Index Terms—gossip algorithms, finite-time convergence, computational complexity

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I. INTRODUCTION

Various gossip algorithms, in which information exchange is always carried out pairwise among the nodes, have been widely used to structure distributed computation, optimization, and signal processing over peer-to-peer, sensor, and social networks [3], [2], [8], [5], [11], [12], [13], [14], [6], [7]. Gossip averaging plays a fundamental role in the study of gossip algorithms due to its simple nature and wide application.

Consider a network with node set \( V = \{1, \ldots, n\} \). Let the value of node \( i \) at time \( k \) be \( x_i(k) \in \mathbb{R}^1 \) for \( k \geq 0 \). Introduce

\[
\mathcal{M} \doteq \left\{ M_{ij} \doteq I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j = 1, \ldots, n \right\},
\]

where \( e_m = (0 \ldots 0 1 0 \ldots 0)^T \) is the \( n \times 1 \) unit vector whose \( m \)th component is 1. Denote \( x(k) = (x_1(k) \ldots x_n(k))^T \). Then a symmetric deterministic gossip algorithm is defined by

\[
x(k + 1) = P_k x(k), \tag{1}
\]

where \( \{P_k\}_{0}^{\infty} \) satisfies \( P_k \in \mathcal{M} \) for all \( k \). Enlarge the set of state-transition matrix by [8], [9]

\[
\mathcal{M}_* \doteq \left\{ I - \frac{(e_i - e_j)(e_i - e_j)^T}{2} : i, j = 1, \ldots, n \right\}
\]

We call Algorithm (1) an asymmetric gossip algorithm given by \( \{P_k\}_{0}^{\infty} \) if instead we have \( P_k \in \mathcal{M}_* \) for all \( k \).

Algorithm (1) and its variations have been extensively studied in the literature for both randomized and deterministic models. Karp et al. [2] derived a general lower bound for synchronous gossiping; Kempe et al. [3] proposed a randomized gossiping algorithm on complete graphs and determined the order of its convergence rate. Then Boyd et al. [5] established both lower and upper bounds for the convergence time of synchronous and asynchronous randomized gossiping algorithms, and developed algorithms for optimizing parameters to obtain fast consensus. Fagnani and Zampieri discussed asymmetric gossiping in [8] and asymmetric update in random setting was further studied in [9]. Liu et al. [10] presented a comprehensive analysis for the asymptotic convergence rates of deterministic averaging, and recently distributed gossip averaging subject to quantization constraints was studied in [13]. Distributed signal processing and estimation algorithms via gossiping were discussed in [11], [12]. A detailed introduction to gossip algorithms can be found in [6].

In this paper, we study the finite-time convergence of gossip algorithms with its precise definition given as follows. **Definition 1.1:** A gossip algorithm in the form of (1) given by \( \{P_k\}_{0}^{\infty} \) achieves finite-time convergence with respect to initial value \( x(0) = x^0 \in \mathbb{R}^n \) if there exists an integer \( T(x^0) \geq 0 \) such that \( x(T) = P_{T-1} \cdots P_0 x(0) \in \text{span}\{1\} \). Global finite-time convergence is achieved if such \( T(x^0) \) exists for every initial value \( x^0 \in \mathbb{R}^n \).

We also introduce the definition on the computational complexity of finite-time convergent gossiping algorithm.

**Definition 1.2:** Let Algorithm (1) given by \( \{P_k\}_{0}^{\infty} \) define a symmetric or asymmetric gossip algorithm. The number of node updates up to \( T \) is given by

\[
C_T := \sum_{k=0}^{T-1} \|I_n - P_k\|_1,
\]

where \( \| \cdot \|_1 \) is the matrix norm defined by \( \|A\|_1 = \sum_{i=1}^{m} \sum_{j=1}^{n} |A_{ij}| \) for any \( A \in \mathbb{R}^{m \times n} \) with \( |\cdot| \) denoting the absolute value. The computational complexity of \( \{P_k\}_{0}^{\infty} \) is indexed by

\[
\max\min_{x^0 \in \mathbb{R}^n} \{ C_T : T = 0 \cdots P_n x^0 \in \text{span}\{1\} \}.
\]

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whenever the above equation defines a finite number.

Reaching a consensus in finite-time pushes the convergence rate optimization of gossip algorithms to the limit [5], and by itself it is a basic and fundamental question for distributed gossip computation. We are interested in the following aspects: (i) Is it possible to reach finite-time convergence for gossip algorithms? (ii) What is the essential difference between symmetric and asymmetric gossiping? (iii) Whenever finite-time convergence is possible, what is its computational complexity?

We present clear answers to these questions in the rest of discussions. Section II and Section III will focus on symmetric and asymmetric gossip algorithms, respectively. Some concluding remarks are given in Section IV.

II. SYMMETRIC GossipING

In this section, we investigate the possibility and complexity of finite-time convergence for symmetric gossiping algorithms.

We present the following main result on the finite-time convergence of gossip algorithms.

Theorem 2.1: There exists a symmetric gossip algorithm \( \{P_k\}_{k=0}^\infty, P_k \in \mathcal{M}, k \geq 0, \) that converges globally in finite time if and only if there exists an integer \( m \geq 0 \) such that \( n = 2^m. \) If \( n = 2^m, \) a fastest symmetric gossip algorithm is reached by \( mn \) node updates.

Theorem 2.1 indicates that if the number of nodes \( n \) is not some power of two, finding a gossip algorithm which converges globally in finite time is impossible. However, in this case, there still might exist a gossip algorithm which converges in finite time for some initial values, say, half of \( \mathbb{R}^n. \) The following result further excludes the possibility of the existence of such algorithms by an indeed stronger claim, which shows that the initial values from which there exists a gossip algorithm converging in finite time form a measure zero set.

Theorem 2.2: Suppose there exists no integer \( m \geq 0 \) such that \( n = 2^m. \) Then for almost all initial values, it is impossible to find a symmetric gossip algorithm \( \{P_k\}_{k=0}^\infty \) with \( P_k \in \mathcal{M}, k \geq 0, \) to reach finite-time convergence.

We give some remarks on randomized algorithms. Most existing works on gossiping algorithms use randomized models [3], [2], [8], [5], [11], [12], [13], [14]. Deterministic gossiping was discussed in [13], [10]. Although we consider deterministic algorithms in this paper, the results can still be easily extended to randomized gossip algorithms.

A. Proof of Theorem 2.1

We prove the necessity, sufficiency, and the fastest convergence statements, respectively.

1) Necessity: Suppose \( n = 2^{m_1}n_2 \) with \( n_1 \geq 0 \) and \( n_2 \geq 3 \) an odd integer. Suppose \( P_0, \ldots, P_{k_*} \in \mathcal{M} \) with \( k_* \geq 0 \) gives an algorithm of (1) that converges in finite time globally.

Take \( x_1, \ldots, x_{2n_1} = 0 \) and \( x_{2n_1+1}, \ldots, x_n = 2^{k_*+1}. \) Then there exists \( c \in \mathbb{R} \) such that \( x_i(k_*+1) = c, i = 1, \ldots, n. \) On the one hand, because each element in \( \mathcal{M} \) is symmetric and therefore doubly stochastic, average is always preserved. Thus, we have

\[
\begin{align*}
c &= \frac{2^{k_*+1}n_1(n_2-1)}{2^{m_1}n_2} = \frac{2^{k_*+1}(n_2-1)}{n_2}.
\end{align*}
\]

On the other hand, it is not hard to see that \( c \) is an integer for the given initial value since pairwise averaging takes place \( k_* + 1 \) times. Consequently, we have \( c = r_22^{r_1} \) with \( 0 \leq r_1 \leq k_* + 1 \) an integer and \( r_2 \geq 1 \) an odd integer.

Therefore, we conclude that

\[
\frac{2^{k_*+1}(n_2-1)}{n_2} = r_22^{r_1},
\]

which implies

\[
2^{k_*+1-r_1}(n_2-1) = r_2n_2.
\]

This is impossible because the left-hand side of Eq. (2) is an even number while the right-hand side odd. Therefore, (1) cannot achieve global finite-time convergence no matter how \( P_0, \ldots, P_{k_*} \) are chosen.

2) Sufficiency: We need to construct a gossip algorithm which converges in finite time globally for \( n = 2^m. \)

We relabel the nodes in a binary system. We use the binary number

\[
B_1 \ldots B_m, B_s \in \{0, 1\}, s = 1, \ldots, m
\]
to mark node \( i \) if \( B_1 \ldots B_m = i - 1 \) as a binary number. The gossip algorithm is derived from the following matrix selection process:

S1. Let \( k = 1. \)

S2. Take \( 2^{m-1} \) matrices from \( \mathcal{M}, \) as the elements in the following set

\[
\mathcal{P}_k = \left\{ I - \frac{(c_{i-1} - c_i)(c_{i-1} - c_i)^T}{2} : \text{in the binary system, the } k^{th} \text{ digit of } i-1 \text{ equals 0, and the } k^{th} \text{ digit of } j-1 \text{ equals 1} \right\}.
\]

In other words, we take all the node pairs \( (i, j), \) where \( i - 1 \) and \( j - 1 \) have identical expressions in the binary system except for the \( k^{th} \) digit. Label the matrices in \( \mathcal{P}_k \) as \( P_{(k-1)2^{m-1}}, \ldots, P_{k2^{m-1}-1} \) with an arbitrary order.

S3. Let \( k = k + 1 \) and go to S2 until \( k = m. \)

Following this matrix selection process, \( P_0, \ldots, P_{m2^{m-1}-1} \) gives a gossip algorithm in the form of (1). It is easy to see that the vector

\[
P_{(k-1)2^{m-1}} \cdots P_{(2^{m-1})} x^0, \quad x^0 \in \mathbb{R}^n, \quad s = 1, \ldots, m
\]

has at most \( 2^{m-s} \) different elements. Thus, convergence is reached after \( m2^{m-1} = (n \log_2 n)/2 \) updates. This completes the proof.

3) Complexity: Assume \( x_i(0) = a_i, \) for \( i = 1, 2, \ldots, 2^n. \) Given any gossip algorithm \( \{P_k\}_{k=0}^\infty. \) After multiplication of \( h \) matrices the value of every point can be written in the form

\[
x_i(h) = \sum_{j=1}^{2^n} \frac{A_{i,h,j}}{2^{h_n,j}a_j}
\]
where $A_{h,j}$ and $B_{h,j}$ are nonnegative integers which depends on $\{P_k\}_{0}^{\infty}$ and $A_{h,i}$ is uniquely determined for all initial values in $R^{2n}$. For any node $i$, denote $s_{i,h}$ as the times node $i$ has been updated for the initial $h$ matrices.

Claim. $A_{h,h} \geq \frac{1}{2^{n+1}}$.

This can be proved by induction on $s_{i,h}$. For $s_{i,h} = 0$, that is to say node $i$ has not been updated for the first $h$ matrices. Then $x_i(h) = a_i$. Thus $A_{h,h} = 2^{n+1}$. Assume $s_{i,h} = l + 1$, assume at the multiplication of the $l$th matrix, node $i$ is updated for the $(l + 1)$th time. Then by the induction hypothesis, $A_{h,l} \geq \frac{1}{2^{n+1}}$. Assume at matrix $P_{h}^{l+1}$, node $i$ and $j$ are updated, i.e. $P_{h}^{l+1} = I - \frac{(e_i - e_j)(e_i - e_j)^T}{2}$. 

The coefficient of $a_i$ is 

$\frac{A_{h,l} + A_{h,l}}{2} \geq \frac{A_{h,l}}{2} \geq \frac{1}{2^{n+1}}$. 

For $s_{i,l} = l + 1$, node $i$ will not be updated in the rest matrices of the initial $h$ matrices. Thus, $x_i(h) = x_i(h')$. 

Thus, if gossip algorithm $\{P_k\}^{\infty}$ converges at finite matrix $P_{T}^{1}$. 

$x_1(T) = x_2(T) = ... = x_{2m}(T) = \sum_{l=1}^{2m} \frac{1}{2m} a_i$. 

According to the claim, $\frac{1}{2m} = A_{h,i} \geq \frac{1}{2^{n+1}}$, for any $i$. Thus, $s_{i,T} \geq m$. That is to say, when all point converges to the same value, each node must have been updated for at least $m$ times. We know that for each multiplication of matrix only two points are updated. Therefore, $T$ is at least $mn/2$ and thus the least number of node updates equals to $mn$.

B. Proof of Theorem 2.2

The proof is built upon an understanding to the finite-time convergence of the general class of averaging algorithms. In fact, (1) is a special case of distributed averaging algorithms defined by products of stochastic matrices [16], [17], [18]:

$x(k + 1) = W_k x(k)$, \hspace{1cm} (3)

where $W_k \in \mathcal{S} = \{ W \in \mathbb{R}^{n \times n} : W$ is a stochastic matrix $\}$. Let $S_0 \subseteq \mathcal{S}$ be a subset of stochastic matrices. We define $\mathcal{X}_{S_0} = \{ x \in \mathbb{R}^n : \exists W_0, ... , W_s \in S_0, s \geq 0 \text{ s.t. } W_s \cdots W_0 x \in \text{span}\{1\} \}$.

Let $M()$ represent the standard Lebesgue measure on $\mathbb{R}^n$. We have the following result for the finite-time convergence of general averaging algorithms.

Proposition 2.1: Suppose $S_0$ is a set with at most countable elements. Then either $\mathcal{X}_{S_0} = \mathbb{R}^n$ or $M(\mathcal{X}_{S_0}) = 0$. In fact, if $\mathcal{X}_{S_0} \neq \mathbb{R}^n$, then $\mathcal{X}_{S_0}$ is a union of at most countably many linear spaces whose dimensions are no larger than $n - 1$.

Remark 2.1: Note that in the definition of $\mathcal{X}_{S_0}$, different initial values can correspond to different averaging algorithms. Even if $S_0$ is finite, there will still be uncountably many different averaging algorithms in the form of (3) as long as $S_0$ contains at least two elements. Therefore, the proof of Proposition 2.1 requires a careful structure characterization of $\mathcal{X}_{S_0}$.

Proof of Proposition 2.1. Define a function $\delta(M)$ of a matrix $M = [m_{ij}] \in \mathbb{R}^{n \times n}$ by (cf. [15])

$$\delta(M) = \max_{j} \max_{i \neq j} |m_{ij} - m_{jj}|.$$ \hspace{1cm} (4)

Given an averaging algorithm (3) defined by $\{W_k\}^\infty_0$ with $W_k \in S_0, k \geq 0$. Suppose there exists an initial value $z^0 \in \mathbb{R}^n$ for which $\{W_k\}^\infty_0$ fails to achieve finite-time convergence. Then obviously $\delta(W_s \cdots W_0) > 0$ for all $s \geq 0$.

Claim. $\text{rank} (W_s \cdots W_0) \geq 2, s \geq 0$.

Let $W_s \cdots W_0 = (\omega_1 \cdots \omega_n)^T$ with $\omega_i \in \mathbb{R}^n$. Since $\delta(W_s \cdots W_0) > 0$, there must be two rows in $W_s \cdots W_0$ that are not equal. Say, $\omega_1 \neq \omega_2$. Note that $W_s \cdots W_0$ is a stochastic matrix because any product of stochastic matrices is still a stochastic matrix. Thus, $\omega_1 \neq \omega_2$ for all $i = 1, \ldots, n$. On the other hand, if $\omega_i = c\omega_2$ for some scalar $c$, we have $1 = \omega_1^T 1 = c\omega_2^T 1 = c$, which is impossible because $\omega_1 \neq \omega_2$. Therefore, we conclude that $\text{rank} (W_s \cdots W_0) \geq \text{rank} (\text{span}\{\omega_1, \omega_2\}) \geq 2$. The claim holds.

Suppose there exists some $y \in \mathbb{R}^n$ such that $y \notin \mathcal{X}_{S_0}$. We see from the claim that the dimension of ker($W_s \cdots W_0$) is at most $n - 2$ for all $s \geq 0$ and $W_s \cdots W_0 \in S_0$.

Now for $s = 0, 1, \ldots$, introduce $\Theta_s = \{ x \in \mathbb{R}^n : \exists W_0, \ldots, W_s \in S_0, s.t. W_s \cdots W_0 x \in \text{span}\{1\} \}$. Then $\Theta_s$ indicates the initial values from which convergence is reached in $s + 1$ steps. For any fixed $W_0, \ldots, W_s \in S_0$, we define

$$\Upsilon_{W_0 \cdots W_s} = \{ z \in \mathbb{R}^n : W_s \cdots W_0 z \in \text{span}\{1\} \}.$$ Clearly $\Upsilon_{W_0 \cdots W_s}$ is a linear space. It is straightforward to see that $\Theta_s = \bigcup_{W_s \cdots W_0 \in S_0} \Upsilon_{W_s \cdots W_0}$, and therefore

$$\mathcal{X}_{S_0} = \bigcup_{s=0}^{\infty} \Theta_s = \bigcup_{s=0}^{\infty} \bigcup_{W_s \cdots W_0 \in S_0} \Upsilon_{W_s \cdots W_0}.$$
Noticing that $z \in \mathcal{Y}_{W_s \ldots W_0}$ implies $(z - W_s \ldots W_0z) \in \ker(W_s \ldots W_0)$, we define a linear mapping

$$f : \mathcal{Y}_{W_s \ldots W_0} \rightarrow \ker(W_s \ldots W_0) \times \text{span}\{1\}$$

s.t. $f(z) = (z - W_s \ldots W_0z, W_s \ldots W_0z)$ (5)

Suppose $z_1, z_2 \in \mathcal{Y}_{W_s \ldots W_0}$ with $z_1 \neq z_2$. It is straightforward to see that either $W_s \ldots W_0z_1 = W_s \ldots W_0z_2$ or $W_s \ldots W_0z_1 \neq W_s \ldots W_0z_2$ implies $f(z_1) \neq f(z_2)$. Hence, $f$ is injective. Therefore, noting that $\ker(W_s \ldots W_0)$ is a linear space with dimension at most $n - 2$, we have $\dim(\mathcal{Y}_{W_s \ldots W_0}) \leq n - 1$, and thus $M(\mathcal{Y}_{W_s \ldots W_0}) = 0$. Consequently, we conclude that

$$M(\mathcal{Y}_{W_s \ldots W_0}) = M\left(\bigcup_{W_0, \ldots, W_r \in S_0} \mathcal{Y}_{W_s \ldots W_0}\right) \leq \sum_{W_0, \ldots, W_r \in S_0} M(\mathcal{Y}_{W_s \ldots W_0}) = 0$$

because any finite power set $S_0 \times \cdots \times S_0$ is still a countable set as long as $S_0$ is countable. This immediately leads to

$$M(\mathcal{Y}_{S_0}) = \sum_{s=0}^{\infty} M(\Theta_s) \leq \sum_{s=0}^{\infty} M(\Theta_s) = 0.$$

Additionally, since every $\Theta_s$ is a union of at most countably many linear spaces, each of dimension no more than $n - 1$, $\mathcal{Y}_{S_0}$ is also an union of countably many linear spaces with dimension no more than $n - 1$. The desired conclusion thus follows.

Noticing that $M$ is a finite set and utilizing Proposition 2.1, Theorem 2.2 follows immediately.

C. Discussion: How Many Algorithms can be Found?

In this subsection, we make some further discussions on essentially how many different finite-time convergent algorithms via symmetric gossiping exist. We present the following result indicating that when $n = 4$, the desired algorithm is indeed unique. Recall that $M_{f_2} = I - \frac{(c_{r-e})(c_{r-e})}{2}$. Since the proof of this proposition is rather technical, we refer [19] for a complete proof.

**Proposition 2.2:** Let $n = 4$. Suppose $P_{T-1} \cdots P_0 = 11^T/4$ with $P_{T-2} \cdots P_0 \neq 11^T/4$. Then there are under certain permutation of index we always have $P_{T-1} = M_{12}$, $P_{T-2} = M_{54}$, $P_{T-3} = M_{13}$ and $P_{T_n} = M_{24}$ for some $0 \leq T_{\alpha} < T - 3$.

III. ASYMMETRIC Gossiping

In this section, we investigate asymmetric gossiping. It turns out that finite-time convergence is always possible despite the number of nodes as long as asymmetric gossiping is allowed. The following conclusion holds.

**Theorem 3.1:** There always exists a deterministic gossip algorithm $\{P_k\}_k^\infty$, $P_k \in M_\ast$, $k \geq 0$, which converges globally in finite time. In fact, for $n = 2^m + r$ with $0 \leq r < 2^m$, a fastest asymmetric gossiping algorithms that converges globally in finite time requires $mn + 2r$ node updates.

A. Complexity

In this subsection, we first establish the least number of node updates for finite-time convergence via asymmetric gossiping. For any $n, m$ can be written as $n = 2^m + r$, where $m$ and $r$ are integers and $0 \leq r < 2^m$. The complexity proof relies on the following lemma, whose proof can be found in [19].

**Lemma 3.1:** Let $n = 2^m + r$ with $0 \leq r < 2^m$. $F$ is a subset of $\mathbb{R}^n$ such that $f = (f_1, \ldots, f_n) \in F$ if and only if

$$1 = \sum_{i=1}^n f_i$$

and $f_i$s have the form $b_i/2^r$ where $b_i$s are positive odd integers and $c_i$s are nonnegative integers, for $i = 1, \ldots, n$. As $b_i$ and $c_i$ are uniquely determined by $f$, we denote them by $b_i(f)$ and $c_i(f)$ respectively. For each $f_i$, there exist a smallest positive integer $n_i(f)$ such that $f_i \geq 1/2^{n_i(f)}$. Define $\hat{n}(f) = \sum_{i=1}^n n_i(f)$. Then,

$$\min_{f \in F} \hat{n}(f) = mn + 2r.$$

B. Existence

We now construct an algorithm that when node states converge to the same value, only $mn + 2r$ node updates have been taken.

Again, we relabel the nodes in a binary system. We use the binary number

$$B_1 \ldots B_{m+1}, B_s \in \{0, 1\}, s = 1, \ldots, m + 1$$

to mark node $i$ if $B_1 \ldots B_{m+1} = i - 1$ as a binary number. The asymmetric gossip algorithm is derived from the following matrix selection process:

**S1.** Take $r$ matrices from $M_\ast$, as the elements in the following set

$$\mathcal{P}_1 = \left\{ I - \frac{(c_{r-e})(c_{r-e})}{2} : i - 1 \text{ and } j - 1 \text{ have identical expressions in the binary system except for the 1st digit.} \right\}.$$ Label the matrices in $\mathcal{P}_1$ as $P_0, \ldots, P_{r-1}$ with an arbitrary order.

**S2.** Let $k = 2$.

**S3.** Take $r$ matrices from $M_\ast$, as the elements in the following set

$$\mathcal{P}_{(1,k)} = \left\{ I - \frac{(c_{r-e})(c_{r-e})}{2} : \text{in the binary system, the 1'th digit of } i - 1 \text{ equals 1, and the 1'th digit of } j - 1 \text{ equals 0}, i - 1 \text{ and } j - 1 \text{ have identical expressions in the binary system except for the 1'st and k'th digits.} \right\}.$$ Label the matrices in $\mathcal{P}_{(1,k)}$ as $P_{0^k}, \ldots, P_{r^k}$ with an arbitrary order.

**S4.** Take $2^{m-1}$ matrices from $M_\ast$, as the elements in the following set

$$\mathcal{P}_{(2,k)} = \left\{ I - \frac{(c_{r-e})(c_{r-e})}{2} : i - 1 \text{ and } j - 1 \text{ have identical expressions in the binary system except for the k'th digit, and the 1'st digits of } i - 1 \text{ and } j - 1 \text{ are both 0} \right\}.$$ Label the matrices in $\mathcal{P}_{(2,k)}$ as $P_{0^k}, \ldots, P_{2^m-1}$ with an arbitrary order.

**S5.** Let $k = k + 1$ and go to S2 until $m = m + 1$. 477
Following this matrix selection process, $P^*_0 \cdots P^*_{m2^m-1+(m+1)r-1}$ gives an asymmetrical gossip algorithm in the form of (1). It is easy to see that the vector $P^*_{r1+(s-1)2^m-1-1} \cdots P^*_{1} x^0$, $x^0 \in \mathbb{R}^n$, $s = 1, \ldots, m+1$ has at most $2^{m+1-s}$ different elements. Note that the matrix selected in $S1$ and $S4$ contribute two updated values, and the matrix selected in $S3$ contribute one updated value. Thus, convergence is reached after $r + 2 + (2^m - 1 + 2 + r) \times m = mn + 2r$ value updates. This completes the proof. □

C. Discussion: Fastest Algorithm in Term of Matrices

Here, we choose the number of node value updates as the efficiency of the asymmetrical gossip algorithm instead of the number of matrices selected. It is still unknown the least number of matrices needed to converge. In fact, the number of matrices selected depends on the efficiency of the asymmetrical gossip algorithm instead of the number of matrices. We showed that there exists a symmetric gossip algorithm that converges in finite time if and only if the number of network nodes is a power of two, while there always exists a globally finite-time convergent gossip algorithm despite the number of nodes if asymmetric gossiping is allowed. In both cases we have constructed desired algorithms explicitly, and we proved that the given algorithms indeed reach fastest convergence. More challenges lie in how to present a precise description on how the graph structure influences the existence and complexity of finite-time convergence via gossiping.

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