21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Decentralised $\mathcal{H}_2$ Norm Estimation and Guaranteed Error Bounds Using Structured Gramians

James Anderson[1] and Aivar Sootla[2]

Abstract— **In this work we consider the problem of obtaining a bound on the error of the $\mathcal{H}_2$-norm of a linear time invariant system when using structured controllability and/or observability Gramians. In particular we consider dynamical systems whose drift matrices are lower block triangular and Gramians that have a block diagonal structure. We motivate the problem by showing that autonomous triangular systems always admit a diagonal Lyapunov function. We then show how the search for block diagonal Gramians can be interpreted in a decentralised manner and provide error bounds on the norm estimation.**

I. INTRODUCTION

Many real world large-scale systems contain an inherent structure when they are modelled. Typical examples include chemical reaction networks [1], vehicle formation control [2], and power system networks to name but a few. Recently there has been a surge of interest in deriving scalable decentralised techniques for both analysis and control of such *networked* systems. In this paper we take a decentralised approach to the analysis of Linear Time Invariant (LTI) systems. We take an intuitive approach of using structured Gramians to obtain estimates of performance (in terms of the $\mathcal{H}_2$-norm) of structured LTI systems. In particular we focus on systems with state-space realisations of the form

$$\begin{aligned} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} &= \begin{pmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} u \\ y &= \begin{pmatrix} C_1 & C_2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \end{aligned}$$

This system description is quite general as it is possible to further partition the input vector u as well as the input and output maps $B^T \triangleq (B_1^T \; B_2^T)^T$ and $C \triangleq (C_1 \; C_2)$ respectively. For example further partitions of B and C one can capture the 2-player system set-ups described in [3], [4]. Indeed the more general class of block triangular systems has been recently studied in [5] in terms of realisability.

Our main objective is to obtain optimal estimates of the $\mathcal{H}_2$-norm of such systems when the computations are carried out with structured controllability or observability Gramians. Furthermore we would like to be able to characterise the magnitude of the error between the actual $\mathcal{H}_2$-norm and the estimate. In particular we will focus on the case where the Gramians have a block diagonal structure. Such approaches have been used in model reduction of interconnected systems [6], but surprisingly little has been done for the case of stability and performance analysis [7].

The paper is structured as follows. In Section II the stability problem is addressed using diagonal Lyapunov functions which provides the motivation for searching for block diagonal Gramians. In particular we show that diagonal systems always have diagonal Lyapunov functions. We then turn our attention to $\mathcal{H}_2$ performance analysis in Section III. In this section we investigate how to compute the $\mathcal{H}_2$-norm in a decentralised manner (using the structured Gramian approach) and then characterise the error between the actual norm and the optimal estimate.

Notation and Nomenclature:

Given a symmetric matrix $A \in \mathbb{R}^{n \times n}$ then A is said to be positive (negative) definite if $x^T A x > 0$ ($x^T A x < 0$) for all non-zero $x \in \mathbb{R}^n$. Positive and negative definite matrices are denoted by $A \succ 0$ and $A \prec 0$ respectively. If the inequality is replaced by its non-strict counterpart and holds for all x then the matrix is said to be positive or negative semidefinite.

A matrix $F \in \mathbb{R}^{n \times n}$ is said to be Hurwitz if all of its eigenvalues have a strictly negative real part. Denote by $\mathcal{RL}_2$ the space of rational, strictly proper transfer functions with no poles on the imaginary axis and $\mathcal{RH}_2$ its stable subspace. For any system $G \in \mathcal{RH}_2$ its state-space realisation is given by

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \triangleq C(sI - A)^{-1}B \qquad (1)$$

[1]JA is with St. John's College and the Department of Engineering Science, University of Oxford, Parks Road, OX1 3JP james.anderson@eng.ox.ac.uk
[2]AS is with the Department of Bioengineering, Imperial College London, UK a.sootla@imperial.ac.uk

ISBN: 978-90-367-6321-9

where $s = j\omega$. The associated norm of G on $\mathcal{H}_2$ is

$$\|G\|_{\mathcal{H}_2} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} \text{tr}[G^*(j\omega)G(j\omega)]d\omega \right)^{\frac{1}{2}} \quad (2)$$

where G^* denotes the conjugate transpose of G. In order for (2) to be finite A must necessarily be Hurwitz.

Throughout this work we will make frequent reference to triangular and block triangular matrices. Consider the $nm \times nm$ (where $n, m \in \mathbb{Z}_+$) partitioned square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A \in \mathbb{R}^{n\times n}$, and $D \in \mathbb{R}^{m\times m}$. We say M is *block lower triangular* if $B = 0_n$ and simply *lower triangular* if $B = 0_{n\times m}$ and

$$A_{ij} = \begin{cases} a_{ij} & \text{for } i \geq j \\ 0 & \text{for } i < j \end{cases}, \quad D_{ij} = \begin{cases} d_{ij} & \text{for } i \geq j \\ 0 & \text{for } i < j \end{cases}.$$

II. STABILITY ANALYSIS

Before proceeding with the performance analysis we consider autonomous Linear Time Invariant (LTI) systems with a triangular structure. The main result from this section is that for systems of this type, searching for a strictly diagonal matrix P (i.e. $P = \text{diag}(p_1, \dots, p_n)$) when constructing a quadratic Lyapunov function of the form $V(x) = x^T Px$ does not introduce any conservatism.

Consider the LTI system

$$\dot{x}(t) = Ax(t), \quad x(0) = x_0 \quad (3)$$

where $x(t) \in \mathbb{R}^n$ and A is a constant $n \times n$ matrix. To streamline notation we will drop the dependence on the variable t. It is well known that the system (3) is asymptotically stable if given a matrix $Q \in \mathbb{R}^{n\times n}$ such that $Q \succ 0$ there exists a matrix $P \in \mathbb{R}^{n\times n}$ such that

$$A^T P + PA = -Q \quad (4)$$

with $P \succ 0$. Specifically in this note we are interested in determining when a diagonal P exists that solves the Lyapunov inequality $A^T P + PA \prec 0$ and later the generalised Gramian $A^T P + PA + Q \prec 0$ when A in (3) is a lower triangular matrix. When such a P exists we refer to V as a Diagonal Lyapunov Function or DLF. Some established results are listed below:

Theorem 1 ([8]): A necessary condition for the system (3) with $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ to admit a DLF is $a_{ii} < 0$ for $i = 1, \dots, n$.

Clearly for A lower (or upper) triangular $a_{ii} < 0$ is required for stability and is thus not a conservative result.

Theorem 2 ([9]): System (3) with $A \in \mathbb{R}^{3\times 3}$ admits a DLF if and only if

1) All principal minors of $-A$ are positive.
2) $\max\{1, \omega_1, \omega_2, \omega_3\} < \frac{1}{2}(1+\omega_1+\omega_2+\omega_3)$ where $\omega_i = \sqrt{a_{ii}(A^{-1})_{ii}}$ for $i = 1, \dots, 3$.

From Theorem 2 it is immediate that for the case when A is lower triangular then condition i) and ii) above are met. This follows from the fact that the eigenvalues of an inverted triangular matrix are exactly the inverse of the eigenvalues of its own inverse. Also, from the definition of stability all principal minors must be positive.

Definition 1: The matrix $F \in \mathbb{R}^{n\times n}$ is said to be Metzler if all off-diagonal elements are non-negative, i.e. $a_{ij}, \geq 0 \ \forall i \neq j$.

LTI systems with vector fields governed by Metzler matrices are referred to as *positive systems*. Recently there has been a great deal of interest in positive systems and distributed control synthesis. The following result relates DLFs to positive systems.

Theorem 3 ([10]): Let $A \in \mathcal{R}^{n\times n}$ be a Metzler matrix. Then the following statements are equivalent:

1) The matrix A is Hurwitz.
2) There exists a diagonal matrix $P \succ 0$ such that the LMI $A^T P + PA \prec 0$ is satisfied.

Before stating our first result we present the following lemma that characterises the location of the eigenvalues when two Hermitian matrices are added together.

Lemma 1 ([11]): Let $C, D \in \mathbb{R}^{n\times n}$ be given matrices and assume that the eigenvalues of C, D and $C + D$ are arranged in increasing order such that $\lambda_{\text{min}} = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n = \lambda_{max}$. Then for every pair of integers j, k such that $1 \leq j,\ k \leq n$ and $j + k \geq n + 1$ we have that

$$\lambda_{j+k-n}(C + D) \leq \lambda_j(C) + \lambda_k(D).$$

We also need the Schur compliment lemma.

Lemma 2 (Schur): Assume the matrices A and C are square and invertible then the following statements are equivalent:

1) The matrix $\begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$ is negative definite.
2) The LMIs $A \prec 0$ and $C - BA^{-1}B^T \prec 0$ are satisfied.

The following theorem shows that for a stable system (3) with $A \in \mathbb{R}^{4\times 4}$ there always exists a diagonal Lyapunov function. After giving the proof it will be shown how this generalises to the $n \times n$ case.

Theorem 4: Consider system (3) where A is lower triangular of dimension 4×4 and Hurwitz. There always exists a DLF such that $P \succ 0$ and $A^T P+PA \prec 0$ where P is diagonal.

Proof: Here the state matrix can be partitioned as follows

$$A = \begin{pmatrix} A_{11} & 0 \\ A_{21} & a_{22} \end{pmatrix} \quad (5)$$

where $A_{11} \in \mathbb{R}^{3\times 3}$ and is lower triangular, $a_{22} \in \mathbb{R}$ and $A_{21} \in \mathbb{R}^{1\times 3}$. Consider also the state vector $x = [x_1^T, x_2]^T$ which is partitioned conformally with A above. Clearly for A to be Hurwitz it must be true that $a_{22} < 0$, similarly the diagonal elements of A_{11} must be negative. By Theorem 2 it must be true that the subsystem $\dot{x}_1 = A_{11}x_1$ has a DLF $V_1(x_1) = x_1^T P_1 x_1$ where $P \succ 0$ and is diagonal. Similarly, the subsystem $\dot{x}_2 = a_{22}x_2$ has a Lyapunov function $V_2(x_2) = x_2^2 p_2$ where $p_2 > 0$. What is left is to show that the Lyapunov function $V(x) = x^T P x$ where $P = \text{diag}(P_1, p_2)$ is indeed a Lyapunov function for the system. Clearly V is positive definite, it will now be shown that the derivative condition can be made negative definite.

The Lie derivative of V is $A^T P + PA$ which when partitioned as above is given by

$$\begin{pmatrix} A_{11}^T P_1 + P_1 A_{11} & A_{21}^T p_2 \\ p_2 A_{21} & 2a_{22}p_2 \end{pmatrix} \tag{6}$$

where the upper left and lower right blocks are by assumption negative definite. Using the Schur compliment (Lemma 2) it follows that negative definiteness of (6) is equivalent to

$$A_{11}^T P_1 + P_1 A_{11} - \frac{p_2}{2a_{22}} A_{21}^T A_{21} \prec 0 \quad \triangleq \quad X + Y \prec 0. \tag{7}$$

Taking care to note the sign change on Y, applying Lemma 1 with $j = 3$, $k = 3$ and $n = 3$ we have that

$$\lambda_3(X + Y) \leq \lambda_3(X) + \lambda_3(Y). \tag{8}$$

It is known that the largest eigenvalue, $\lambda_3(X)$ is negative, furthermore as $A_{21}^T A_{21}$ is a rank one matrix we have that $\lambda_1(Y) = \lambda_2(Y) = 0$, however, $\lambda_3(Y) > 0$ which implies that $\lambda_3(X) + \lambda_3(Y)$ may be positive and thus cannot conclude stability via the DLF. Let us re-examine the Lyapunov function V_1. If instead of $V_1 = x_1^T P_1 x_1$ we select $\widehat{V}_1 = \alpha x_1^T P_1 x_1$ where P is still diagonal and satisfies $A_{11}^T P_1 + P_1 A_{11}$ then for any choice of $\alpha > 0$ $\widehat{V}_1$ is still a valid Lyapunov function. Thus by increasing α we can decrease $\lambda_3(X)$ as much as needed in order to satisfy (8). From this we conclude that $V(x)$ is a diagonal Lyapunov function which completes the proof. ∎

The results of Theorem 4 can be generalised to the case where $A \in \mathbb{R}^{n\times n}$.

Theorem 5: All stable systems of the form (3) with $A \in \mathbb{R}^{n\times n}$ and lower triangular admit a diagonal Lyapunov function.

Proof: The proof is via recursive application of a generalised version of Theorem 4. Note that the system matrix A can always be partitioned into

$$A = \begin{pmatrix} A_{1,1} & 0 & 0 & 0 & 0 \\ A_{2,1} & A_{2,2} & 0 & 0 & 0 \\ A_{3,1} & A_{3,2} & \ddots & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_{n,1} & A_{n,2} & \dots & A_{n,n-1} & A_{n,n} \end{pmatrix}$$

where $A_{i,i}$ is a triangular Hurwitz matrix of dimension $n_i \times n_i$ where $n_i \in \{1, 2, 3\}$ for $i = 1, \dots, n$ and the matrices $A_{i,j}$ for $i \neq j$ are of conformal dimension. The result of Theorem 4 is easily seen to be extended to the case where the for the matrix (5) the diagonal elements are $A_{11} \in \mathbb{R}^{m\times m}$ and $a_{22} \in \mathbb{R}^{p\times p}$ where $m, p \in \{1, 2, 3\}$. Thus we can iteratively compute the a DLF of increasing dimension as there is always a feasible solution to LMI (7) regardless of dimension and thus via a scaling factor α and Lemma 1 we can ensure that the largest eigenvalue of the derivative condition is negative. ∎

A similar result with a non-constructive proof can be found in [12].

III. PERFORMANCE ANALYSIS

In this section we shall focus on the study of block diagonal Gramians and their use in $\mathcal{H}_2$-norm computation via Lyapunov equations. For all systems G of the form (1) that are in $\mathcal{H}_2$ there exist unique positive definite matrices P and Q that solve:

$$AP + PA^T + BB^T = 0, \tag{9a}$$
$$A^T Q + QA + C^T C = 0. \tag{9b}$$

P and Q are referred to as the controllability and observability Gramians respectively. Recalling the definition of the $\mathcal{H}_2$-norm (2) it is easily shown that $\|G\|_{\mathcal{H}_2}$ can be computed from either the controllability or observability Gramians. The equation (2) can be written in terms of the system matrices:

$$\begin{aligned} \|G\|_{\mathcal{H}_2}^2 &= \int_0^\infty \text{tr}\left[B^T e^{A^T t} C^T C e^{At} B\right] dt \\ &= \text{tr}\left[B^T Q B\right], \end{aligned}$$

where Q solves (9b). Equivalently it follows that $\|G\|_{\mathcal{H}_2}^2 = \text{tr}\left[B^T Q B\right] = \text{tr}\left[CPC^T\right]$, where P solves (9a).

A. Problem Formulation

The focus of this paper is to obtain bounds on the error between the exact $\mathcal{H}_2$-norm of an LTI system (which may be computed as described above) and an estimate for the norm arising from the use of *generalised*

Gramians $\tilde{P}$ and $\tilde{Q}$ with a sparsity pattern that satisfy the Linear Matrix Inequalities (LMIs):

$$A\tilde{P} + \tilde{P}A^T + BB^T \prec 0, \quad \tilde{P} \succ 0, \tag{10a}$$
$$A^T\tilde{Q} + \tilde{Q}A + C^TC \prec 0, \quad \tilde{Q} \succ 0. \tag{10b}$$

It is easy[1] to show that $\tilde{Q} \succeq Q$ and $\tilde{P} \succeq P$ and hence estimates of the $\mathcal{H}_2$-norm of G that use (10) will be suboptimal upper bounds unless $\tilde{Q} = Q$. It is assumed throughout the rest of the paper that G has the structure:

$$G = \left[\begin{array}{cc|c} A_{11} & 0 & B_1 \\ A_{21} & A_{22} & B_2 \\ \hline C_1 & C_2 & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right] \tag{11}$$

with A Hurwitz and $A_{11} \in \mathbb{R}^{n\times n}$ and $A_{22} \in \mathbb{R}^{m\times m}$.

We can now state the problem we address in this paper: Given a stable LTI system G, what is the smallest achievable error in the estimate of G's $\mathcal{H}_2$-norm computed using the generalised Gramians (10) with a given sparsity structure. Formally this can be expressed by the optimisation problem:

$$\begin{array}{ll} \inf\limits_{\tilde{P}} & \|G\|^2_{\mathcal{H}_2} - \|G\|^2_{\widetilde{\mathcal{H}}_2} \\ \text{subject to:} & A\tilde{P} + \tilde{P}A^T + BB^T \preceq 0 \\ & \tilde{P} \succ 0, \quad \tilde{P} \in \mathcal{S}_{\{n,m\}} \end{array} \tag{12}$$

where $\mathcal{S}$ defines a sparsity pattern and $\|\widehat{G}\|_{\widetilde{\mathcal{H}}_2}$ denotes a (suboptimal) estimate of the $\mathcal{H}_2$-norm of G computed using the generalized controlability Gramian (10a). Note that we have specified in (12) that the controllability Gramian will be used, of course it is possible that the observability Gramian could be used. Indeed, there is no reason to suggest which Gramian should be used as $\|G\|_{\widetilde{\mathcal{H}}_2}$ will vary depending upon the choice. *A priori* selection methods of the Gramian will be a focus of future work. We narrow our attention to the case where $\mathcal{S}$ defines a block diagonal subspace, i.e.

$$\mathcal{S}_{\{n,m\}} = \left\{ \tilde{P} \mid \tilde{P} \in \begin{bmatrix} * & * & * & & \\ * & * & * & & \\ * & * & * & & \\ & & & * & * \\ & & & * & * \end{bmatrix} \right\}$$

where in this case $n = 3$ and $m = 2$. We will frequently omit the n, m subscripts when it obvious from context what the dimensions are.

The following result motivates the assumption that searching for block diagonal Gramians is a reasonable objective. Assume that:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}, \quad B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}, \tag{13}$$

[1]Simply substitute $-C^TC = A^TQ + QA$ into (10b) and rearrange.

then there always exists a coordinate transformation such that (9a) is satisfied with a block diagonal Gramian Q.

Theorem 6: Let (A, B) given by (13) be a controllable pair and Q the corresponding controllability Gramian. Then there exists a transformation $(A, B, Q) \mapsto (TAT^{-1}, TB, TQT^T) \triangleq (\tilde{A}, \tilde{B}, \tilde{Q})$ such that $\tilde{A}\tilde{Q} + \tilde{Q}\tilde{A}^T = -\tilde{B}\tilde{B}^T$ where

$$\tilde{Q} = \begin{pmatrix} \tilde{Q}_{11} & 0 \\ 0 & \tilde{Q}_{22} \end{pmatrix} \quad \text{and} \quad \tilde{A} = \begin{pmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ 0 & \tilde{A}_{22} \end{pmatrix}.$$

Furthermore, if A_{22} is Hurwitz then (A_{22}, B_2) is controllable and Q_{22} is positive definite.

Proof: We first show that if A_{22} is Hurwitz then (A_{22}, B_2) is controllable and $Q_{22} \succ 0$. Substitution of A and B from (13) into (9a) gives

$$\begin{pmatrix} * & * \\ * & A_{22}Q_{22} + Q_{22}A_{22}^T \end{pmatrix} = -\begin{pmatrix} B_1B_1^T & B_1B_2^T \\ B_2B_1^T & B_2B_2^T \end{pmatrix}$$

where $*$ denotes elements we are not interested in. Clearly the $(2, 2)$ block satisfies the Lyapunov equation and if A_{22} is Hurwitz then Q_{22} must be positive definite (and unique). Controllability of (A_{22}, B_2) is shown via a PBH test (c.f. [13]). Define the vector $v = \begin{pmatrix} 0 \\ v_2 \end{pmatrix}$ where v_2 is any left eigenvector of A_{22}. It then it follows that

$$v^TA = \lambda v^T$$

and the PBH test in conjunction with the assumption that (A, B) is controllable implies that $v^TB = v_2^TB_2 \neq 0$ thus we can conclude controllability. Finally we show that a transformation exists that gives the desired structured Q. Consider the matrix

$$T = \begin{pmatrix} I & -Q_{12}Q_{22}^{-1} \\ 0 & I \end{pmatrix}$$

clearly TAT^T gives the desired structure of $\tilde{Q}$, furthermore $\tilde{Q}_{22} = Q_{22}$ and $\tilde{Q}_{11} = Q_{11} - Q_{12}Q_{22}^{-1}Q_{12}^T$ and TAT^{-1} gives

$$\begin{aligned} \tilde{A} &= \begin{pmatrix} A_{11} & \tilde{A}_{12} \\ 0 & A_{22} \end{pmatrix} \\ \tilde{A}_{12} &= A_{12} - Q_{12}Q_{22}^{-1}A_{22} + A_{11}Q_{12}Q_{22}^{-1} \end{aligned}$$

which has the desired block triangular structure. ■

B. Decentralisation

In this section the main results on the minimum achievable error of the $\mathcal{H}_2$-norm of a stable LTI system using a structured Gramian are given. The presentation of these results is derived in a manner that highlights the decentralised nature of the problem.

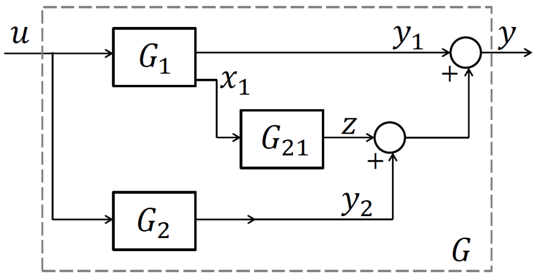

Fig. 1. The composition of G_1, G_2 and G_{21} such $y = Gu$.

Given the system G as defined by (11), define two *subsystems* of G as:

$$G_1 = \left[\begin{array}{c|c} A_{11} & B_1 \\ \hline C_1 & 0 \end{array}\right], \quad G_2 = \left[\begin{array}{c|c} A_{22} & B_2 \\ \hline C_2 & 0 \end{array}\right],$$

where $A_{11} \in \mathbb{R}^{n\times n}$ and $A_{22} \in \mathbb{R}^{m\times m}$ and the matrices B_i, C_i for $i \in \{1,2\}$ are of conformal dimension with respect to G. Observe that these definitions are made for convenience and there is no implication that the cascade, feedback or parallel connection of G_1, with G_2 is equivalent to the original system G. What is important to note is that the two systems are only coupled through the input signal u. In order to obtain some intuition of these subsystems we need to define a third subsystem G_{21} and then refer to Figure 1. Define

$$G_{21} = \left[\begin{array}{c|c} A_{22} & A_{21} \\ \hline C_2 & 0 \end{array}\right], \quad x_2(0) = 0;$$

which maps $[x_2^T, x_1^T]^T \mapsto z$ according to

$$z(t) = C_2 e^{A_{22}t} \int_0^t e^{-A_{22}\tau} A_{21} x_1(\tau) d\tau.$$

Thus it can be seen that $y = Gu = y_1(t) + y_2(t) + z(t)$.

Let P and Q be the observability and controllability Gramians of G, respectively (i.e. solutions of (9b)–(9a)). Partition P and Q blockwise as follows

$$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{pmatrix}, \quad Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix}.$$

Note that the submatrices P_{11} and Q_{22} are the controllability Gramian of G_1 and observability Gramian of G_2 respectively. Now consider the optimisation programme:

$$\begin{aligned} \|G\|^2_{\widetilde{\mathcal{H}}_2} &= \min_{X,Y} \text{tr}\left[C \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} C^T\right] \\ \text{s.t.} &\quad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \succeq 0 \qquad (14) \\ &\quad A \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} + \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} A^T + BB^T \preceq 0 \end{aligned}$$

The solution to (14), $\|G\|^2_{\widetilde{\mathcal{H}}_2}$ is an upper bound for $\|G\|^2_{\mathcal{H}_2}$. Due to the block triangular structure of A there always exists a feasible solution to (14). Now let us expand upon our definition of X and Y above. Let

$$\begin{aligned} X &= K + P_{11} \\ -M &= A_{22}Y + YA_{22}^T + B_2B_2^T \end{aligned}$$

where K and M are positive semidefinite. The objective function from (14) can be decomposed as follows:

$$\text{tr}\left[C^T \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} C\right] = \text{tr}\left[C_1^T X C_1\right] + \text{tr}\left[C_2^T Y C_2\right].$$

The right hand term above can be expanded using the definitions of X and Y and the properties of the trace operator:

$$\begin{aligned} \text{tr}\left[C_2^T Y C_2\right] &= \text{tr}\left[C_2 C_2^T Y\right] \\ &= -\text{tr}\left[(A_{22}Q_{22} + Q_{22}A_{22}^T)Y\right] \\ &= -\text{tr}\left[Q_{22}(YA_{22} + A_{22}^T Y)\right] \\ &= \text{tr}\left[Q_{22}(M + B_2B_2^T)\right]. \end{aligned}$$

We are now ready to present one of the main results of the paper.

Proposition 1: The optimal estimate of $\|G\|^2_{\mathcal{H}_2}$ obtainable using block diagonal structured controllability Gramian is found by solving the optimisation programme

$$\begin{gathered} \min_{K,M} \text{tr}\left[C_1 K C_1^T\right] + \text{tr}\left[M Q_{22}\right] \\ \text{subject to:} \quad K \succeq 0, \quad M \succeq 0 \\ \begin{pmatrix} A_{11}K + KA_{11}^T & KA_{21}^T + P_{11}A_{21}^T + B_1B_2^T \\ * & -M \end{pmatrix} \preceq 0 \end{gathered} \tag{15}$$

from which we obtain

$$\|G\|^2_{\widetilde{\mathcal{H}}_2} = \text{tr}\left[C_1 K C_1^T\right] + \text{tr}\left[M Q_{22}\right] + \|G_1\|^2_{\mathcal{H}_2} + \|G_2\|^2_{\mathcal{H}_2}.$$

Proof: The proof is a direct consequence of substituting the derived identities above into optimization problem (14). ■

Remark 1: Observe that the cost function is completely decoupled between the two subsystems G_1 and G_2. The only coupling present occurs in the LMI constraint (15) in the off diagonal block. Also note that $\|G_i\|_{\mathcal{H}_2}$ are fixed values and can thus be computed off line and independently.

The feasibility of LMI (15) can be equivalently cast as checking for positive-realness of the interconnection.

Proposition 2: Define the LTI systems

$$\begin{aligned} H &= \left[\begin{array}{c|c} A_{11} & P_{11}A_{21}^T + B_1B_2^T \\ \hline -A_{21} & M/2 \end{array}\right], \\ J &= \left[\begin{array}{c|c} A_{11} & A_{12}P_{12} + P_{12}A_{22}^T \\ \hline A_{21} & M/2 \end{array}\right]. \end{aligned}$$

Then the feasibility of LMI (15) is equivalent to both $H(j\omega) + H^*(j\omega) \geq 0$, $J(j\omega) + J^*(j\omega) \geq 0$ $\forall \omega \in [0, \infty)$, i.e. they are positive real.

Proof: The result follows immediately from a slightly modified version of the KYP Lemma which is presented in the Appendix. ∎

Another way to compute the bound is through the dual optimization programme, which is as follows:

$$\begin{gathered} \max_Z \ \mathrm{tr}[B_1^T Z_{11} B_1] + 2 \cdot \mathrm{tr}[B_1^T Z_{12} B_2] + \mathrm{tr}[B_2^T Z_{22} B_2] \\ \text{subject to:} \\ \mathrm{Sym}\left(\begin{pmatrix} Z_{11} & Z_{12} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \right) + C_1^T C_1 = 0 \\ \mathrm{Sym}(Z_{22} A_{22}) + C_2^T C_2 = 0 \\ \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{pmatrix} \succeq 0 \end{gathered} \tag{16}$$

where Sym$(\cdot)$ denotes the symmetric completion. Note that (16) is a relaxation to the programme

$$\begin{gathered} \max_Z \ \mathrm{tr}[B_1^T Z_{11} B_1] + 2 \cdot \mathrm{tr}[B_1^T Z_{12} B_2] + \mathrm{tr}[B_2^T Z_{22} B_2] \\ \text{subject to:} \\ \mathrm{Sym}\left(\begin{pmatrix} Z_{11} & Z_{12} \end{pmatrix} \begin{pmatrix} A_{11} \\ A_{21} \end{pmatrix} \right) + C_1^T C_1 = 0 \\ A_{11}^T Z_{12} + A_{21}^T Z_{22} + Z_{12} A_{22} + C_1^T C_2 = 0 \\ \mathrm{Sym}(Z_{22} A_{22}) + C_2^T C_2 = 0 \\ Z = \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{12}^T & Z_{22} \end{pmatrix} \succeq 0 \end{gathered} \tag{17}$$

which computes the norm of G. Indeed, (17) has a unique Z satisfying the constraints, because the constraints describe the Lyapunov equation $A^T Z + ZA + C^T C = 0$. Hence there is no maximisation and $Z = Q$, where Q is a full observability Gramian.

C. Error Bounds

We will now derive a bound on the error between the actual $\mathcal{H}_2$-norm of G and the estimate $\|G\|^2_{\widetilde{\mathcal{H}}_2}$ obtained using a structured Gramian. The result is derived using the a structured controllability Gramian. Similar results follow naturally using the observability Gramian but due to the limited space we focus on the first case. Define the error $\mathcal{E}$ to be the difference between the actual $\mathcal{H}_2$-norm and an estimate using a structured Gramian:

$$\mathcal{E} = \|G\|^2_{\widetilde{\mathcal{H}}_2} - \|G\|^2_{\mathcal{H}_2} = \mathrm{tr}[C\tilde{P}C^T] - \mathrm{tr}[CPC^T]$$

where $\tilde{P} \in \mathcal{S}$. We denote by $\mathcal{E}^*$ the optimal (i.e. the smallest) error estimate. As was done in (14) we define the generalised structured Gramian $P = \mathrm{diag}(X, Y) \in \mathcal{S}$. Then the error can be computed via the following convex optimisation programme:

$$\begin{aligned} \mathcal{E}^* = \min_{X,Y} \quad & \mathrm{tr}\left[C \begin{pmatrix} X - P_{11} & -P_{12} \\ -P_{12}^T & Y - P_{22} \end{pmatrix} C^T \right] \\ \text{subject to:} \quad & X \succeq P_{11}, \quad Y \succeq P_{22} \\ & \mathrm{sym}\left(A \begin{pmatrix} X - P_{11} & -P_{12} \\ -P_{12}^T & Y - P_{22} \end{pmatrix} \right) \preceq 0 \end{aligned} \tag{18}$$

Let us now define the variables

$$\begin{aligned} K &= X - P_{11}, \\ M &= -(A_{22}(Y - P_{22}) + (Y - P_{22})A_{22}^T), \\ N &= C_2(Y - P_{22})C_2^T. \end{aligned}$$

Note that N is obtained by expanding out the cost function from (18). Using the definitions above gives

$$\begin{aligned} \mathrm{tr}[N] &= -\mathrm{tr}[(A_{22}^T Q_{22} + Q_{22} A_{22})(Y - P_{22})] \\ &= -\mathrm{tr}[(A_{22}(Y - P_{22}) + (Y - P_{22})A_{22}^T)Q_{22}] \\ &= \mathrm{tr}[MQ_{22}]. \end{aligned}$$

Applying some simple algebraic manipulations it is straightforward to obtain the following programme:

$$\begin{aligned} \mathcal{E}^* = \min_{K,M} \quad & \mathrm{tr}(C_1 K C_1^T) - 2\mathrm{tr}(C_1 P_{12} C_2^T) + \mathrm{tr}(MQ_{22}) \\ \text{subject to:} \quad & K \succeq 0, \quad M \succeq 0 \\ & \begin{pmatrix} A_{11}K + KA_{11}^T & KA_{21}^T - \tilde{B} \\ * & -\tilde{D} - \tilde{D}^T - M \end{pmatrix} \preceq 0, \end{aligned}$$

where $\tilde{B} = A_{11}P_{12} + P_{12}A_{22}^T$ and $\tilde{D} = A_{21}P_{12}$.

Note that

$$C_2^T C_1 = -Q_{12}A_{11}^T - A_{22}Q_{12}^T - A_{22}Q_{22}.$$

Now, if $Q_{12} = 0$, then $C_2^T C_1 = -A_{22}Q_{22}$ and it can be shown that:

$$\begin{aligned} \mathcal{E}^* \ = \ & \min_{K,M} \ \mathrm{tr}(C_1 K C_1^T) + \mathrm{tr}(MQ_{22}) \\ \text{subject to:} \quad & K \succeq 0, \quad M \succeq 0 \\ & \begin{pmatrix} A_{11}K + KA_{11}^T & KA_{21}^T - \tilde{B} \\ * & -M \end{pmatrix} \preceq 0 \end{aligned} \tag{19}$$

Note also that

$$A_{11}P_{12} + P_{12}A_{22}^T = -P_{11}A_{21}^T - B_1 B_2^T$$

due to the fact that P is a controllability Gramian of G. Hence the LMI in (19) is exactly the same as in (15).

Remark 2: Computing $\mathcal{E}^*$ in the closed form has also implications for structured model reduction. Indeed, $\tilde{P}$ can be used to provide stable reduced order models of subsystems, while preserving the structure of the interconnection. Let $\tilde{Q}$ be a generalised controllability Gramian satisfying the condition for $\tilde{Q} \in S$. It is known that the singular values of $\tilde{P}\tilde{Q}$ are conservative with respect to the singular values of PQ. Hence these singular values provide conservative estimates of the approximation accuracy, however, it is not clear how much conservatism is gained by imposing the structure $\mathcal{S}$ on the generalised Gramians $\tilde{P}$ and $\tilde{Q}$. This programme can be considered as a first step towards answering this question.

IV. Conclusion

In this work we have presented some initial results on the accuracy of $\mathcal{H}_2$-norm estimation of LTI systems with block triangular structure using structured Gramians. It has been shown how the optimal cost functions have a clear decentralised interpretation. Furthermore bounds on the error between the estimates have been derived in terms of LMIs. Further results upon existence of solutions, decentralised controller synthesis (state-feedback) and links to model reduction will appear in forthcoming publications.

Acknowledgment

JA acknowledges funding through a junior research fellowship from St. John's College, Oxford and thanks Antonis Papachristodoulou for helpful discussions related to this work. AS is supported by the EPSRC Science and Innovation Award EP/G036004/1.

References

[1] M. Arcak and E. D. Sontag, "Diagonal stability of a class of cyclic systems and its connection with the secant criterion," *Automatica*, vol. 42, no. 9, pp. 1531–1537, 2006.

[2] N. Matni and M. Oishi, "Stability of switched block upper-triangular linear systems with bounded switching delay: Application to large distributed systems," in *Proc. of the American Control Conference*, 2012.

[3] A. Lamperski and J. C. Doyle, "On the structure of state-feedback LQG controllers for distributed systems with communication delays," in *Proc. of the 50th IEEE Conference on Decision and Control and European Control Conference*. IEEE, 2011, pp. 6901–6906.

[4] L. Lessard and S. Lall, "Optimal controller synthesis for the decentralized two-player problem with output feedback," in *Proc. of the American Control Conference*. IEEE, 2012, pp. 6314–6321.

[5] L. Lessard, M. Kristalny, and A. Rantzer, "On structured realizability and stabilizability of linear systems," in *Proc. of the American Control Conference*, 2012, pp. 5694–5700.

[6] H. Sandberg and R. M. Murray, "Model reduction of interconnected linear systems using structured gramians," in *Proc. of the 17th IFAC World Congress*, 2008, pp. 8725–8730.

[7] S. Boyd and Q. Yang, "Structured and simultaneous Lyapunov functions for system stability problems," *International Journal of Control*, vol. 49, no. 6, pp. 2215–2240, 1989.

[8] E. Kaszkurewicz and E. Bhaya, *Matrix diagonal stability in systems and communication*. Boston: Birkhauser, 2000.

[9] J. Kraaijevanger, "A characterization of Lyapunov diagonal stability using Hadamard products," *Linear Algebra and its Applications*, vol. 151, pp. 245 – 254, 1991.

[10] A. Rantzer, "Optimizing positively dominated systems," in *IEEE 51st Annual Conference on Decision and Control*, 2012, pp. 272–277.

[11] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge University Press, 1990.

[12] G. Barker, A. Berman, and R. Plemmons, "Positive diagonal solutions to the Lyapunov equations," *Linear and Multilinear Algebra*, vol. 5, no. 4, pp. 249–256, 1978.

[13] K. Zhou, J. C. Doyle, and K. Glover, *Robust and optimal control*. Prentice Hall New Jersey, 1996.

Appendix

In deriving Proposition 2 we require a version of the positive real lemma that is presented in a non-standard form. For completeness we include it here. Recall that for a system with a state-space realization

$$H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = C(sI - A)^{-1}B + D$$

its transpose is defined by

$$H = \left[\begin{array}{c|c} A^T & C^T \\ \hline B^T & D^T \end{array}\right] = B^T(sI - A^T)^{-1}C^T + D^T,$$

and its conjugate transposition is

$$H^* = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] = B^T(-sI - A^T)^{-1}C^T + D^T.$$

Lemma 3: Let G be a stable, strictly proper, transfer function, with a state-space realisation (not necessarily minimal)

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right].$$

Suppose there exist a positive semidefinite matrix K such that

$$\begin{pmatrix} AK + KA^T & KC^T - B \\ * & -D - D^T \end{pmatrix} \preceq 0, \tag{20}$$

then $G + G^T$ is a positive-real transfer function. Moreover, $G + G^T$ is equal to HH^*, where

$$H = \left[\begin{array}{c|c} A & Q \\ \hline C & W \end{array}\right],$$

where $WW^T = D + D^T$, and $QW^T = KC^T - B^T$.
Sketch of the Proof: If (20) is valid then there exist matrices Q and W such that:

$$\begin{aligned} AK + KA^T &= -QQ^T \\ CK - B^T &= WQ^T \\ D + D^T &= WW^T \end{aligned} \tag{21}$$

Let $\tilde{A} = A^T$, $\tilde{B} = C^T$, $\tilde{C} = B^T$, $\tilde{Q} = Q^T$, $\tilde{W} = W^T$, and rewrite the equations as:

$$\begin{aligned} \tilde{A}^T K + K\tilde{A} &= -\tilde{Q}^T\tilde{Q} \\ \tilde{B}^T K - \tilde{C} &= \tilde{W}^T\tilde{Q} \\ D + D^T &= \tilde{W}^T\tilde{W} \end{aligned} \tag{22}$$

Note that the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$, D^T constitute a state-space realisation of the transfer function G^T. By applying Theorem 13.25 from [13] to G^T, we show that G^T (and hence G) is a positive real function. Moreover, we can compute the right spectral factor N, that is $G^T + (G^T)^* = N^* N$, as:

$$N = \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{Q} & \tilde{W} \end{array}\right] = \left[\begin{array}{c|c} A^T & C^T \\ \hline Q^T & W^T \end{array}\right].$$

If we transpose the expression $G^T + (G^T)^*$, then we obtain the left spectral factor H as $H = N^T$, that is

$$H = \left[\begin{array}{c|c} A & Q \\ \hline C & W \end{array}\right].$$

This concludes the proof.