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Optimal charging/discharging and commutativity properties of ARE solutions for RLC circuits

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***Abstract*— We consider multi-port RLC circuits and study the problems of charging the circuit to a specified state with the minimum supply of energy and that of discharging the circuit from a specified state with maximum energy extraction. These are known to be respectively the anti-stabilizing and stabilizing solutions of the associated Algebraic Riccati Equation (ARE). For the special case of states being *physical states*, i.e. capacitor voltages and inductor currents, several interesting properties are shown. Using Hamiltonian matrix arguments, we prove that for multi-port RLC circuits, the state space realizations of the impedance $Z(s)$ and the admittance $Y(s)$ are related such that they both admit the same stabilizing solution of the ARE. The same holds for the anti-stabilizing solution too. We next show that the corresponding 'closed loop state transition matrices' computed using $Z(s)$ or $Y(s)$ are equal too.**

We next consider single-port RLC circuits, for which we provide capacitor/inductor loop/cut-set conditions with respect to the port that result in a pole at the origin or a pair of purely imaginary poles. In this case we show, using network topological arguments, that all the ARE solutions and the corresponding closed loop transition matrices share common eigenvectors. We give physical RLC-circuit based insights for these results. These results have potential implications for port-controlled Hamiltonian matrices.

***Keywords:* passivity, storage functions, impedance, admittance, cutset, loop, open-circuit, short-circuit, Hamiltonian matrix**

I. INTRODUCTION

This paper deals with energy considerations (optimal charging/discharging) of multi-port RLC circuits that have a biproper impedance and admittance transfer matrices. We use the relation of the optimal energies with the extremal storage functions, which are also the extremal solutions of the Algebraic Riccati Equation (ARE) to infer various interesting properties, like independence of the maximal ARE solution with respect to the state space realizations of the admittance or impedance of the RLC circuit: as long as states correspond to the physical states, i.e. capacitor voltages and inductor currents. We use these states throughout the paper. We further show that the 'closed loop dynamics' too is the same as obtained from either the admittance and impedance state space descriptions. While these results appear reasonable using intuitive arguments based on the behavioral approach as noted in Remark 4.2, we use Hamiltonian matrix arguments to prove these properties. We next show that for single-port networks, under suitable locations of the capacitor/inductor with respect to the port, all the ARE solutions and the corresponding closed loop state transition matrices share certain common eigenvectors related to capacitor voltages and inductor currents. We formulate and prove this using network topology arguments.

In order to describe the main results, we define the relevant matrices here. Suppose $x(t)$ denotes the vector of all capacitor voltages and inductor currents at time t: the states of the RLC system. Let n be the total of the number of capacitors and inductors. The following five square matrices, each of size $n \times n$, play a central role in this paper.

1) $K_{\rm a}$, the diagonal and positive definite matrix such that $x_0^T K_{\rm a} x_0/2$ is the *actual* (physical) energy in the system when the state x is $x_0 \in \mathbb{R}^n$: the diagonal elements are the capacitances and inductances.
2) $K_{\max}$: the symmetric and positive definite matrix such that $x_0^T K_{\max} x_0/2$ is the minimum energy required to charge the (initially discharged) circuit to state x_0.
3) $K_{\min}$: the symmetric and positive semi-definite matrix such that $x_0^T K_{\min} x_0/2$ is the maximum energy that can be extracted from the circuit from state x_0 (to finally discharged state $x = 0$).
4) $A_{K_{\max}}$: the 'closed loop' state transition matrix that 'achieves'[1] the minimum energy required while charging: see equation (5) for the definition of A_K.
5) $A_{K_{\min}}$: the closed loop state transition matrix that achieves the maximum energy extractable while discharging.

Of course, these matrices are well-studied (in, for example, [13], [16]) using the theory of ARE, LMIs and Hamiltonian matrices: these matrices depend on the state space realization of the system and the transfer function. The earlier part of this paper concerns proving independence of these matrices whether the state space realization is constructed from the RLC system's admittance[2] $Y(s)$ or from its impedance $Z(s)$. The later part of this paper concerns proving that certain locations of capacitors/inductors with respect to the port result in common eigenvectors of the above five matrices.

While these results are interesting in their own right, they have applications in more general port-controlled Hamiltonian systems, of which RLC circuits form a central structured class of systems: see [12]. Closely related is the approach followed in bond-graphs where the power is a product of a 'flow' variable and an 'across' variable: see [2] and the references therein. Energy storage elements are often of the type where energy is a constant times the square of either the

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[1]Note that we are aiming at only an infimum and one might not be able to achieve the minimum: see equations (6), (7) and (5). This is indeed the situation for the problems addressed in this paper. This is elaborated on in Section VIII.

[2]One may note that, although we work with just admittance or impedance in this paper, using [17, Proposition 2, Page 71], it is possible to state all our results in more general hybrid transfer matrices also.

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flow-variable or the across-variable of the storage element. Our results are extendable to this class of systems too.

The paper is organized as follows. The rest of this section deals with notation. The following section contains the necessary preliminaries about network topology, Hamiltonian matrices and the Algebraic Riccati Equation. Section III contains various assumptions for the results in this paper and also their system-theoretic justifications. Section IV contains our main results about independence of the solution to the ARE and the closed loop transition matrix with respect to the state space realizations of the admittance and the impedance. Section V contains our main results on conditions under which all the ARE solutions, together with the corresponding closed loop transition matrices, share a few common eigenvectors. Section VI contains an example demonstrating the results of this paper. While we provide results for the case of imaginary axis open-loop poles in Section V, we investigate in Section VII certain limiting case behavior as the dominant pole (i.e. the pole that is closest to the imaginary axis) approaches the origin. We end the paper with a few concluding remarks and scope for future work in Section VIII.

The notation we use is standard. The set of real numbers is denoted by $\mathbb{R}$ and that of complex numbers by $\mathbb{C}$. The set of n-tuples of real numbers is denoted by $\mathbb{R}^n$ and we assume an element $v \in \mathbb{R}^n$ is a *column* vector. The set of matrices with entries from $\mathbb{R}$ and having m rows and n columns is denoted by $\mathbb{R}^{m\times n}$. The $n \times n$ identity matrix is denoted by I_n and its i-th column by e_i. For a matrix $K \in \mathbb{R}^{n\times n}$, K_{ij} refers to the entry in the i-th row and j-th column. The block diagonal matrix $\operatorname{diag}(A_1, A_2)$ contains as its diagonal blocks the square matrices A_1 and A_2 (of possibly unequal sizes).

II. Preliminaries

This section first covers preliminaries of network topology and later briefs regarding solutions of the Algebraic Riccati Equation (ARE) and the Hamiltonian matrix.

A. Network topology

We consider an electrical network whose topology is described by a directed graph G with vertex set V and edge set E. The graph G has no self-loops (i.e. both terminals of a device connected to the same node), but can have multiple edges across a pair of nodes; this is typical of electrical networks. One or more of these edges is a 'port', and the remaining edges are either resistors, inductors or capacitors. While there can be any number of resistors, there is an upper bound on the number of capacitors and on the number of inductors: this bound gets imposed by further assumptions on the network and the associated state space realization. This is elaborated on in Section III.

A *loop* is defined as a collection of edges that form a cycle: the directions of the edges are ignored for this purpose. In this paper, we deal with only connected graphs. A cutset C is a subset of edges such that C has the property that C's removal causes the graph to become disconnected, and further, no proper subset of C has this property.

When analyzing the situation at DC, i.e. at $s = 0$, we consider all the capacitors as open and all the inductors as short. When analyzing the situation at $s = \infty$, i.e. at very high frequencies, we assume all capacitors are short and the inductors are open.

B. Hamiltonian matrix and ARE solutions

Dissipative systems theory and the link between storage functions and Algebraic Riccati Inequality is well-studied in the literature: see [16], [13], for example. We cover only the very essential preliminaries here. Consider an RLC system with minimal input/state/output representation

$$\dot{x} = Ax + Bu, \qquad y = Cx + Du, \tag{1}$$

with port variables v (voltage) and i (current) and $A \in \mathbb{R}^{n\times n}$ and $B, C^T \in \mathbb{R}^{n\times p}$ where p is the number of ports. Thus, $v(t), i(t) \in \mathbb{R}^p$ and one of these is an input u, and the other is output y: this depends on whether the state-space realization is of the impedance or the admittance. Define the supply rate, i.e. power $P(v,i) := u^T y = v^T i$. For the RLC system, because it is 'dissipative' with respect to this supply rate, there exists a real symmetric solution $K = K^T \in \mathbb{R}^{n\times n}$ to the following Linear Matrix Inequality (LMI)

$$\begin{bmatrix} (A^T K + KA) & (KB - C^T) \\ (B^T K - C) & -(D + D^T) \end{bmatrix} \leqslant 0. \tag{2}$$

The set of all trajectories (u,x,y) that satisfy the state space description (1) is defined as the *behavior* $\mathfrak{B}$ of the system. For this system, with K satisfying LMI (2), the state function $x^T K x$ is a *storage function*, i.e. $\frac{d}{dt} x^T K x \leqslant 2u^T y$ for all x, u and y that satisfy equation (1). Assuming $D + D^T > 0$, the Schur complement with respect to $D + D^T$ in the matrix of equation (2) gives the *Algebraic Riccati Inequality* (ARI)

$$(A^T K + KA) + (KB - C^T)(D + D^T)^{-1}(B^T K - C) \leqslant 0. \tag{3}$$

Instead of the inequality in (3), we need the *equality* more frequently: the *Algebraic Riccati Equation* (ARE); we will refer to equation (3) as the ARE (3). Corresponding to the ARE (3), the *Hamiltonian matrix* $H \in \mathbb{R}^{2n\times 2n}$ is defined as

$$H := \begin{bmatrix} A - B(D + D^T)^{-1}C & B(D + D^T)^{-1}B^T \\ -C^T(D + D^T)^{-1}C & -\left(A - B(D + D^T)^{-1}C\right)^T \end{bmatrix}. \tag{4}$$

It is straightforward that if two state space realizations have the same Hamiltonian matrix, then all the corresponding ARE solutions are the same: we use this fact later. Each solution K of the ARE gives rise to what we will call a *closed loop state transition matrix* A_K corresponding to an n-dimensional invariant subspace of H; we define A_K as

$$A_K := A - B(D + D^T)^{-1}C + B(D + D^T)^{-1}B^T K. \tag{5}$$

The set of ARE solutions (for the controllable/observable case) is known to be a bounded set with a maximum $K_{\max}$ and a minimum $K_{\min}$ with respect to the partial ordering defined by sign-definiteness on the real symmetric matrices. Further, the maximum and minimum ARE solutions have the following significance. For a given $a \in \mathbb{R}^n$, consider $\mathfrak{B}_a$, the set of all continuous system trajectories (u,x,y) satisfying equation (1) with $x(0) = a$. Then,

$$a^T K_{\max} a = \inf_{\substack{(u,x,y) \in \mathfrak{B}_a, \\ x(-\infty) = 0}} \int_{-\infty}^{0} 2uy \, dt \tag{6}$$

and

$$a^T K_{\min} a = \sup_{\substack{(u,x,y) \in \mathfrak{B}_a, \\ x(\infty)=0}} \int_0^\infty -2uy \, \mathrm{d}t. \qquad (7)$$

See [16, Section 6] for a detailed treatment and the proofs.

III. Network assumptions and significances

We make a few assumptions on the RLC network: while most of these can be justified as reasonable system-theoretic properties, some of them are restrictive and have been assumed for simplicity of exposition. *These assumptions hold for the rest of this paper.* See [3], [8] for a detailed exposition on the link between electrical networks and graph theory.

1) We assume there are a total of n components consisting of capacitors and inductors, and there are p ports. Hence $A \in \mathbb{R}^{n\times n}$, $B, C^T \in \mathbb{R}^{n\times p}$ and $D \in \mathbb{R}^{p\times p}$. Resistors do not affect the state space matrices' sizes. In order to provide an energy interpretation, capacitor voltages and inductor currents are used as states.
2) We assume all inductances and capacitances are of value one. This helps by $K_{\mathrm{a}} = I_n$. Many of the results are unaffected by this assumption since this amounts to normalizing the inductor currents and capacitor voltages. Each resistor has an arbitrary but positive and finite resistance.
3) We assume that the multi-port network is such that the port edges do not form a cutset nor a loop. This ensures that both the admittance and the impedance transfer matrices exist.
4) We assume that the capacitors do not form a loop. Also we assume that the inductors do not form a cut-set. Further, the capacitances and the port edges together are assumed to not form loops nor do the inductors and port edges form cutsets. This ensures existence of a *regular* state space realization, instead of a singular/descriptor state space form. These two assumptions also impose restrictions on the maximum number of capacitors and the maximum number of inductors. See also Assumption 6 below.
5) We assume that none of the capacitor edges form a cutset and none of the inductors form a loop. This is necessary for the eigenvalues at the origin, if any, to be controllable. Further, again for controllability, we assume that the edges comprising of just the capacitor edges and the port edges are such that, if any cutsets are formed by them, then there exists an independent set of cutsets such that each cutset contains at least one port edge. We assume the same for the inductor edges and the port edges in the context of loops. This is equivalent to controllability at the origin.
6) While constructing the ARE, since we take the Schur complement of the LMI with respect to $D + D^T$, we need its nonsingularity. For RLC networks, due to reciprocity, D is a symmetric matrix and hence nonsingularity of $(D + D^T)$ translates to biproperness of the transfer matrix $Y(s)$ or $Z(s)$. This is equivalent to the condition that when all the capacitors are shorted and the inductors are opened (i.e. $s = \infty$), then there is a finite and nonsingular resistance matrix across the port. This nonsingularity corresponds to the port becoming neither open nor short (along any direction) when all capacitors are shorted and inductors are opened. In particular, this rules out one or more capacitors across the port and also rules out one or more inductors in series with the port. This condition is related to Assumption 4 above.
7) We assume that a finite amount of energy is required to charge the circuit to any given state $a \in \mathbb{R}^n$, from the initially discharged state. This is equivalent to controllability.
8) We assume that a nonzero amount of energy can be extracted out from any nonzero state $a \in \mathbb{R}^n$ while discharging the circuit. This is equivalent to observability.

IV. Independence of optimal energies and closed loop transition matrices from $Y(s)$ and $Z(s)$

This section contains our first main result: Theorem 4.1, which states that for an arbitrary multi-port RLC circuit, whether one uses a state space realization of the admittance $Y(s)$ or the impedance $Z(s)$, the maximum ARE solution is the same. Similarly, the minimum ARE solution is also the same. Further, though the realizations are different, each of the resulting closed loop transition matrices (equation (5)) is the same whichever state space description is used.

Theorem 4.1: Consider an RLC circuit which has a biproper admittance transfer matrix $Y(s)$. Suppose (A_y, B_y, C_y, D_y) and (A_z, B_z, C_z, D_z) are minimal state-space realizations of the admittance $Y(s)$ and the impedance $Z(s)$ respectively. Let the closed loop state transition matrices (as defined in equation (5)) during charging be $A^Y_{K_{\max}}$ and $A^Z_{K_{\max}}$ with respect to the above realizations and let those during discharging be $A^Y_{K_{\min}}$ and $A^Z_{K_{\min}}$. Suppose $K^Y_{\max}$, $K^Y_{\min}$, $K^Y_{\max}$ and $K^Z_{\max}$ are the corresponding positive definite matrices indicating the optimal energies. Then, the following hold.

(a) $K^Y_{\max} = K^Z_{\max}$, (b) $K^Y_{\min} = K^Z_{\min}$,

(c) $A^Y_{K_{\max}} = A^Z_{K_{\max}}$ and (d) $A^Y_{K_{\min}} = A^Z_{K_{\min}}$.

The proof of the above result requires further results that we first state and prove below. After this we prove Theorem 4.1 by constructing the Hamiltonian matrix and the closed loop transition matrix corresponding to the two state space realizations. The following remark notes some significance.

Remark 4.2: Thinking of the RLC system from a behavioral viewpoint, Theorem 4.1 merely says that the optimal charging/discharging energies are independent of the input/output partition, though intermediate matrices in the calculation procedure do depend on the partition. Further, the optimal trajectories also depend on just the system (and the supply rate $u^T y$). In fact, the set of storage functions is also known to depend only on the system and the supply rate. Given these observations, the above theorem merely formalizes this and is proved using Hamiltonian matrix arguments.

The following lemma relates the state space realizations of the admittance and the impedance of an RLC circuit. Of course, this is also the relation between the state space realizations of a system transfer matrix and its inverse:

assuming both are proper. Since the states are intended to be the same (due to their physical meaning), we state this in the context of an RLC circuit.

Lemma 4.3: Consider (A_z, B_z, C_z, D_z) and (A_y, B_y, C_y, D_y), the state space realizations of biproper $Z(s)$ and $Y(s)$. Then,

$$A_y = A_z - B_z D_z^{-1} C_z, \qquad B_y = B_z D_z^{-1},$$
$$C_y = -D_z^{-1} C_z \quad \text{and} \quad D_y = D_z^{-1}.$$

Proof of Lemma 4.3: The state space representation of the system in terms of (A_z, B_z, C_z, D_z) is:

$$\begin{bmatrix} sI - A_z & -B_z & 0 \\ C_z & D_z & -I \end{bmatrix} \begin{bmatrix} x \\ i \\ v \end{bmatrix} = 0.$$

Perform the row operations on the above equations that correspond to premultiplying the above matrix with the following square and nonsingular matrix of size $n+m$:

$$\begin{bmatrix} I_n & B_z D_z^{-1} \\ 0 & D_z^{-1} \end{bmatrix} \text{ to get } \begin{bmatrix} sI - A_z + B_z D_z^{-1} C_z & 0 & -B_z D_z^{-1} \\ D_z^{-1} C_z & I & -D_z^{-1} \end{bmatrix} \begin{bmatrix} x \\ i \\ v \end{bmatrix} = 0. \tag{8}$$

The admittance realization for this system with input v and output i is

$$\begin{bmatrix} sI - A_y & -B_y & 0 \\ C_y & D_y & -I \end{bmatrix} \begin{bmatrix} x \\ v \\ i \end{bmatrix} = 0. \tag{9}$$

Comparing (8) with (9) we get

$$A_y = A_z - B_z D_z^{-1} C_z, \qquad B_y = B_z D_z^{-1},$$
$$C_y = -D_z^{-1} C_z \quad \text{and} \quad D_y = D_z^{-1}.$$

This proves Lemma 4.3. ☐

Equipped with the above lemma, we now prove Theorem 4.1 .

Proof of Theorem 4.1: The proof proceeds by showing that, though the state space realizations of the admittance $Y(s)$ and the impedance $Z(s)$ are different, their inter-relation is such that the Hamiltonian matrix is the same. Construct the Hamiltonian matrix from (A_y, B_y, C_y, D_y):

$$H_y = \begin{bmatrix} A_y - B_y(D_y + D_y^T)^{-1} C_y & B_y(D_y + D_y^T)^{-1} B_y^T \\ -C_y^T (D_y + D_y^T)^{-1} C_y & -\left(A_y - B_y(D_y + D_y^T)^{-1} C_y\right)^T \end{bmatrix}.$$

We will show that each of the four blocks are the same for the corresponding block of H_z. Note that due to reciprocity of the RLC network, D_y and D_z are symmetric matrices: we use this symmetry below. Using Lemma 4.3

$$\begin{aligned} A_y - B_y(D_y + D_y^T)^{-1} C_y &= A_z - B_z D_z^{-1} C_z + B_z (2D_z)^{-1} C_z \\ &= A_z - B_z (2D_z)^{-1} C_z = A_z - B_z (D_z + D_z^T)^{-1} C_z. \end{aligned}$$

Similarly, Lemma 4.3 helps simplify $B_y(D_y + D_y^T)^{-1} B_y^T$:

$$B_y(D_y + D_y^T)^{-1} B_y^T = B_z (D_z + D_z^T)^{-1} B_z^T,$$

and

$$-C_y^T (D_y + D_y^T)^{-1} C_y = -C_z^T (D_z + D_z^T)^{-1} C_z. \tag{10}$$

Hence all the submatrices of the Hamiltonian with respect to impedance $Z(s)$ and admittance $Y(s)$ realizations are the same. This proves that the set of solutions K to the ARE is the same whether the state space realization is constructed from $Z(s)$ or $Y(s)$. Hence the set of corresponding closed loop state transition matrices A_K is the same too. ☐

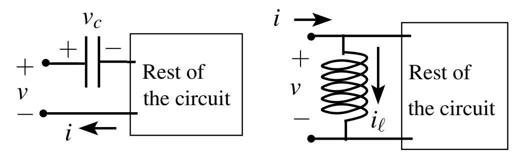

Fig. 1. Pole or zero at the origin

V. Common eigenvectors of extremal ARE solutions and closed loop state-transition matrices

In this section we formulate conditions on single-port RLC circuits under which all the ARE solutions share some common eigenvectors. In this case, the closed loop transition matrices too turn out to share those common eigenvectors. Figure 1 is relevant to the following theorem about a capacitor in series with the port; the dual result of an inductor across the port can be easily stated and proved and is hence skipped.

Theorem 5.1: Consider a single-port RLC circuit with a capacitor forming a cut-set with the port. Suppose (A_z, B_z, C_z, d_z) is a state-space description of the impedance $Z(s)$ and let x_1, the first component of the state, be v_C: the voltage across the capacitor. Then, the following hold.

1) Every ARE solution K can be written as $\operatorname{diag}(1, \tilde{K})$, with $\tilde{K}$ a symmetric matrix of size $(n-1) \times (n-1)$.
2) e_1 is an eigenvector of each ARE solution K and also of the corresponding closed loop state transition matrix A_K, with corresponding eigenvalue 1 and 0 respectively.

Theorem 5.1 is proved later below in this section. Our next main result is about an LC tank (also called the LC resonant circuit), instead of a capacitor, in series with the port: this is the case of purely imaginary axis eigenvalues also causing common eigenvectors. Figures 2 and 3 illustrate these situations: here too we skip the dual result corresponding to Figure 2.

Theorem 5.2: Consider a single-port RLC circuit with an LC forming a cut-set with the port. Let x_1 be v_C and x_2 be i_L (the states corresponding to the LC tank). Let (A_z, B_z, C_z, d_z) be a state-space description of the impedance $Z(s)$. Then, the following hold.

1) Every ARE solution K can be written as $\operatorname{diag}(I_2, \tilde{K})$, with $\tilde{K}$ a symmetric matrix of size $(n-2) \times (n-2)$.
2) e_1 and e_2 are eigenvectors of each ARE solution K with corresponding eigenvalue 1.
3) The vectors e_1 and e_2 span an invariant subspace of each closed loop state transition matrix A_K with eigenvalues $\pm i$.

The proofs of the above two theorems require further auxiliary results: we state these and prove them first.

Lemma 5.3: Suppose a single-port RLC circuit satisfies the assumptions listed in Section III. Assume there is a capacitor in series with the port and let the first state correspond to this

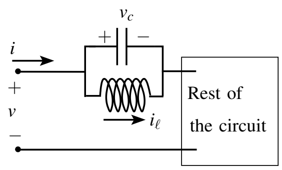

Fig. 2. LC tank in series with the port

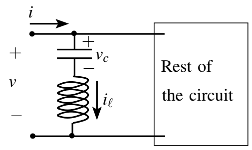

Fig. 3. LC tank formed upon shorting of the port

capacitor's voltage. Then A_z has its first row and first column identically zero. Further, the first entry of each of B_z and C_z equals 1.

Analogously for an inductor across the port with the first state as the current through this inductor: A_y has its first row and first column identically zero. We do not delve further into this dual situation.

Proof of Lemma 5.3: This is proved using the superposition principle. Consider Figure 1 where a capacitor is in series with the port (and the rest of the circuit). In order to prove the property of A_z, we assume the input i is zero (i.e the port is open). The first column of A_z being zero is equivalent to the statement that the rates of change of all states are zero, when all states, except possibly the first state v_c, are zero: this equivalence follows from the meaning of A_z and from the superposition principle. Since the port is open, no current flows through the capacitor and hence v_c being nonzero cannot cause a nonzero rate of change in any of the other states. This proves that the first column of A_z is zero. We now apply a slightly different argument to prove that that the first *row* of A_z is zero too. Notice that, again with the port being open, the first row being zero is equivalent to v_c being constant even if one or more of the states are nonzero: this follows again from the meaning of A_z. Since the port is open and no current can flow through the capacitor, v_c is constant for arbitrary values of the other states. This proves that the first row of A_z is zero too.

We next prove that the first component of B_z is 1. Notice that $\frac{\mathrm{d}}{\mathrm{d}t}Cv_c = i$, and since v_c is the first state and since the capacitance is unity, this equation proves that B_z has first component equal to 1.

It remains to show that the first component of C_z is 1. Consider again Figure 1 with the port open. In order to see the value of the first component of C_z, assume all other states are zero: i.e. all other capacitors are shorted and inductors are opened. Since no current flows through the rest of the network, $v = v_c$ and hence the first component of C_z equals 1. We crucially used here the fact that the rest of the circuit is *not open* and has zero potential across it. (See Assumption 6 in Section III.) This proves Lemma 5.3. □

The next result is for the purely imaginary eigenvalues case.

Lemma 5.4: Suppose an RLC circuit satisfies the assumptions listed in Section III. Assume there is a capacitor and inductor tank in series with the port and let the first state correspond to this capacitor's voltage and the second state correspond to the inductor current: see Figure 2. Then the top left 2×2 leading principal submatrix of A_z equals $J := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and remaining entries in those two rows and two columns are all zero. The first and second elements of B_z are 1 and 0 respectively. The first and second elements of C_z also are 1 and 0.

Proof of Lemma 5.4: We use the superposition principle to prove this lemma too. Consider Figure 2 which has an LC tank in series with the rest of the circuit. Consider the impedance realization with input as the current through the circuit and output as the voltage across the port. Suppose the input and all the states, except those corresponding to the LC tank circuit, are zero. As the input is zero, we consider the port as open. Hence only the LC tank's inductor and capacitor states can affect each other. This implies that all entries of the first and second rows of A_z are zero except the leading principal 2×2 minor of A_z. Using Figure 2 and that the capacitance/inductance values are unity, due to the rest of the circuit being open, we obtain that this minor equals J as defined in the Lemma statement.

We next show that the first and second columns of A_z are zero except the top 2×2 minor. Assume again the port is open, i.e. the input $i = 0$. The required structure on A_z follows due to the first two states (of the LC tank) not affecting any of the other states in the circuit for lack of a discharging path through the other elements.

We now show that B_z has the first two entries as 1 and 0. Investigating B is same as studying the rates of change of states assuming all the states are zero. In particular, the inductor in the LC tank is open. Hence the input current i affects only the capacitor voltage in the LC tank. Exactly as proved in Lemma 5.3 the first element of B_z is 1. As i does not affect the inductor state, the second element is 0.

It remains to show that the first component of C_z is 1 and second component is 0. Consider again Figure 2 and assume the input $i = 0$, i.e. the port is open. In order to see the value of the first component of C_z, assume all other states are zero: i.e. all other capacitors are shorted and inductors are opened. Again using the principle of superposition, since no current flows through the rest of the network, $v = v_c$ and hence the first component of C_z equals 1. As no current flows through the rest of the circuit the inductor state will not affect the voltage at the port. Hence the second component of C_z is 0. We again used here the fact that the rest of the circuit is *not open* and has zero potential across it. (See Assumption 6 in Section III above.) This proves Lemma 5.4. □

Using the above lemmas, we next prove Theorems 5.1-5.2.

Proof of Theorem 5.1: The proof proceeds by finding a general structure for an arbitrary ARE solution K. We obtain e_1 as an eigenvector of K. We then show that e_1 lies in the nullspace of each A_K. In the sequel, we use $\frac{d_y}{2} = (d_z + d_z^T)^{-1}$.

Using Lemma 5.3, the vector e_1 lies in the null space of A_z. Premultiplying and postmultiplying the ARE in (3) by e_1^T

and e_1 respectively, we get

$$(A_ze_1)^TKe_1+e_1^TKA_ze_1+\tfrac{d_y}{2}e_1^T(KB_z-C_z^T)(B_z^TK-C_z)e_1=0,$$
$$\text{which gives}\quad \tfrac{d_y}{2}e_1^T(KB_z-C_z^T)(B_z^TK-C_z)e_1=0,$$
$$\text{and hence,}\quad (B_z^TK-C_z)e_1=0.$$

Post-multiplying the ARE (3) by e_1 we get

$$A_z^TKe_1+K(A_ze_1)+\tfrac{d_y}{2}(KB_z-C_z^T)(B_z^TK-C_z)e_1=0,$$
$$\text{and hence,}\quad A_z^TKe_1=0.$$

Since the rest of the matrix A_z can have an arbitrary structure, invoking the controllability assumption, we conclude that the first column of K is of the form ce_1. By symmetry of K, the first row of K is ce_1^T.

This proves that any ARE solution has its first row and first column identically zero except K_{11}. While for $K_{\rm a}$, the *actual* energy matrix, unit-capacitance ensures $c=1$, it remains to show that $c=1$ for *all* ARE solutions. This is shown using Lemma 5.3 and equation (2) as follows. The first row and the first column of the top-left block matrix $A_z^TK+KA_z$ of the matrix in LMI in (2) is zero. Due to semidefiniteness of that matrix, this causes the first component of $KB_z-C_z^T$ to be 0. Using Lemma 5.3, we infer $K_{11}=1$ for all ARE solutions.

We now show that e_1 is an eigenvector of not just every ARE solution K but also each corresponding closed loop state transition matrix A_K. Expressing A_K in terms of the impedance realization of the system, we get

$$A_K=A_z-\frac{d_y}{2}B_zC_z+\frac{d_y}{2}B_zB_z^TK$$

Postmultiplying the above equation by e_1 we obtain

$$A_Ke_1=A_ze_1-\tfrac{d_y}{2}B_zC_ze_1+\tfrac{d_y}{2}B_zB_z^TKe_1,$$
$$\text{which gives}\quad A_Ke_1=-\tfrac{d_y}{2}B_zC_ze_1+\tfrac{d_y}{2}B_zC_ze_1,$$

and hence $A_Ke_1=0$. This proves that e_1 is an eigenvector of each A_K with eigenvalue 0, and thus proves Theorem 5.1. □

The proof of Theorem 5.2 is similar: only that we have purely imaginary axis eigenvalues instead of at the origin.

Proof of Theorem 5.2: The proof proceeds by finding a general structure for K. From this structure we deduce that e_1 and e_2 span a K-invariant subspace for every ARE solution K. Exploiting this structure of K, we eventually show that e_1 and e_2 span the eigenspace of A_K with $\pm i$ as the eigenvalues.

Using Lemma 5.4, for $V\in\mathbb{R}^{n\times 2}$ with $V:=[e_1\ e_2]$, we have an A_z-invariant subspace, $A_zV=VJ$.

Consider the ARE in (3). Premultiplying and postmultiplying by V^T and V respectively we get:

$$(A_zV)^TKV+V^TKA_zV+\tfrac{d_y}{2}V^T(KB_z-C_z^T)(B_z^TK-C_z)V=0,$$
which simplifies to
$$J^TV^TKV+V^TKVJ+\tfrac{d_y}{2}(B_z^TKV-C_zV)^T(B_z^TKV-C_zV)=0.$$

Define $Q\in\mathbb{R}^{2\times 2}$ as $\frac{d_y}{2}(V^TKB_z-(C_zV)^T)(B_z^TKV-C_zV)$, and notice that Q is symmetric and $Q\geqslant 0$. Denote the leading principal 2×2 submatrix of K by K_{22}. This gives

$$J^TK_{22}+K_{22}J+Q=0. \qquad (11)$$

Now using the Lemma 5.5 (stated and proved below), we conclude that $Q=0$. Using the same lemma, we conclude that K_{22} equals cI_2 for some $c>0$. It remains to show that $c=1$ for all ARE solutions: this is true for $K_{\rm a}$, the *actual* energy matrix. We use the argument exactly like in the case of a single capacitor in series with the port. Note that $A_z^TK+KA_z$ has its first 2 rows and first 2 columns identically zero due to $J+J^T=0$. Using this fact within the LMI (2), we obtain that the first two components of $KB_z-C_z^T$ equal zero for every ARE solution K. Use Lemma 5.4 now to conclude that $c=1$ and thus $K_{22}=I_2$. Thus e_1 and e_2 form eigenvectors of K with eigenvalues 1. This proves Statements 1 and 2 of Theorem 5.2.

In order to prove the Statement 3, we express A_K in terms of the impedance realization of the system:

$$A_K=A_z-\frac{d_y}{2}B_zC_z+\frac{d_y}{2}B_zB_z^TK.$$

Postmultiplying the above equation by V we obtain

$$A_KV=A_zV-\tfrac{d_y}{2}B_zC_zV+\tfrac{d_y}{2}B_zB_z^T(KV),$$
$$\text{which gives}\quad A_KV=VJ-\tfrac{d_y}{2}B_zC_zV+\tfrac{d_y}{2}B_zC_zV,$$
$$\text{and hence,}\quad A_KV=VJ.$$

This proves that $\pm i$ are eigenvalues of A_K (Statement 3) and hence proves Theorem 5.2. □

In the above proof, we used the following auxiliary result.

Lemma 5.5: Suppose $A\in\mathbb{R}^{2\times 2}$ has purely imaginary and nonzero eigenvalues. Assume $Q\in\mathbb{R}^{2\times 2}$ and $Q=Q^T\geqslant 0$. If $A^TP+PA=-Q$ has a positive definite symmetric solution P, then $Q=0$. Further, if $A=J=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$, then $P=cI$ for some c.

Proof of Lemma 5.5: We prove this by contradiction. Suppose $Q\neq 0$. Assume further that (Q,A) is observable. Since $P>0$, using standard Lyapunov arguments, we obtain that A should be Hurwitz. This contradiction implies that either (Q,A) is unobservable and/or $Q=0$. (Of course, $Q=0$ implies unobservability too.) Suppose (Q,A) is unobservable. Then at least one of the eigenvectors of A is in the nullspace of Q. Since eigenvalues of A occur in complex conjugate pairs, the other eigenvector is independent and is unobservable too. This implies that nullspace of Q is of dimension 2. This proves that the 2×2 matrix $Q=0$. The special case when $A=J$ and $Q=0$ is proved by straightforward substitution. □

Remark 5.6: We comment here about the significance of 1 being an eigenvalue of matrices $K_{\max}$ and $K_{\min}$ (and of course, all ARE solutions). Note that the *actual* storage function $K_{\rm a}$ satisfies $K_{\min}\leqslant K_{\rm a}\leqslant K_{\max}$. Due to our normalization of all inductances and capacitances to value one, this means $K_{\min}\leqslant I_n\leqslant K_{\max}$. This means each eigenvalue of $K_{\min}$ is at most one and each eigenvalue of $K_{\max}$ is at least one. In such a situation, an eigenvalue of $K_{\max}$ or $K_{\min}$ being one implies 'conservedness' or 'losslessness'. Indeed, suppose a is the eigenvector corresponding to eigenvalue 1, then the minimum energy required to charge the circuit to state a equals the actual energy, and this further equals the maximum energy that can be extracted from this state. No resistors are 'encountered' during the charging or discharging process. While this is obvious for an LC-circuit, our results reveal that for the RLC case too, there are limiting cases where a capacitor in series with the port, and possibly with resistances, also admits such lossless states. Theorems 5.1 and 5.2 correspond to such lossless states with zero 'i^2R' losses in the limit. It is essential that these states are also 'invariant' directions with respect to closed loop dynamics (i.e. of A_K): this requires that the corresponding eigenvalue is on the imaginary axis and has no 'damping'. These intuitive expectations have been made formal in the above theorems.

VI. Example

In this section we consider an example of an RLC circuit for which Theorem 5.1 is applicable (see Figure 4). We extensively use the recently developed tool described in [6] to automatically generate the state space realizations using Python and Scilab from the circuit schematic drawn using OpenModelica's graphical editor.

Consider the RLC circuit shown in Figure 4 where capacitor C_1 forms a cutset with the source. The state variables for the circuit are $x = \begin{bmatrix} v_{c3} & i_{\ell 1} & v_{c2} & v_{c1}\end{bmatrix}^T$ and the parameters are $C_1 = C_2 = C_3 = 1$F, $L_1 = 1$H, $R_1 = R_2 = R_3 = 1\Omega$. The state-space matrices for these parameters are

$$A_z = \begin{bmatrix} -0.333 & -0.667 & -0.333 & 0 \\ 0.667 & -0.667 & -0.333 & 0 \\ -0.333 & 0.333 & -0.333 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad B_z = \begin{bmatrix} 0.333 \\ 0.333 \\ 0.333 \\ 1 \end{bmatrix},$$

$C_z = \begin{bmatrix} 0.333 & -0.333 & 0.333 & 1 \end{bmatrix}$ and $D_z = 0.667$.

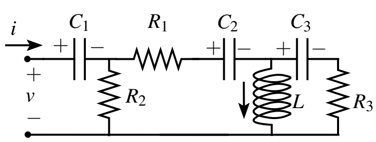

Fig. 4. Circuit for example

The extremal ARE solutions $K_{\max}$ and $K_{\min}$ for this realization are respectively:

$$\begin{bmatrix} 57.55 & -33.55 & 9.80 & 0 \\ -33.55 & 33.51 & -9.77 & 0 \\ 9.80 & -9.77 & 9.86 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ and } 10^{-4} \times \begin{bmatrix} 417.4 & -417.4 & -1.757 & 0 \\ -417.4 & 836.6 & -413.1 & 0 \\ -1.757 & -413.1 & 1425 & 0 \\ 0 & 0 & 0 & 10^4 \end{bmatrix},$$

and they give the closed loop state transition matrices $A_{K_{\max}}$ and $A_{K_{\min}}$ respectively as

$$\begin{bmatrix} 2.40 & -1.40 & 0.41 & 0 \\ 3.40 & -1.40 & 0.41 & 0 \\ 2.40 & -0.40 & 0.41 & 0 \\ 8.20 & -2.20 & 2.22 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} -0.42 & -0.58 & -0.41 & 0 \\ 0.58 & -0.58 & -0.41 & 0 \\ -0.42 & 0.42 & -0.41 & 0 \\ -0.25 & 0.25 & -0.22 & 0 \end{bmatrix}.$$

Notice that the voltage across the capacitor in series with the port is the fourth state, and hence we have the fourth row/column of A_z as zero. Other properties of the fourth row/column are visible for the matrices $K_{\min}$, $K_{\max}$, $A_{K_{\min}}$ and $A_{K_{\max}}$.

VII. Dominant pole and limiting case phenomenon

In this section we investigate whether the commonality of the eigenvector corresponding to a pole at the origin (or a pair of purely imaginary axis poles) is because of this eigenvalue being 'dominant'(i.e. the pole closest to the imaginary axis). More precisely, keeping in mind the points noted in Remark 5.6, the pole at the origin of $A_{K_{\max}}$ and $A_{K_{\min}}$ also correspond to the *conserved* states: no energy is lost (in resistors) while optimal charging to this state and all the actual stored energy can be discharged out of the port while optimal discharging. Of course, by the Courant-Fischer-Weyl min-max principle of eigenvalues of symmetric matrices, eigenvalue 1 being the least of all eigenvalues of $K_{\max}$ and the maximum of all eigenvalues of $K_{\min}$, this common eigenvector is clearly the most *efficient* state of all states with the same *actual* energy. What we investigate in this section is whether the 'most efficient' aspect of this state is related to the imaginary-axis poles being 'dominant' too. In order to investigate this relation, we study the circuit in Figure 5, and in particular, the role that the 'shunt resistor' R_{sh} plays in the extent of commonality of eigenvector.

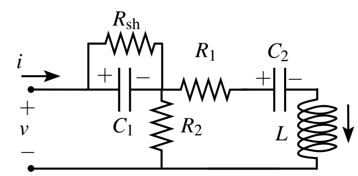

Fig. 5. A capacitor with a shunt R_{sh}

Consider the shunt resistor R_{sh} across the capacitor C_1. In the absence of this resistor, we have a pole at the origin for A_z and each A_K. The pole at the origin is shifted slightly into the negative half complex plane $\mathbb{C}^-$ for A_z when R_{sh} is large but finite. In this situation, we also study how the smallest eigenvalue $\lambda_{\min}$ of $K_{\max}$ and the largest eigenvalue $\lambda_{\max}$ of $K_{\min}$ converge to 1 as $R_{\text{sh}} \to \infty$.

For the circuit of Figure 5, the chosen parameters are $C_1 = C_2 = 1$F, $L_1 = 1$H, $R_1 = 10\ \Omega$, $R_2 = 1\ \Omega$, $R_{\text{sh}} = 1$ KΩ and the state-variable $x = \begin{bmatrix} v_{c1} & i_{\ell 1} & v_{c2}\end{bmatrix}^T$. We get

$$A_z = \begin{bmatrix} -0.001 & 0 & 0 \\ 0 & -11 & -1 \\ 0 & 1 & 0 \end{bmatrix}, B_z = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, C_z = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix}, D_z = 1.$$

Of course, the ARI solution corresponding to *actual* storage is $K_{\text{a}} = I_3$. The eigenvalues of the Hamiltonian matrix H are $\pm 0.03144, \pm 0.09681$ and ± 10.3918. The minimum and maximum ARE solutions are respectively

$$\begin{bmatrix} 1.112 & -0.045 & -1.936 \\ -0.045 & 42.02 & 0.052 \\ -1.936 & 0.052 & 78.27 \end{bmatrix} \text{ and } \frac{1}{10}\begin{bmatrix} 9.39 & -0.007 & 0.23 \\ -0.007 & 0.238 & 0.0014 \\ 0.23 & 0.0014 & 0.134 \end{bmatrix}.$$

Eigenvalue of $K_{\max}$ that is closest to one is 1.0638 and that of $K_{\min}$ is 0.9400. The corresponding eigenvectors of $K_{\max}$ and $K_{\min}$ are respectively

$$\begin{bmatrix} 0.9997 \\ 0.0008 \\ 0.025 \end{bmatrix} \text{ and } \begin{bmatrix} 0.9997 \\ -0.0008 \\ 0.025 \end{bmatrix}$$

which are no longer dependent. One can check that none of these are eigenvectors of $A_{K_{\max}}$ or $A_{K_{\min}}$. The dominant eigenvalue of $A_{K_{\min}}$ is -0.0314.

Of course, as the shunt resistance R_{sh} approaches ∞, we obtain a case where that resistance is absent, and Theorem 5.1 is applicable and an eigenvalue each of $K_{\max}$ and $K_{\min}$ approach one. (Also see Remark 5.6.) A plot of $\lambda_{\min}$ of $K_{\max}$ and $\lambda_{\max}$ of $K_{\min}$ versus the shunt resistance R_{sh} is shown in Figure 6.

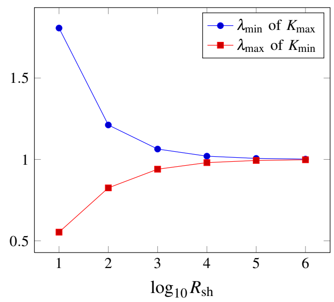

Fig. 6. Plot of $\lambda_{\min}$ and $\lambda_{\max}$ (respectively, the smallest eigenvalue of $K_{\max}$ and the largest eigenvalue of $K_{\min}$) versus the shunt resistance R_{sh} (in Ω)

The conclusion we draw from this example is that the dominance of an eigenvector of $A_{K_{\max}}$ or $A_{K_{\min}}$ is not a reason for causing them to have a common eigenvector with each other or with $K_{\max}$ and/or $K_{\min}$.

VIII. Conclusion

We summarize the results obtained in this paper. Theorem 4.1 formulated the independence of the optimal energies and the optimal trajectory dynamics with respect to the state space realizations of the impedance or admittance: the states being inductor currents and capacitor voltages. From a purely systems viewpoint, this is reasonable since the optimal charging/discharging specification is independent of the input/output classification of the port variables: voltage and current. Similarly, it is reasonable that the optimal trajectory dynamics is independent of the input/output classification. This 'behaviorally' expected result was also revealed by the Hamiltonian matrix: in fact, the state space realizations of the admittance and impedance were related in such a way that the Hamiltonian matrix is the same!

Two of our other main results: Theorems 5.1 and 5.2 formulated how a pole or zero at the origin of the immittance functions causes the corresponding state to result in a common eigenvector of every ARE solution K, every corresponding closed loop transition matrix A_K, and the suitable *one* of A_z or A_y (all defined in Section I). It must be noted that imaginary axis poles at 'optimal' charging or discharging is a limiting case situation and not implementable in practice. Intuitively, to avoid 'i^2R losses' while charging a capacitor in series with the port and resistances elsewhere (thus causing the open loop impedance to have a pole at the origin), it is reasonable that the current is very low, thus causing the charging-time (in order to charge to a specified capacitor voltage) to be arbitrarily large. This causes the optimal charging dynamics (not the maximum/minimum but the supremum/infimum) to have a pole at the origin. A similar argument holds for the optimal discharging. The same arguments also hold for imaginary axis poles, for example, due to an LC tank. Obviously, these results can be restated for an inductor across the port and an LC tank formed across the port upon shorting the port (Figures 1 and 3).

While it is known that extremal ARE solutions are related to optimal charging and discharging, and the link between ARE solutions and Hamiltonian matrix is also known, this paper brings out key properties about common eigenvector and eigenvalues across ARE solutions. Further, we relate the common eigenvector/eigenvalues to the location of the capacitor/inductor with respect to the port. We have provided physical interpretation to these results in the context of optimal charging/discharging of RLC circuits.

We considered an example to demonstrate Theorem 5.1. When the dominant pole is not at the origin but very close, we demonstrated in Section VII that the commonality of the eigenvectors of $K_{\max}$ and $K_{\min}$ is no longer the case.

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