21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Converse Lyapunov Theorems for Discrete-Time Systems: an Alternative Approach

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***Abstract*— This paper presents an alternative approach for obtaining a converse Lyapunov theorem for discrete-time systems. The proposed approach is constructive, as it provides an explicit Lyapunov function. The developed converse Lyapunov theorem establishes existence of global Lyapunov functions for globally exponentially stable (GES) systems and for globally asymptotically stable systems, Lyapunov functions on a set $[a, b] \subset \mathbb{R}_+$ are derived. In particular, we discuss our result for both continuous and discontinuous dynamics.**

keyword: Discrete-time systems, Stability analysis, Converse Lyapunov theorems

I. Introduction

Lyapunov functions are a powerful tool for establishing stability properties of dynamical systems, such as global asymptotic stability (GAS). The existence of a Lyapunov function is guaranteed by converse Lyapunov theorems under the assumption that the system is GAS. Classical results in this direction are presented in the seminal books [1], [2]. Extensions of this theory can be found e.g. in [3], [4], [5], [6], [7], [8], [9], [10], [11], and the references therein. For the discrete-time case, which is of interest in this work, the abstract construction of a global Lyapunov function for a converse theorem is performed by taking infinite series [5] or the supremum over all solutions and all times [10], [7]. As such, these approaches require the knowledge of solutions for all positive times. For certain classes of dynamics there are classes of Lyapunov functions that preserve necessity, such as quadratic functions [6] for linear difference equations and polyhedral functions [3], [12], [13] for linear difference inclusions.

Existing results on constructive converse Lyapunov theorems for general nonlinear systems are scarce and come with certain limitations, as discussed next. In the monograph [14] converse Lyapunov theorems for *continuous-time* systems are obtained *via* piecewise linear Lyapunov functions and linear programming. Another relevant result for continuous-time systems can be found in [15], where the authors show the relation between control Lyapunov functions and solutions to generalized Zubov equations, i.e., a first order partial differential equation. A result relevant for *discrete-time* systems was given in [16], where it was shown that for a globally exponentially stable (GES) discrete-time system a Lyapunov function can be constructed by a finite sum of solutions. This was established under the assumption that the system dynamics is continuous and locally Lipschitz.

In this paper, an alternative approach to the construction of Lyapunov functions for discrete-time systems is proposed. Continuity of the system dynamics is not required and the proposed approach is shown to hold for a larger class of systems than GES systems. Nevertheless, if the system dynamics is continuous then our approach yields a continuous Lyapunov function. This is important as for discrete-time dynamics continuous Lyapunov functions yield inherent robustness, see [17].

The first ingredient of the proposed approach consists of a relaxation of the Lyapunov function concept, which was originally introduced in [18]: the Lyapunov function is required to decrease along the system solutions after a finite number of time steps, and not at every time step. It is shown that this relaxation, termed finite-step Lyapunov function in this work, still yields sufficient conditions for establishing GAS of the underlying system, without requiring continuity of the system dynamics. Secondly, a converse finite-step Lyapunov theorem is derived. This converse theorem is constructive in that it yields a global finite-step Lyapunov function that is explicitly given. Then, a way to construct a standard Lyapunov function based on the knowledge of a finite-step Lyapunov function and a corresponding natural number is given. The construction depends only on a finite sum and hence, it is straightforward to implement.

Combining the above results yields the main contribution of this paper, i.e., a novel converse Lyapunov theorem that provides an explicit construction of a Lyapunov function for a large class of discrete-time systems. More specifically, global Lyapunov functions for GES discrete-time systems, and Lyapunov functions on a set $[a, b] \subset \mathbb{R}_+$ for GAS discrete-time systems can be constructed. The Lyapunov function construction hinges of finding a suitable natural number a priori. Of course, finding such a suitable number may be undecidable or computationally intractable, as it is well known that stability analysis and hence, also construction of Lyapunov functions, is an NP-hard problem in general [19].

The remainder of this paper is structured as follows. The necessary preliminaries are given in Section II and the definition of global (finite-step) Lyapunov functions as well as the problem statement are given in Section III. Section IV contains the main results of this paper. In Section IV-A global finite-step Lyapunov functions are shown to be sufficient

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for the system being GAS, whereas in Section IV-B also necessity is proven under an appropriate assumption. In Section IV-C it is shown how to construct a global Lyapunov function from the knowledge of a global finite-step Lyapunov function, which combined with the results from Section IV-B gives a converse Lyapunov theorem in Section IV-D. A discussion of our results for systems with continuous dynamics is given in Section V. Finally, an example shows how the results can be applied, see Section VI.

II. Preliminaries

By $\mathbb{N}$ we denote the natural numbers and we assume $0 \in \mathbb{N}$. Let $\mathbb{R}$ denote the field of real numbers, $\mathbb{R}_+$ the set of nonnegative real numbers and $\mathbb{R}^n$ the vector space of real column vectors of length n.

By $\|\cdot\|$ we denote any arbitrary norm on $\mathbb{R}^n$. To state the stability results, we use standard *comparison functions*. We call a function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ a *function of class* $\mathcal{K}$ (denoted by $\alpha \in \mathcal{K}$), if it is strictly increasing, continuous, and satisfies $\alpha(0) = 0$. In particular, if $\alpha \in \mathcal{K}$ is unbounded, it is said to be of class $\mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is called a *function of class* $\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if it is of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second argument. A continuous function $\eta : \mathbb{R}_+ \to \mathbb{R}_+$ is called *positive (semi-)definite*, if $\eta(0) = 0$ and $\eta(s) > 0$ (resp. $\eta(s) \geq 0$) for all $s > 0$. For two functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, we denote $\alpha_1 < \alpha_2$ (resp. $\alpha_1 \leq \alpha_2$) if $\alpha_2 - \alpha_1$ is positive (semi)-definite. Furthermore, $\alpha_1 \circ \alpha_2$ denotes the composition, and $\alpha_1^k := \alpha_1 \circ \ldots \circ \alpha_1$ is the k-th iterate of α_1. For $\rho \in \mathcal{K}_\infty$, $j \in \mathbb{Z}$, $M \in \mathbb{N}$, $M > 0$, we denote by $\gamma := \rho^{j/M}$ a solution of the equation $\gamma^M = \rho^j$, that exists by [20, Proposition 3.1]. A function $G : \mathbb{R}^n \to \mathbb{R}^n$ is called $\mathcal{K}$*-bounded*, if there exists an $\omega \in \mathcal{K}$ such that

$$\|G(x)\| \leq \omega(\|x\|), \qquad \forall x \in \mathbb{R}^n.$$

III. Problem statement

We consider discrete-time systems of the form

$$x(k+1) = G(x(k)), \qquad k \in \mathbb{N}, \tag{1}$$

where $G : \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following standing assumption.

Assumption 1: The function G in (1) is $\mathcal{K}$-bounded.

Compared to the typical assumptions employed in converse Lyapunov theorems, Assumption 1 is not restrictive, as it does not require continuity of the map $G(\cdot)$ (except[1] at $x = 0$). On the other hand, any continuous map $G : \mathbb{R}^n \to \mathbb{R}^n$ with $G(0) = 0$ is $\mathcal{K}$-bounded. Also, for any GES system (1) the map $G(\cdot)$ is $\mathcal{K}$-bounded with a linear function ω, as worked out further on in this section.

By $x(k, \xi) \in \mathbb{R}^n$ we denote the solution of system (1) at time instance $k \in \mathbb{N}$ with initial condition $x(0) = \xi \in \mathbb{R}^n$.

Definition 2: The origin of system (1) is called *globally asymptotically stable* (GAS) if there exists a $\mathcal{KL}$-function β such that for all $\xi \in \mathbb{R}^n$ and all $k \in \mathbb{N}$

$$\|x(k, \xi)\| \leq \beta(\|\xi\|, k). \tag{2}$$

If the $\mathcal{KL}$-function in (2) can be chosen as

$$\beta(r, t) = C\mu^t r \tag{3}$$

with $C \geq 1$ and $\mu \in [0, 1)$, then the origin of system (1) is called globally exponentially stable (GES).

Remark 3: The definition of GES is somehow misleading, since *global* only indicates that (2) with β as in (3) holds for all $\xi \in \mathbb{R}^n$. In particular, systems in which all solutions have an exponential rate of decay may fail to be GES. Since $C \geq 1$ in (3) is chosen globally, it does not reflect the local behavior of a particular solution near 0. Note that this property is also often called *exponentially stable in the whole*, see e.g., [1, Sec. 2] and [21, Sec. 6.3].

Note that for a GES system the $\mathcal{K}$-bound ω on G in (1) can always be chosen to be linear. This follows directly, since $\|G(\xi)\| = \|x(1, \xi)\| \leq C\mu^1 \|\xi\|$ for all $\xi \in \mathbb{R}^n$. Similarly, any GAS system is $\mathcal{K}$-bounded; the $\mathcal{K}$-bound may be chosen as $\omega(s) = \beta(s, 1)$.

Definition 4: A function $W : \mathbb{R}^n \to \mathbb{R}_+$ is a *global Lyapunov function* for system (1) if

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq W(\xi) \leq \alpha_2(\|\xi\|),$$

(ii) there exists a positive definite function ρ satisfying $\rho < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$

$$W(x(1, \xi)) \leq \rho(W(\xi)).$$

Remark 5: In many prior works (e.g., [5]) the definition of a Lyapunov function requires the existence of a positive definite function α_3 such that $W(x(1, \xi)) - W(\xi) \leq -\alpha_3(\|\xi\|)$ holds for all $\xi \in \mathbb{R}^n$. Let us briefly explain, that this requirement is equivalent to Definition 4. Note that by following similar steps as in [22, Theorem 2.3.5] we conclude

$$\begin{aligned} W(x(1, \xi)) &\leq W(\xi) - \alpha_3(\|\xi\|) \\ &\leq (\mathrm{id} - \alpha_3 \circ \alpha_2^{-1})(W(\xi)) = \rho(W(\xi)) \end{aligned}$$

with $\rho := (\mathrm{id} - \alpha_3 \circ \alpha_2^{-1})$ positive definite. We further have $0 \leq W(x(1, \xi)) \leq (\alpha_2 - \alpha_3)(\|\xi\|)$, so $\alpha_2 \geq \alpha_3$ and therefore $\rho < \mathrm{id}$. On the other hand for a given $\rho < \mathrm{id}$ we obtain $W(x(1, \xi)) - W(\xi) \leq -\alpha_3(\|\xi\|)$ for $\alpha_3 := (\mathrm{id} - \rho) \circ \alpha_1$.

Next, the assumptions on the global Lyapunov function given in Definition 4 are relaxed as follows.

Definition 6: A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is a *global finite-step Lyapunov function* for system (1) if

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|),$$

(ii) there exists a finite $M \in \mathbb{N}$ and a positive definite function $\rho < \mathrm{id}$ such that for all $\xi \in \mathbb{R}^n$

$$V(x(M, \xi)) \leq \rho(V(\xi)).$$

It is worth pointing out that the concept of a global finite-step Lyapunov function was originally introduced in [18],

[1] Recall that continuity at $x = 0$ is a necessary condition for Lyapunov stability of the origin.

which dealt with stability analysis of time-varying systems, although the term finite-step was not used therein. Clearly, any global Lyapunov function is a particular global finite-step Lyapunov function.

Observe that if V is a global finite-step Lyapunov function for system (1) with suitable $M \in \mathbb{N}$ then V is a global Lyapunov function for the system

$$\bar{x}(k+1) = G^M(\bar{x}(k)), \quad k \in \mathbb{N}.$$

Thus, global finite-step Lyapunov functions can be seen as global Lyapunov functions of the iterated system.

IV. Main results

The aim of this section is to derive a constructive converse Lyapunov theorem for systems of the form (1). To this aim we proceed by showing that any global finite-step Lyapunov function guarantees GAS of the origin of the underlying dynamical system. We show that the converse, i.e,. the existence of a global finite-step Lyapunov function for a GAS system, also holds and, moreover, that under an appropriate assumption the existence is guaranteed for any scaled norm as a global finite-step Lyapunov function. In particular, we show that the assumption is satisfied for any GES system of the form (1). We further prove that any scaled norm is at least an $[a, b]$ finite-step Lyapunov function (see Definition 17) for a GAS system of the form (1). To obtain a converse Lyapunov theorem, an explicit construction of a global Lyapunov function from a global finite-step Lyapunov function is provided.

For brevity, we have omitted the proofs of most of the results. For the proofs we refer to the preprint [23]. Note that in this preprint finite-step Lyapunov functions are called finite-time Lyapunov functions, which has been changed to avoid confusion with the term *finite-time stability*.

A. Finite-step Lyapunov functions

This section proceeds by showing that the existence of a global finite-step Lyapunov function is sufficient to conclude GAS of origin. Recall, that our assumptions imply continuity of G at the origin.

Theorem 7: The existence of a global finite-step Lyapunov function implies that the origin of system (1) is GAS.

The proof of this theorem essentially follows the standard proof that the existence of a global Lyapunov function implies GAS of the origin of system (1). Assumption 1 ensures boundedness of the solution for each initial point $\xi \in \mathbb{R}^n$. A detailed proof can be found in [23, Theorem 7].

If we impose stronger conditions on the global finite-step Lyapunov function then we can conclude GES of the origin.

Corollary 8: Let the $\mathcal{K}$-bound on G be $\omega(s) = ws$ for all $s \geq 0$ and $w > 0$. Then the existence of a global finite-step Lyapunov function V satisfying Definition 6 with

$$\alpha_1(s) = a_1 s^\lambda, \ \alpha_2(s) = a_2 s^\lambda, \ \rho(s) = cs \tag{4}$$

with $0 < a_1 \leq a_2$, $\lambda > 0$ and $c \in [0, 1)$ implies that the origin of system (1) is GES.

This result is shown by constructing the constants $C \geq 1$ and $\mu \in [0, 1)$ as

$$C = \max_{i \in \{1,\ldots,M-1\}} \left(\tfrac{a_2}{a_1 c}\right)^{1/\lambda} \omega^i, \quad \mu = c^{1/M\lambda} \in [0, 1),$$

see [23, Corollary 8].

Note that the assumption on ω to be linear is necessary for the origin of the system being GES as previously indicated.

Since any global finite-step Lyapunov function with $M = 1$ is even a global Lyapunov function, Lyapunov's theorem can also be obtained as a corollary of Theorem 7.

Corollary 9: The existence of a global Lyapunov function implies that the origin of system (1) is GAS.

The same holds true for the GES case by means of Corollary 8.

Clearly, if G is a contraction, then $W : \mathbb{R}^n \to \mathbb{R}_+$ defined by $W(\xi) = \|\xi\|$ is a global Lyapunov function, yielding GAS of the origin.

Corollary 10: If the map G in (1) is $\mathcal{K}$-bounded with $\mathcal{K}$-function $\omega < \mathrm{id}$, then the origin of system (1) is GAS.

B. A converse finite-step Lyapunov function theorem

This section proceeds by showing that the existence of a global finite-step Lyapunov function for GAS systems (1) is also necessary.

As global Lyapunov functions are in particular global finite-step Lyapunov functions, we can use standard converse Lyapunov theorems as e.g. [5, Theorem 1] (for continuous G) resp. [10, Lemma 4] (for discontinuous G), to ensure the existence of global finite-step Lyapunov functions for GAS systems (1). Indeed, a converse result was not pursued in [18].

Next, a constructive converse finite-step Lyapunov theorem is stated, which does not rely on a standard Lyapunov function, but on an appropriate assumption that is discussed in the following.

Assumption 11: There exists a $\mathcal{KL}$-function β satisfying (2) for system (1) and

$$\beta(r, M) < r \tag{5}$$

for some $M \in \mathbb{N}$ and all $r > 0$.

Under this assumption a global finite-step Lyapunov function can be given explicitly. For a proof we refer to [23, Theorem 13].

Theorem 12: If Assumption 11 is satisfied, then for any function $\eta \in \mathcal{K}_\infty$ the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$V(\xi) := \eta(\|\xi\|) \tag{6}$$

for all $\xi \in \mathbb{R}^n$ is a global finite-step Lyapunov function for system (1).

Let us briefly discuss Assumption 11. First of all, by definition, any GES system is bounded by a $\mathcal{KL}$-function $\beta(r, t) = C\mu^t r$ with $C \geq 1, \mu \in [0, 1)$, see (3). So we can find an $M \in \mathbb{N}$ such that $C\mu^M < 1$ by simply taking $M \in \mathbb{N}$ with $M > \log_\mu(1/C)$, which immediately yields the following proposition.

Proposition 13: If the origin of system (1) is GES, then Assumption 11 is satisfied.

Proposition 13 and Theorem 12 immediately imply that for a GES system (1) any scaled norm is a global finite-step Lyapunov function.

Corollary 14: If the origin of system (1) is GES, then for any function $\eta \in \mathcal{K}_\infty$ the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined in (6) is a global finite-step Lyapunov function for this system.

Note that the converse implication of Proposition 13, i.e., Assumption 11 implies that the origin of system (1) is GES, doesn't hold in general, which is shown in the following example.

Example 15: Consider the system

$$x(k+1) = \begin{cases} \left|x(k)\right| - x^2(k) & \text{if } \left|x(k)\right| \leq \frac{1}{2} \\ \frac{1}{2}\left|x(k)\right| & \text{if } \left|x(k)\right| > \frac{1}{2} \end{cases} \qquad k \in \mathbb{N}. \tag{7}$$

The right-hand site is $\mathcal{K}$-bounded with $\omega(s) = \max\{s - s^2, 0.5s\}$. Thus by defining $\beta(s,k) := \omega^k(s)$ we see that the system is GAS, and satisfies Assumption 11 with $M = 1$. On the other hand, system (7) is not GES in the sense of Definition 2 as the decrease rate of any solution approaches 1, i.e., $\lim_{k\to\infty} \frac{\|x(k+1)\|}{\|x(k)\|} = \lim_{k\to\infty} 1 - \|x(k)\| = 1$.

If we assume the origin of system (1) to be GAS only, then (5) doesn't have to hold globally, i.e., for all $r > 0$. But we can at least show that Assumption 11 is satisfied in a semi-global practical sense.

Lemma 16: Let $\beta \in \mathcal{KL}$. Then for any $0 < a < b < \infty$ there exists an $M \in \mathbb{N}$, such that (5) holds for all $r \in [a,b]$. For a proof see [23, Lemma 16].

This now implies that for any GAS system of the form (1), any scaled norm is a finite-step Lyapunov function for a set $[a,b]$ as defined next.

Definition 17: A function $V : \mathbb{R}^n \to \mathbb{R}_+$ is an (a,b) *finite-step Lyapunov function* for system (1) with $0 < a < b < \infty$ if

(i) there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for all $\xi \in \mathbb{R}^n$

$$\alpha_1(\|\xi\|) \leq V(\xi) \leq \alpha_2(\|\xi\|),$$

(ii) there exists a finite $M \in \mathbb{N}$ and a positive definite function $\rho < \mathrm{id}$ such that for all $\xi \in V^{-1}([a,b]) := \{z \in \mathbb{R}^n : V(z) \in [a,b]\}$ we have

$$V(x(M,\xi)) \leq \rho(V(\xi)),$$

and for all $\xi \in V^{-1}([0,a])$ we have

$$V(x(M,\xi)) \leq a.$$

Definition 17 implies that the function $V : \mathbb{R}^n \to \mathbb{R}_+$ is decreasing at least any M steps towards the set $[0,a]$ as long as $\xi \in V^{-1}([a,b])$. Finally, along solutions $x(k,\xi)$ starting in $\xi \in V^{-1}([0,a])$ we see that $V(x(k,\xi))$ is within $[0,a]$ at least for any $k = lM$ with $l \in \mathbb{N}$.

Corollary 18: If the origin of system (1) is GAS, then for any function $\eta \in \mathcal{K}_\infty$ and any $0 < a < b < \infty$, the function $V : \mathbb{R}^n \to \mathbb{R}_+$ defined in (6) is an (a,b) finite-step Lyapunov function for this system.

For a proof we refer to our upcoming journal version.

We stress that the constant $M \in \mathbb{N}$ chosen in Corollary 18 depends on the interval $[a,b]$. In general, a larger interval requires a larger M.

The meaning of Corollary 18 is that if we are not aware of a global Lyapunov function for system (1), we can nevertheless construct a Lyapunov function in (6) to ensure practical asymptotic stability of the origin (or ultimate boundedness with respect to the set $V^{-1}([0,a])$). In this respect, $M \in \mathbb{N}$ can be interpreted as a tuning parameter that regulates the size of the interval $[a,b]$.

C. Construction of a global Lyapunov function from a global finite-step Lyapunov function

In this section we construct a global Lyapunov function for system (1) from the knowledge of a global finite-step Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ as defined in Definition 6. We want to emphasize that this construction does not depend on the particular choice of the global finite-step Lyapunov function in (6), but rather holds in general. The construction is as follows:

$$W(\xi) := \sum_{j=0}^{M-1} V(x(j,\xi)). \tag{8}$$

The idea behind this construction is that the summands for $W(\xi)$ are the same as for $W(x(1,\xi))$ except that $V(\xi)$ is replaced by $V(x(M,\xi)) < V(\xi)$. Hence, W is decreasing along trajectories of (1), which yields the following result.

Theorem 19: **(Construction of a global Lyapunov function I)** If $V : \mathbb{R}^n \to \mathbb{R}_+$ is a global finite-step Lyapunov function for system (1) with $M \in \mathbb{N}$ satisfying Definition 6-(ii), then $W : \mathbb{R}^n \to \mathbb{R}_+$ defined in (8) is a global Lyapunov function for system (1).

A proof of Theorem 19 can be found in [23, Theorem 19].

An alternative construction to the one given in (8) is given by

$$W(\xi) := \max_{j\in\{0,\dots,M-1\}} \rho^{j/M}\big(V(x(M-1-j,\xi))\big). \tag{9}$$

Note that if $\rho \in \mathcal{K}_\infty$, which we can assume without loss of generality, then $\rho^{1/M}$ exists by [20, Proposition 3.1]. With this construction we obtain an analogue to Theorem 19.

Theorem 20: **(Construction of a global Lyapunov function II)** If $V : \mathbb{R}^n \to \mathbb{R}_+$ is a global finite-step Lyapunov function for system (1) with $M \in \mathbb{N}$ satisfying Definition 6-(ii), then $W : \mathbb{R}^n \to \mathbb{R}_+$ defined in (9) is a global Lyapunov function for system (1).

D. Converse Lyapunov theorems

Now we can state the main result of this work as a corollary of Theorem 12 and Theorem 19 resp. Theorem 20.

Corollary 21: **(Converse Lyapunov theorem)** Let $M \in \mathbb{N}$ satisfy Assumption 11. Then for any function $\eta \in \mathcal{K}_\infty$ the function $W : \mathbb{R}^n \to \mathbb{R}_+$ defined by

$$W(\xi) := \sum_{j=0}^{M-1} \eta(\|x(j,\xi)\|) \tag{10}$$

resp.

$$W(\xi) := \max_{j\in\{0,\ldots,M-1\}} \rho^{j/M}\left(\eta(\|x(M-1-j,\xi)\|)\right) \quad (11)$$

for all $\xi \in \mathbb{R}^n$ is a global Lyapunov function for system (1).

Remark 22: The main difference of the construction of the Lyapunov functions in Corollary 21 in contrast to the constructions of Lyapunov functions in other converse theorems, is that we use a finite sum of solutions instead of an infinite series [5], and we use the maximum over a finite set instead of the supremum over all solutions and all times [10], [7].

Since any GES system satisfies Assumption 11 (see Proposition 13), we obtain the following converse Lyapunov theorem for GES systems of the form (1).

Corollary 23: If the origin of system (1) is GES then for any $\eta \in \mathcal{K}_\infty$ there exists an $M \in \mathbb{N}$ such that the function W defined in (10) resp. (11) is a global Lyapunov function for system (1).

Combining Corollary 18 and Theorem 19 resp. Theorem 20 we immediately obtain an (a, b) Lyapunov function.

Corollary 24: If the origin of system (1) is GAS, then for $0 < a < b < \infty$ there exists an $M \in \mathbb{N}$ such that the function W defined in (10) resp. (11), is an (a, b) Lyapunov function for system (1).

For several classes of systems the results presented in this section have particular consequences of relevance. In particular, we show in [23] that the existence of conewise linear Lyapunov functions is also necessary for GES conewise linear systems. Furthermore, we can use the construction of the Lyapunov functions in (10) and (11) to derive tractable construction of polyhedral Lyapunov functions for linear systems, see [23]. For more general systems the following procedure enables us to check the stability at least on a set $B_{[a,b]} := \{\xi \in \mathbb{R}^n : \|\xi\| \in [a, b]\}$. Note that this procedure may not be successful for any system.

Procedure 25: The stability analysis we proposed so far to construct a Lyapunov function (at least on a set $B_{[a,b]}$) can be summarized as follows. Let a system of the form (1) be given.

[0] Set $k = 1$.
[1] Check $\left\|G^k(\xi)\right\| < \|\xi\|$ for all $\xi \in \mathbb{R}^n$ with $\|\xi\| \in [a, b]$, and $\left\|G^k(\xi)\right\| \leq a$ for all $\xi \in \mathbb{R}^n$ with $\|\xi\| \in [0, a]$. If these inequalities hold proceed with step [2]; else set $k = k + 1$ and repeat.
[2] Define $W : \mathbb{R}^n \to \mathbb{R}_+$ by (10) or (11) with $M = k$.

If this procedure is successful then W is an (a, b) finite-step Lyapunov function for the overall system (1), and it in particular ensures practical asymptotic stability. If $a = 0$ then we obtain a Lyapunov function and if, additionally, $b = \infty$, then the Lyapunov function is a global one.

Computation of a suitable M can be done by iteratively checking the condition $\|x(M, \xi)\| < \|\xi\|$ while increasing the value of M, which needs to be verified either globally or for a subset of $\mathbb{R}^n$. At least, when the origin is GES, there always exists an M large enough for which the condition holds globally. The difficulty of checking this condition depends on the particular map G. Systematic methods for obtaining an M for linear and conewise linear maps are given in [23]. Note that there are other possibilities to construct Lyapunov functions numerically as e.g. [24], [25].

V. Further insights

We start with a result that is already known (see e.g. [5]). Nevertheless, using the construction of the global Lyapunov function W in (10) respectively (11) in Corollary 21, the proof is simpler because Assumption 11 can be employed.

Proposition 26: Let system (1) satisfy Assumption 11. If G in (1) is continuous then there exists a continuous global Lyapunov function for system (1).

Proof: From (10) in Corollary 21 we directly obtain the global Lyapunov function

$$W(\xi) := \sum_{j=0}^{M-1} \eta(\left\|G^j(\xi)\right\|). \quad (12)$$

As the composition of continuous functions (η, ρ, G) yields a continuous function, W is a continuous global Lyapunov function. ■

Remark 27: The importance of Proposition 26 lies in the fact that, even for discontinuous dynamics, a continuous Lyapunov function already yields inherent robustness, see [17]. So smoothness of the Lyapunov function, which can be achieved using smoothing techniques (see e.g. [5],[7, Sec. 3]) is not required for guaranteeing inherent robustness when discrete-time systems are considered. Additionally, if the system dynamics are discontinuous it is not possible to guarantee the existence of a continuous Lyapunov function for GAS systems, see also [10].

Proposition 26 requires the same continuity conditions on the system dynamics as the results in [5] and obtains a continuous global Lyapunov function as opposed to the smooth Lyapunov function obtained in [5]. However, the method of proof is significantly simpler. On the other hand, the reference [7] considers set-valued dynamics. However, when the set-valued map is single valued, the upper semi-continuity assumption in [7] reduces to continuity.

Proposition 28: If $G : \mathbb{R}^n \to \mathbb{R}^n$ in (1) is a polynomial function satisfying Assumption 11 then there exists a polynomial global Lyapunov function for system (1).

This proposition follows by taking $\eta(s) := s^2$ in the global Lyapunov function in (10), and noticing that the sum and the composition of polynomial functions yields a polynomial function.

The following result can be obtained for another relevant type of map G, which was considered in [26], [27], [28], [29]. Again, as in the two previous proposition, the proof more or less argues that the composition of homogeneous function of degree one yields a homogeneous function of degree one. For a detailed proof, see [23, Proposition 29].

Proposition 29: Let G in (1) be positively homogeneous of degree one[2] and let the origin of system (1) be GAS. Then there exists a global Lyapunov function for system (1) that

[2] I.e., for any $\xi \in \mathbb{R}^n$, $G(c\xi) = cG(\xi)$ for any $c > 0$.

is positively homogeneous of degree one.
If G is in addition continuous then there exists a continuous homogeneous of degree one global Lyapunov function.

VI. Example

In this section we study the stability of the following discrete-time system

$$x(k+1) = \begin{pmatrix} [x(k)]_1 - 0.3[x(k)]_2 \\ 0.7[x(k)]_1 + 0.2\frac{[x(k)]_2^2}{1+[x(k)]_2^2} \end{pmatrix}. \tag{13}$$

We will first show that the map

$$G(\xi) := \begin{pmatrix} [\xi]_1 - 0.3[\xi]_2 \\ 0.7[\xi]_1 + 0.2\frac{[\xi]_2^2}{1+[\xi]_2^2} \end{pmatrix}$$

satisfies Assumption 1. Therefore, we take the infinity norm $\|\xi\|_\infty := \max\{[\xi]_1, [\xi]_2\}$, and observe that that for all $t \in \mathbb{R}$, we have

$$\frac{t^2}{1+t^2} \le \frac{|t|}{2}. \tag{14}$$

Thus,

$$\begin{aligned} \|G(\xi)\|_\infty &= \max\left\{\left|[\xi]_1 - 0.3[\xi]_2\right|, \left|0.7[\xi]_1 + 0.2\frac{[\xi]_2^2}{1+[\xi]_2^2}\right|\right\} \\ &\le \max\left\{\left|[\xi]_1\right| + 0.3\left|[\xi]_2\right|, 0.7\left|[\xi]_1\right| + 0.1\left|[\xi]_2\right|\right\} \\ &\le 1.3\,\|\xi\|_\infty \end{aligned}$$

i.e., G is $\mathcal{K}$-bounded with linear $\omega(s) = 1.3s \in \mathcal{K}_\infty$ (with respect to the norm $\|\cdot\|_\infty$).

To show GAS of the origin of system (13), we follow the methodology given in Procedure 25. Computing solutions $x(k,\xi)$, i.e., iterating the dynamics map G, we see that for $k = 3$ we obtain (15).

Again, exploting (14), we obtain

$$\begin{aligned} \|x(3,\xi)\|_\infty \le \max\{&0.601\left|[\xi]_1\right| + 0.397\left|[\xi]_2\right|, \\ &0.6335\left|[\xi]_1\right| + 0.253\left|[\xi]_2\right|\} \\ \le\ & 0.998\,\|\xi\|_\infty\,. \end{aligned}$$

Thus, $V : \mathbb{R}^n \to \mathbb{R}_+$ defined by $V(\xi) = \|\xi\|_\infty$ is a global finite-step Lyapunov function for system (13) yielding GAS of the origin by Theorem 7. Furthermore, as the $\mathcal{K}$-bound is linear, and V satisfies (4) with $a_1 = a_2 = \lambda = 1$ and $c = 0.998$ the origin of system (13) is even GES by Corollary 8.

Finally, a global Lyapunov function can be constructed from the global finite-step Lyapunov function as in (10) resp. (11). In Figure 1 we show a contour plot of the global Lyapunov function constructed as in (10) with $\eta = \text{id}$. We furthermore give the plot of the trajectory starting in $\xi = (3, -3)$.

VII. Conclusions

This paper presented a novel converse Lyapunov theorem for discrete-time systems under a mild assumption. The developed theorem is constructive, as it provides an explicit Lyapunov function, whereas other converse Lyapunov theorems only show existence of a Lyapunov function.

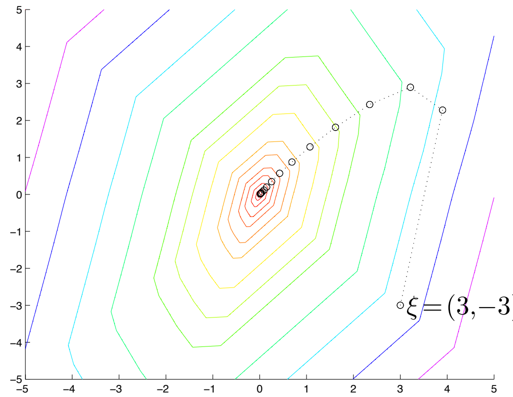

Fig. 1. Contour plot of the Lyapunov function W for system (13) constructed as in (10) with $\eta = \text{id}$, and the trajectory of system (13) starting in $\xi = (3, -3)$.

Furthermore, the converse Lyapunov theorem applies to both continuous and discontinuous dynamics. An example is presented which shows how the results can be applied.

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$$x(3,\xi)=\begin{pmatrix} 0.58[\xi]_1-0.237[\xi]_2-0.06\frac{[\xi]_2^2}{1+[\xi]_2^2}-0.06\frac{(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2} \\ 0.553[\xi]_1-0.21[\xi]_2-0.042\frac{[\xi]_2^2}{1+[\xi]_2^2}+0.2\frac{\left(0.7[\xi]_1-0.21[\xi]_2+0.2\frac{(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}\right)^2}{1+\left(0.7[\xi]_1-0.21[\xi]_2+0.2\frac{(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}{1+(0.7[\xi]_1+0.2\frac{[\xi]_2^2}{1+[\xi]_2^2})^2}\right)^2} \end{pmatrix}. \tag{15}$$

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