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H_∞- and H_2-synthesis for nested interconnections: A direct state-space approach by linear matrix inequalities

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***Abstract*— Recently we proposed a state-space solution of the structured H_∞-synthesis problem for nested interconnections. These are characterized by generalized plants with a block-triangular structure of the control channel and the search for optimal controllers whose transfer matrix shares this structure. The existing approach is based on eliminating the structured controller matrices and results in LMI existence conditions for H_∞-controllers. The present paper serves to develop an alternative approach with a novel structured convexifying controller parameter transformations which allows the extension of the design procedure to H_2- and mixed H_2/H_∞-cost criteria.**

***Index Terms*— Structured H_2-control, mixed H_2/H_∞-synthesis, linear matrix inequalities**

NOTATION

If the transfer matrix $G(s)$ is realized as $C(sI-A)^{-1}B+D$ we write $G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$, while matrix partitions are denoted by dashed lines. We use $\mathrm{He}(M) := M^T + M$ and $I, 0$ or $I_n, 0_{n\times m}$ denote the identity, zero matrix of the given dimensions.

I. PROBLEM FORMULATION

Let us consider the linear time-invariant generalized plant

$$\begin{pmatrix} e \\ \hdashline y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} P_{00} & P_{01} & P_{02} \\ \hdashline P_{10} & P_{11} & 0 \\ P_{20} & P_{21} & P_{22} \end{pmatrix} \begin{pmatrix} d \\ \hdashline u_1 \\ u_2 \end{pmatrix} \quad (1)$$

with a control channel that is described by a lower block-triangular transfer matrix of dimension $k \times m$ that carries a partition into 2 block-rows and block-columns according to the dimensions $k = k_1 + k_2$ and $m = m_1 + m_2$. We target at designing an internally stabilizing controller

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} K_{11} & 0 \\ K_{21} & K_{22} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \quad (2)$$

for (1) in the partition $m = m_1 + m_2$ and $k = k_1 + k_2$ such that some H_∞- or H_2-cost criterion imposed on the performance channel $d \to e$ is satisfied.

Such structured synthesis problems have been shown to be tractable by convex optimization techniques through a structured version of the classical Youla parametrization [11], [6]. The resulting infinite dimensional optimization problem is handled with a Galerkin-type approach, by reducing the search for the structured Youla parameter to a sequence of finite-dimensional subspaces of increasing dimension [11], [7], [6], [2]. For close-to-optimal controllers, this might require to use high-dimensional subspaces which results in large McMillan degrees of the related controllers; low speed of convergence and numerical instabilities could prevent the application of this approach to large-scale systems. In particular, the latter technique excludes the possibility to impose a priori bounds on the degree of (close to) optimal controllers or on the dimension of the corresponding optimization problem.

For the H_∞-cost, we recently managed to reduce the synthesis problem to solving a system of LMIs of a fixed dimension [8], resulting in controllers whose degree can be bounded a priori in terms of the problem formulation. In [8], convexification is achieved by eliminating the controller parameters. The present paper serves to develop an approach in parallel to [5], [9] and based on a structured convexifying controller parameter transformation. This allows to arrive at an LMI-solution of the structured H_2-synthesis problem, which extends the results [4] formulated with coupled Riccati equations and only apply for regular problems.

Triangular structures appear, e.g., from nested interconnections as discussed and motivated in [11], [6]. In a concrete numerical example in Section IV, we consider a car-platoon with $N = 3$ as depicted in Figure I. We follow [10] and assume that the i-th car is modeled as

$$\xi_i = H_i u_i \quad \text{with} \quad H_i(s) = \frac{1}{m_i s^2(\tau_i s + 1)}, \quad m_i > 0, \quad \tau_i > 0$$

for $i = 0, 1, \ldots, N$. Here ξ_i is the absolute position and u_i the control input. If viewing ξ_0 as a given signal, the goal is to keep the spacing error $\xi_i - \xi_{i-1}$ close to some reference input r_i, which amounts to choosing the performance outputs $e_i = \xi_i - \xi_{i-1} - r_i$ for $i = 1, \ldots, N$. With $H = \mathrm{diag}(H_1, \ldots, H_N)$ and $d_1 = -\xi_0 - r_1$ and $d_i = -r_i$, synthesis will be based on the un-weighted generalized plant

$$\begin{pmatrix} e \\ u \\ \hdashline y \end{pmatrix} = \begin{pmatrix} I & \Delta H \\ 0 & I \\ \hdashline I & \Delta H \end{pmatrix} \begin{pmatrix} d \\ \hdashline u \end{pmatrix} \quad \text{with} \quad \Delta = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & -1 & 1 \end{pmatrix},$$

where e_i, d_i, u_i are collected into the vectors e, d, u and $y = e$ is the measurement output. Note that ΔH is lower-triangular. Our main result allows us to design triangular controllers

$$\begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix} = \begin{pmatrix} K_{11} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ K_{N1} & \cdots & K_{NN} \end{pmatrix} \begin{pmatrix} y_1 \\ \vdots \\ y_N \end{pmatrix}$$

which optimize the H_∞- or H_2-norm of $d \to e$ or a dynamically weighted versions thereof. If l_i is the length

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of the i-th car and $\xi_0 > l_1 + \xi_1 > \cdots > l_N + \xi_N$, this controller structure means that the i-th car is measuring only its own error signal and those of all leading cars.

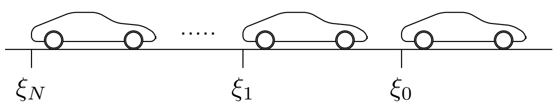

Fig. 1. Car platoon

The paper is organized as follows. In Section II we review the known LMI solution of the H_∞-problem by variable transformation and present structured extension. In Section III we briefly address H_2-synthesis while Section IV contains a concrete design example. Proofs are given in the appendix.

II. H_∞-SYNTHESIS

We choose a realization of (1) that is denoted as

$$\begin{pmatrix} e \\ y \end{pmatrix} = \left[\begin{array}{c|cc} A & B_0 & B \\ \hline C_0 & D_0 & E \\ C & F & D \end{array}\right] \begin{pmatrix} d \\ u \end{pmatrix} \tag{3}$$

and where the matrices in (A, B, C, D) admit the structure

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{cc|cc} A_{11} & 0 & B_{11} & 0 \\ A_{21} & A_{22} & B_{21} & B_{22} \\ \hline C_{11} & 0 & D_{11} & 0 \\ C_{21} & C_{22} & D_{21} & D_{22} \end{array}\right) \tag{4}$$

with D sharing its partition with the transfer matrix $u \to y$ in (1). Furthermore, $A \in \mathbb{R}^{n\times n}$ is partitioned according to $n = n_1 + n_2$ which fixes the structures of $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{k\times n}$; w.l.o.g. we assume $D = 0$. The controller is described as

$$u = \left[\begin{array}{c|c} A^c & B^c \\ \hline C^c & N \end{array}\right] y = \left[\begin{array}{cc|cc} A^c_{11} & 0 & B^c_{11} & 0 \\ A^c_{21} & A^c_{22} & B^c_{21} & B^c_{22} \\ \hline C^c_{11} & 0 & N_{11} & 0 \\ C^c_{21} & C^c_{22} & N_{21} & N_{22} \end{array}\right] y. \tag{5}$$

Note that the choice of the partition $n^c = n^c_1 + n^c_2$ of A^c is part of the design problem, while the ones of B^c, C^c and N are then determined through $n^c \times k$, $m \times n^c$ and $m \times k$ respectively. Such a controller is said to be structured, in contrast to unstructured controllers defined with A^c, B^c, C^c, N having no specific sparsity pattern.

The controlled system, the interconnection of (3) and (5), is then described by the LTI system defined with

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} := \left(\begin{array}{cc|c} A & 0 & B_0 \\ 0 & 0 & 0 \\ \hline C_0 & 0 & D_0 \end{array}\right) + \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & E \end{array}\right) \begin{pmatrix} A^c & B^c \\ C^c & N \end{pmatrix} \left(\begin{array}{cc|c} 0 & I & 0 \\ C & 0 & F \end{array}\right). \tag{6}$$

For some level $\gamma > 0$, the H_∞-design problem consists of finding a controller (5) which renders $\mathcal{A}$ Hurwitz and such that $\|\mathcal{C}(sI - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}\|_\infty < \gamma$ is satisfied. With the classical bounded real lemma [3, Lemma 4.1], these two closed-loop properties are equivalently translated into the existence of some $\mathcal{X} = \mathcal{X}^T$ which satisfies

$$\mathcal{X} > 0 \quad \text{and} \quad \mathrm{He}\begin{pmatrix} \mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B} & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \mathcal{C} & \mathcal{D} & -\frac{\gamma}{2}I \end{pmatrix} < 0. \tag{7}$$

A. Unstructured controllers

We start by deriving necessary conditions for the existence of an H_∞-controller. In a partition according to $n + n^c$, let $\begin{pmatrix} X \\ U \end{pmatrix}$ and $\begin{pmatrix} Y \\ V \end{pmatrix}$ denote the first block columns of $\mathcal{X}$ and $\mathcal{X}^{-1}$ respectively. It can be assumed w.l.o.g. that U is tall (i.e. $n^c \geq n$) and has full column rank [9]. We obtain

$$\mathcal{X}\mathcal{Y} = \mathcal{Z} \quad \text{for} \quad \mathcal{Y} := \begin{pmatrix} Y & I_n \\ V & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{Z} := \begin{pmatrix} I_n & X \\ 0 & U \end{pmatrix}. \tag{8}$$

Since $\mathcal{Z}$ has full column rank, the same holds for $\mathcal{Y}$. With a congruence transformation involving $\mathcal{Y}$, (7) implies

$$\mathcal{Y}^T\mathcal{Z} > 0 \quad \text{and} \quad \mathrm{He}\begin{pmatrix} \mathcal{Z}^T\mathcal{A}\mathcal{Y} & \mathcal{Z}^T\mathcal{B} & 0 \\ 0 & -\frac{\gamma}{2}I & 0 \\ \mathcal{C}\mathcal{Y} & \mathcal{D} & -\frac{\gamma}{2}I \end{pmatrix} < 0. \tag{9}$$

The first inequality can be expressed as

$$\begin{pmatrix} Y^T & Y^TX + V^TU \\ I & X \end{pmatrix} > 0. \tag{10}$$

Since X, Y are square and by symmetry, the right upper block equals I which leads to the standard coupling condition between X and Y. Despite $X^T = X$, $Y^T = Y$, we keep the transposes in the sequel since these matrices will be non-symmetric in the solution of the structured problem. The second inequality in (9) reads with (6) as

$$\mathrm{He}\left[\left(\begin{array}{cc|cc} AY & A & B_0 & 0 \\ X^TAY & X^TA & X^TB_0 & 0 \\ \hline 0 & 0 & -\frac{\gamma}{2}I & 0 \\ C_0Y & C_0 & D_0 & -\frac{\gamma}{2}I \end{array}\right) + \right.$$
$$\left. + \left(\begin{array}{cc} 0 & B \\ U^T & X^TB \\ \hline 0 & 0 \\ 0 & E \end{array}\right)\begin{pmatrix} A^c & B^c \\ C^c & N \end{pmatrix}\left(\begin{array}{cc|cc} V & 0 & 0 & 0 \\ CY & C & F & 0 \end{array}\right)\right] < 0. \tag{11}$$

For compactness of notation we introduce the abbreviation

$$Q(X,Y) := \left(\begin{array}{cc|cc} AY & A & B_0 & 0 \\ 0 & X^TA & X^TB_0 & 0 \\ \hline 0 & 0 & -\frac{\gamma}{2}I & 0 \\ C_0Y & C_0 & D_0 & -\frac{\gamma}{2}I \end{array}\right) \tag{12}$$

which depends affinely on X, Y. After performing the variable substitution

$$\begin{pmatrix} K & L \\ M & N \end{pmatrix} := \begin{pmatrix} X^TAY & 0 \\ 0 & 0 \end{pmatrix} +$$
$$+ \begin{pmatrix} I & X^TB \\ 0 & I \end{pmatrix}\begin{pmatrix} U^TA^cV & U^TB^c \\ C^cV & N \end{pmatrix}\begin{pmatrix} I & 0 \\ CY & I \end{pmatrix}, \tag{13}$$

the inequality (11) reads as

$$\mathrm{He}\left[Q(X,Y) + \left(\begin{array}{cc} 0 & B \\ I & 0 \\ \hline 0 & 0 \\ 0 & E \end{array}\right)\begin{pmatrix} K & L \\ M & N \end{pmatrix}\left(\begin{array}{cc|cc} I & 0 & 0 & 0 \\ 0 & C & F & 0 \end{array}\right)\right] < 0 \tag{14}$$

which is affine in X, Y and K, L, M, N. This proves one direction of the following well-known result [5], [9].

Theorem 1: There exists an unstructured controller such that the closed-loop system satisfies (7) for some symmetric $\mathcal{X}$ iff there exist matrices $X = X^T$, $Y = Y^T$ and K, L, M, N with (14) and

$$\begin{pmatrix} Y & I \\ I & X \end{pmatrix} \succ 0. \tag{15}$$

If these LMIs are feasible, one can construct an unstructured controller with McMillan degree of at most n which solves the H_∞-problem.

If (14)-(15) are feasible, a suitable controller is obtained by setting $U := I$ and $V := I - X^TY$ and solving (13) for A^c, B^c, C^c. Indeed, by (15), $I - X^TY$ is invertible. Due to the choice of U and V, the inequality (15) leads back to (10). Moreover, due to (13), (14) implies (11). With (6), all this shows the validity of (9), and by a congruence transformation using $\mathcal{Y}^{-1}$ we obtain (7) for $\mathcal{X} := \mathcal{Z}\mathcal{Y}^{-1}$.

B. Structured controllers

A simple restriction of K, L, M in (14) to lower block-diagonal matrices does not lead to the correct synthesis conditions for structured controllers. The lower triangular parts of these matrices will be free, but the upper right blocks have to be specified suitably. This motivates to introduce, for some 2×2 block partition of any matrix H, the decomposition $H = \mathscr{L}(H) + \mathscr{U}(H)$ into the lower-block triangular/strict upper triangular parts $\mathscr{L}(H)$ and $\mathscr{U}(H)$ with the projections

$$\mathscr{L}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} := \begin{pmatrix} H_{11} & 0 \\ H_{21} & H_{22} \end{pmatrix}$$
$$\text{and} \quad \mathscr{U}\begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix} := \begin{pmatrix} 0 & H_{12} \\ 0 & 0 \end{pmatrix}.$$

The underlying partition will always be clear from the context. For (4) we will also use the abbreviations

$$\begin{pmatrix} B_1 & B_2 \end{pmatrix} := \left(\begin{array}{c|c} B_{11} & 0 \\ B_{21} & B_{22} \end{array}\right), \quad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \left(\begin{array}{cc} C_{11} & 0 \\ \hline C_{21} & C_{22} \end{array}\right)$$
$$\text{and} \quad B_e := \begin{pmatrix} 0_{n\times m_1} & B_2 \\ 0_{n\times m_1} & 0_{n\times m_2} \end{pmatrix}, \quad C_e := \begin{pmatrix} 0_{k_1\times n} & C_1 \\ 0_{k_2\times n} & 0_{k_2\times n} \end{pmatrix}.$$

The synthesis LMIs for the structured problem involve the block-triangular decision variables

$$\begin{pmatrix} \tilde{K} & \tilde{L} \\ \tilde{M} & N \end{pmatrix} = \left(\begin{array}{cc|cc} \tilde{K}_{11} & 0_{n\times n} & \tilde{L}_{11} & 0_{n\times m_2} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{L}_{21} & \tilde{L}_{22} \\ \hline \tilde{M}_{11} & 0_{k_1\times n} & N_{11} & 0_{k_1\times m_2} \\ \tilde{M}_{21} & \tilde{M}_{22} & N_{21} & N_{22} \end{array}\right) \tag{16}$$

of dimension $(n+n+k_1+k_2)\times(n+n+m_1+m_2)$ as well as the rectangular unknowns X and Y of dimension $n\times(n+n)$ and partitioned as

$$X := \begin{pmatrix} X_2 & X_3 \end{pmatrix} \quad \text{and} \quad Y := \begin{pmatrix} Y_1 & Y_2 \end{pmatrix}; \tag{17}$$

here $X_3 = X_3^T$ and $Y_1 = Y_1^T$ are unstructured, while X_2, Y_2 carry the structures

$$X_2 := \begin{pmatrix} \hat{X}_2 & \hat{Z}_2^T \\ 0 & I_{n_2} \end{pmatrix} \quad \text{and} \quad Y_2 = \begin{pmatrix} I_{n_1} & 0 \\ -\hat{Z}_2 & \hat{Y}_2 \end{pmatrix} \tag{18}$$

with symmetric $\hat{X}_2 \in \mathbb{R}^{n_1\times n_1}$ and $\hat{Y}_2 \in \mathbb{R}^{n_2\times n_2}$. In total X and Y hence comprise $3n(n+1)/2$ scalar decision variables.

We are now ready to formulate the solution of the H_∞-synthesis problem for structured controllers by a controller variable transformation, the main result of this paper.

Theorem 2: There exists a structured controller and some $\mathcal{X} = \mathcal{X}^T$ such that the closed-loop system satisfies (7) iff there exist $\tilde{K}$, $\tilde{L}$, $\tilde{M}$, N and X, Y that are structured as in (16)-(18) and satisfy the LMIs

$$\begin{pmatrix} Y_1 & Y_2 & I \\ Y_2^T & Y_2^TX_2 & X_2^T \\ I & X_2 & X_3 \end{pmatrix} \succ 0 \tag{19}$$

together with (14) after the substitution

$$K = \tilde{K} + \mathscr{U}(X^TAY) + \mathscr{U}(B_e\tilde{M}) + \mathscr{U}(\tilde{L}C_e) + B_eNC_e, \tag{20}$$

$$L = \tilde{L} + \mathscr{U}(B_eN) \quad \text{and} \quad M = \tilde{M} + \mathscr{U}(NC_e). \tag{21}$$

If this LMI system is feasible, one can construct a structured controller with McMillan degree of at most $2n$ which solves the H_∞-synthesis problem.

The proof is deferred to the Appendix. Note that Theorem 2 easily generalizes to the setting in [8], with (1) having a lower-triangular control channel $u \to y$ with p blocks and u, y partitioned into p components.

We emphasize that

$$X_2^TY_2 = \begin{pmatrix} \hat{X}_2 & 0 \\ \hat{Z}_2 & I \end{pmatrix}\begin{pmatrix} I & 0 \\ -\hat{Z}_2 & \hat{Y}_2 \end{pmatrix} = \begin{pmatrix} \hat{X}_2 & 0 \\ 0 & \hat{Y}_2 \end{pmatrix}$$

and hence the l.h.s. of (19) is affine the unknowns. Similarly, we have

$$\mathscr{U}(X^TAY) = \mathscr{U}\begin{pmatrix} X_2^TAY_1 & X_2^TAY_2 \\ X_3^TAY_1 & X_3^TAY_2 \end{pmatrix} = \begin{pmatrix} 0 & X_2^TAY_2 \\ 0 & 0 \end{pmatrix}$$

and

$$X_2^TAY_2 = \begin{pmatrix} \hat{X}_2A_{11} & 0 \\ \hat{Z}_2A_{11} - A_{22}\hat{Z}_2 + A_{21} & A_{22}\hat{Y}_2 \end{pmatrix},$$

which does indeed imply that the right-hand sides in (20)-(21) are also affine linear in the decision variables.

As in [8], Theorem 2 allows to compute the optimal achievable H_∞-synthesis level for structured controllers by solving a semi-definite program of fixed dimension and with the guaranteed a priori bound $2n$ on the degree of close-to-optimal controllers. This is in stark contrast to prior techniques that emerged in [11], [6], see also [2]. If compared to [8] based on eliminating the controller parameters, the proposed alternative approach relies on a structured controller parameter transformation that is reminiscent of [5], [9] for unstructured controllers. As well-know, the key benefit is the possibility to solve mixed synthesis problems with structured controllers in a straightforward fashion, with the same guaranteed bound of $2n$ on the controller's degree. However, the inclusion of rank constraints for reduced order synthesis is not straightforward (see [9]). A specialization leads to a new LMI-solution for the structured H_2-synthesis problem, which extends the solution in [4] based on Riccati equations for

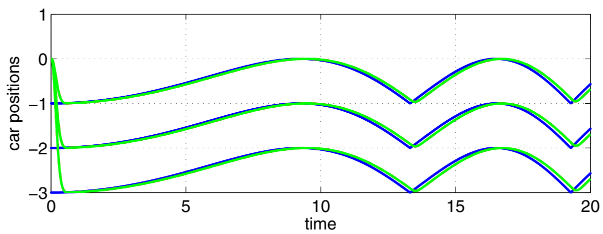

Fig. 2. References (dashed blue) and car positions (full green)

regular plants. Other extensions to gain-scheduling or robust synthesis with structured controllers for nested generalized plants become routine as well.

III. H_2-SYNTHESIS

Instead of repeating [5], [9], we now illustrate the general design procedure for H_2-synthesis only. As well-known, the controlled system satisfies $\|\mathcal{C}(sI-\mathcal{A})^{-1}\mathcal{B}+\mathcal{D}\|_2<\gamma$ iff $\mathcal{D}=0$ and there exist $\mathcal{X}=\mathcal{X}^T$, Z with

$$\mathrm{He}\begin{pmatrix}\mathcal{X}\mathcal{A} & \mathcal{X}\mathcal{B}\\ 0 & -\frac{\gamma}{2}I\end{pmatrix}<0,\quad \begin{pmatrix}\mathcal{X} & \mathcal{C}^T\\ \mathcal{C} & Z\end{pmatrix}>0,\quad \mathrm{Trace}(Z)<\gamma.$$

The corresponding synthesis conditions read as $\boldsymbol{D}=0$ and

$$\mathrm{He}\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ 0 & -\frac{\gamma}{2}I\end{pmatrix}<0,\quad \begin{pmatrix}\boldsymbol{X} & \boldsymbol{C}^T\\ \boldsymbol{C} & Z\end{pmatrix}>0,\quad \mathrm{Trace}(Z)<\gamma$$

with the substitutions

$$\begin{pmatrix}\boldsymbol{A} & \boldsymbol{B}\\ \boldsymbol{C} & \boldsymbol{D}\end{pmatrix}=\left(\begin{array}{cc|c}AY+BM & A+BNC & B_0+BNF\\ K & X^TA+LC & X^TB_0+LF\\ \hline C_0Y+EM & C_0+ENC & ENF\end{array}\right),$$

$$\boldsymbol{X}:=\begin{pmatrix}Y_1 & Y_2 & I\\ Y_2^T & Y_2^TX_2 & X_2^T\\ I & X_2 & X_3\end{pmatrix}$$

and (20)-(21). It would be interesting but goes beyond the scope of this paper to establish the link to the results of [4].

IV. A NUMERICAL EXAMPLE

Let us design (with [1]) a triangular controller for the vehicle string in Section I with $N=3$ and $m_i=i$, $\tau_i=0.1$, $i=1,\ldots,3$. With the weights $w_e(s)=0.5(s+10)/(s+10^{-5})$ on each error e_i and $w_u(s)=10^{-3}$ on each control input u_i, we compare an unstructured H_∞-design with a structured version. Both optimal levels are approximately equal to 1.07, which implies that structured controllers cause in this sense no performance loss over unstructured ones. In the sequel results for the unstructured controller (with degree 12) are plotted with dashed red lines, while those for the structured controller (with degree 36) are depicted by a full green line.

A time-domain plot for the closed-loop system is shown in Figure 2, with the reference signals for each car indicated by a dotted blue line; at the given scale, the structured and unstructured controller lead to undistinguishable responses.

The magnitude plots of the transfer functions $d\to e$ for the controlled systems in Figure 3, which includes a plot of the error weight by a dotted blue line, confirm the fact that the tracking properties of the structured and unstructured controllers are very similar; note that the unstructured controller creates some mild cross-coupling in $d_i\to e_j$ for $i>j$.

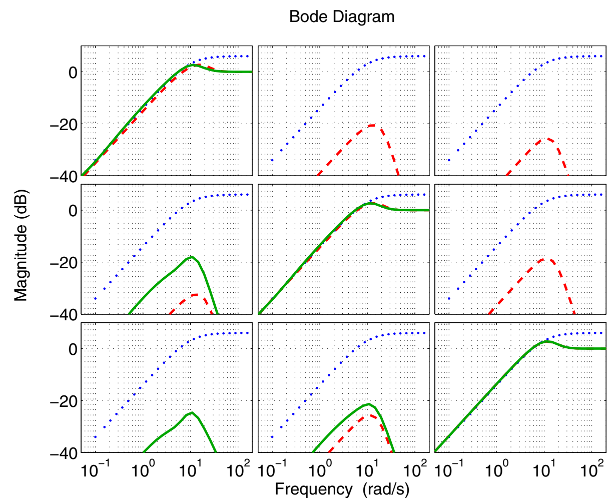

Fig. 3. Frequency magnitude responses of weight (dotted blue) and of $d\to e$ for unstructured (dashed red) and structured (full green) controller.

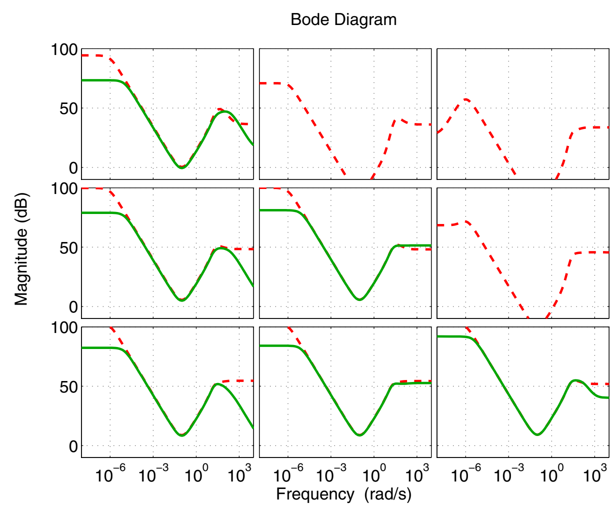

Fig. 4. Frequency magnitude responses of the structured (full green) and unstructured (dashed red) controller.

The magnitude plots of both controllers in Figure 4 match each other in those frequency ranges that are constraining the weighted H_∞-cost; triangularity of the structured controller is clearly discernible. Finally, the plots do not change visibly if replacing the full structured controller by one that is reduced to order 16 with the standard (unstructured) model reduction tool as accessible in Matlab through the command `reduce`.

V. CONCLUSIONS

In this paper we have developed an exact LMI solution for the optimal H_∞-design of block-triangular controllers for a generalized plant whose control channel matches this structure. In contrast to a recently proposed algorithm, the presented one is based on a new convexifying structured controller parameter transformation. This opens the way to directly generalize structured controller synthesis to a mixture of H_∞- and H_2-cost-criteria, among others, in complete parallel to what is know for unstructured controller synthesis.

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APPENDIX

A. *Proof of necessity in Theorem 2*

Suppose that there exists a structured controller (5) such that (7) holds for some $\mathcal{X} = \mathcal{X}^T$. As in the unstructured case we assume w.l.o.g. that $n_1^c \geq n$ and $n_2^c \geq n$. We intend to construct a factorization (8) with upper block-triangular U and lower block-triangular V in the partition $(n_1^c + n_2^c) \times (n+n)$, where X and Y are rectangular and structured as in (17)-(18). To this end we introduce the partition

$$\mathcal{X} = \begin{pmatrix} X_3 & U_{13}^T & U_{23}^T \\ U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{pmatrix} \in \mathbb{R}^{(n+n_1^c+n_2^c)\times(n+n_1^c+n_2^c)}.$$

Since the right-lower 2×2 block of $\mathcal{X}$ is positive definite and U_{13} and U_{23} are tall, it is possible to perturb these latter matrices such that

$$H_1 := \begin{pmatrix} I & 0 \end{pmatrix} \begin{pmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{pmatrix}^{-1} \begin{pmatrix} U_{13} \\ U_{23} \end{pmatrix} \quad \text{and} \quad H_2 := Z_{22}^{-1} U_{23} \tag{22}$$

have full column rank (f.c.r.) and (7) still holds. We then define

$$\begin{pmatrix} Y_1 \\ V_{11} \\ V_{21} \end{pmatrix} := \begin{pmatrix} X_3 & U_{13}^T & U_{23}^T \\ U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_n \\ 0 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{Y}_2 \\ \tilde{V}_{22} \end{pmatrix} := \begin{pmatrix} X & U_{23}^T \\ U_{23} & Z_{22} \end{pmatrix}^{-1} \begin{pmatrix} I_n \\ 0 \end{pmatrix}.$$

Clearly Y_1 and $\tilde{Y}_2$ are symmetric and positive definite. By the block-inversion formula we have $V_{11} = -H_1 Y_1$ and $\tilde{V}_{22} = -H_2 \tilde{Y}_2$ which shows that V_{11} and $\tilde{V}_{22}$ have f.c.r.. The definitions obviously imply, for some suitable $\tilde{U}_{12}$,

$$\begin{pmatrix} X_3 & U_{13}^T & U_{23}^T \\ U_{13} & Z_{11} & Z_{12} \\ U_{23} & Z_{21} & Z_{22} \end{pmatrix} \begin{pmatrix} Y_1 & \tilde{Y}_2 & I_n \\ V_{11} & 0 & 0 \\ V_{21} & \tilde{V}_{22} & 0 \end{pmatrix} = \begin{pmatrix} I_n & I_n & X_3 \\ 0 & \tilde{U}_{12} & U_{13} \\ 0 & 0 & U_{23} \end{pmatrix}. \tag{23}$$

In a next step we search X_2, Y_2 with the structure (18) that satisfy $\tilde{Y}_2 X_2 = Y_2$; in the partition

$$\tilde{Y}_2 := \begin{pmatrix} \tilde{Y}_{11} & \tilde{Y}_{12} \\ \tilde{Y}_{21} & \tilde{Y}_{22} \end{pmatrix} \in \mathbb{R}^{(n_1 \times n_2)\times(n_1\times n_2)}$$

it is easy to check by computation that the blocks

$$\hat{X}_2 := \tilde{Y}_{11}^{-1}, \ \hat{Z}_2 := -\tilde{Y}_{21}\hat{X}_2 \quad \text{and} \quad \hat{Y}_2 := \tilde{Y}_{22} + \hat{Z}_j \tilde{Y}_{12}$$

do indeed serve the desired purpose. We emphasize that X_2 and Y_2 are invertible. By right-multiplying the second block column of (23) with X_2 and if setting $V_{22} := \tilde{V}_{22} X_2$, $U_{12} := \tilde{U}_{12} X_2$, we arrive the the desired triangular factorization:

$$\mathcal{X} \begin{pmatrix} Y_1 & Y_2 & I \\ V_{11} & 0 & 0 \\ V_{21} & V_{22} & 0 \end{pmatrix} = \begin{pmatrix} I & X_2 & X_3 \\ 0 & U_{12} & U_{13} \\ 0 & 0 & U_{23} \end{pmatrix}. \tag{24}$$

Due to $\det(X_2) \neq 0$ we conclude that V_{11} and V_{22} have full column rank. Hence the l.h.s. of (24) has full column rank, which implies the same for the r.h.s.; consequently, U_{12} and U_{23} have full column rank. Furthermore, (8) holds with

$$\begin{pmatrix} X \\ U \end{pmatrix} := \begin{pmatrix} X_2 & X_3 \\ U_{12} & U_{13} \\ 0 & U_{23} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} Y \\ V \end{pmatrix} := \begin{pmatrix} Y_1 & Y_2 \\ V_{11} & 0 \\ V_{21} & V_{22} \end{pmatrix}. \tag{25}$$

Since U has full column rank, we can argue as for the unstructured problem in order to infer the validity of (10) and (14) for (13).

To prove the coupling condition (19), we note that

$$\begin{pmatrix} W_{12} & W_{13} \\ 0 & W_{23} \end{pmatrix} := V^T U \tag{26}$$

is upper block-triangular. Hence (10) reads explicitly as

$$\begin{pmatrix} Y_1^T & Y_1^T X_2 + W_{12} & Y_1^T X_3 + W_{13} \\ Y_2^T & Y_2^T X_2 & Y_2^T X_3 + W_{23} \\ I & X_2 & X_3 \end{pmatrix} \succ 0 \tag{27}$$

which is identical to (19) by symmetry.

With

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & N \end{pmatrix} := \begin{pmatrix} U^T A^c V & U^T B^c \\ C^c V & N \end{pmatrix}, \tag{28}$$

the variable transformation (13) amounts to

$$M = \tilde{C} + NCY, \quad L = \tilde{B} + X^T B N \quad \text{and} \tag{29}$$

$$K = X^T AY + \tilde{A} + X^T BM + LCY - X^T BNCY. \quad (30)$$

Since U^T, V and the controller matrices are lower block-triangular, we infer from (28) that $\mathscr{U}(\tilde{A}) = 0$, $\mathscr{U}(\tilde{B}) = 0$ and $\mathscr{U}(\tilde{C}) = 0$. In the sequel we will exploit

$$X_2^T B_2 = \begin{pmatrix} \hat{X} & 0 \\ \hat{Z} & I \end{pmatrix} \begin{pmatrix} 0 \\ B_{22} \end{pmatrix} = B_2 \ \text{ and, similarly, } \ C_1 Y_2 = C_1.$$

This leads to the following essential relations. First,

$$\mathscr{U}(NCY) = \mathscr{U}\left(\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} C_1Y_1 & C_1Y_2 \\ C_2Y_1 & C_2Y_2 \end{pmatrix}\right) =$$
$$= \mathscr{U}\left(\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} 0 & C_1 \\ 0 & 0 \end{pmatrix}\right) = \mathscr{U}(NC_e) \quad (31)$$

and

$$\mathscr{U}(X^TBN) = \mathscr{U}\left(\begin{pmatrix} X_2^TB_1 & X_2B_2 \\ X_3^TB_1 & X_3^TB_2 \end{pmatrix}\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\right) =$$
$$= \mathscr{U}\left(\begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\right) = \mathscr{U}(B_eN). \quad (32)$$

Due to (29) this implies

$$\mathscr{U}(M) = \begin{pmatrix} 0 & N_{11}C_1 \\ 0 & 0 \end{pmatrix} \ \text{ and } \ \mathscr{U}(L) = \begin{pmatrix} 0 & B_2N_{22} \\ 0 & 0 \end{pmatrix}.$$

Therefore, we get

$$\mathscr{U}(X^TB\mathscr{U}(M)) + \mathscr{U}(\mathscr{U}(L)CY) - \mathscr{U}(X^TBNCY) =$$
$$= \mathscr{U}\left(\begin{pmatrix} X_2^TB_1 & X_2B_2 \\ X_3^TB_1 & X_3^TB_2 \end{pmatrix}\begin{pmatrix} 0 & N_{11}C_1 \\ 0 & 0 \end{pmatrix}\right) +$$
$$+ \mathscr{U}\left(\begin{pmatrix} 0 & B_2N_{22} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} C_1Y_1 & C_1 \\ C_2Y_1 & C_2Y_2 \end{pmatrix}\right) -$$
$$- \mathscr{U}\left(\begin{pmatrix} X_2^TB_1 & X_2^TB_2 \\ X_3^TB_1 & X_3^TB_2 \end{pmatrix}\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} C_1Y_1 & C_1Y_2 \\ C_2Y_1 & C_2Y_2 \end{pmatrix}\right) =$$
$$= \begin{pmatrix} 0 & X_2^TB_1N_{11}C_1 + B_2N_{22}C_2Y_2 \\ 0 & 0 \end{pmatrix} -$$
$$- \begin{pmatrix} X_2^TB_1 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} 0 & C_1 \\ 0 & C_2Y_2 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & B_2N_{21}C_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} N_{11} & 0 \\ N_{21} & N_{22} \end{pmatrix}\begin{pmatrix} 0 & C_1 \\ 0 & 0 \end{pmatrix} =$$
$$= B_eNC_e.$$

Moreover,

$$\mathscr{U}(X^TB\mathscr{L}(M)) =$$
$$= \mathscr{U}\left(\begin{pmatrix} X_2^TB_1 & X_2^TB_2 \\ X_3^TB_1 & X_3^TB_2 \end{pmatrix}\begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}\right) =$$
$$= \mathscr{U}\left(\begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} M_{11} & 0 \\ M_{21} & M_{22} \end{pmatrix}\right) = \mathscr{U}(B_e\tilde{M})$$

and, by the very same computation,

$$\mathscr{U}(\tilde{L}CY) = \mathscr{U}(\tilde{L}C_e).$$

If combined with the trivial decompositions $\mathscr{U}(X^TBM) = \mathscr{U}(X^TB\mathscr{U}(M)) + \mathscr{U}(X^TB\mathscr{L}(M))$ and $\mathscr{U}(LCY) = \mathscr{U}(\mathscr{U}(L)CY) + \mathscr{U}(\mathscr{L}(L)CY)$, we thus arrive at

$$\mathscr{U}(X^TBM + LCY - X^TBNCY) =$$
$$= \mathscr{U}(B_e\tilde{M}) + \mathscr{U}(\tilde{L}C_e) + B_eNC_e. \quad (33)$$

This latter relation is the technical key for the present paper, since the seemingly non-linear term on the l.h.s. is actually affine in the decision variables, as seen on the right.

Now the proof can be concluded quickly. Define

$$\tilde{K} := \mathscr{L}(K), \quad \tilde{L} := \mathscr{L}(L) \ \text{ and } \ \tilde{M} := \mathscr{L}(M)$$

with the structure as in (16). Due to (31)-(32) we have

$$M - \tilde{M} = \mathscr{U}(\tilde{C}) + \mathscr{U}(NCY) = \mathscr{U}(NC_e),$$
$$L - \tilde{L} = \mathscr{U}(\tilde{B}) + \mathscr{U}(X^TBN) = \mathscr{U}(B_eN),$$

i.e., M and L do have indeed the structure as claimed in (21). For (20) this is verified with the help of (33) as

$$K - \tilde{K} - \mathscr{U}(X^TAY) =$$
$$= \mathscr{U}(\tilde{A}) + \mathscr{U}(X^TBM + LCY - X^TBNCY) =$$
$$= \mathscr{U}(B_e\tilde{M}) + \mathscr{U}(\tilde{L}C_e) + B_eNC_e.$$

This finishes the proof of “only if” in Theorem 2.

B. Proof of sufficiency in Theorem 2

Let $\tilde{K}$, $\tilde{L}$, $\tilde{M}$ and X, Y satisfy the LMIs as described. We choose

$$\begin{pmatrix} U_{12} & U_{13} \\ 0 & U_{23} \end{pmatrix} := \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$$

and

$$\begin{pmatrix} V_{11}^T & V_{21}^T \\ 0 & V_{22}^T \end{pmatrix} := \begin{pmatrix} Y_2 - Y_1^TX_2 & I - Y_1^TX_3 \\ 0 & X_2^T - Y_2^TX_3 \end{pmatrix},$$

just in order to make sure that (19) equals (27) with (26). Note that (19) implies invertibility of $Y_1 - Y_2(Y_2^TX_2)^{-1}Y_2^T = Y_1^T - Y_2X_2^{-1}$ and $X_3 - X_2(Y_2^TX_2)^{-1}X_2^T = X_3 - Y_2^{-T}X_2^T$, which shows the same for V_{11} and V_{22}. We can hence simply define $\mathcal{Y}$, $\mathcal{Z}$ and also $\mathcal{X}$ by (8). Then (27) equals (10) and the l.h.s. is symmetric. This implies $\mathcal{Z}^T\mathcal{Y} = \mathcal{Y}^T\mathcal{Z} \succ 0$ and thus $\mathcal{Y}^{-T}\mathcal{Z}^T = \mathcal{Z}\mathcal{Y}^{-1} \succ 0$, which in turn shows that $\mathcal{X} = \mathcal{Z}\mathcal{Y}^{-1}$ is symmetric and positive definite.

Next define K, L, M through (20)-(21). Motivated by (29)-(30) we take

$$\tilde{C} := \mathscr{L}(M - NCY), \quad \tilde{B} := \mathscr{L}(L - X^TBN) \ \text{ and}$$
$$\tilde{A} := \mathscr{L}(\tilde{K} - X^TAY - X^TBM - LCY + X^TBNCY)$$

which certainly admit the structure (16). In view of (31)-(33) and due to (20)-(21), we conclude that the relations (29)-(30) hold for the complete matrices. Since U^T and V are lower block-triangular and invertible, we can solve (28) for lower block-triangular controller realization matrices A^c, B^c, C^c. This renders (13) valid. Then (14) is identical to (11) and, hence, to the second inequality in (9). Since $\mathcal{Y}$ is square and invertible, we can back-transform to (7) by congruence. All this proves that the constructed structured controller solves the H_∞-problem.