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Controllability for positive discrete–time linear systems with positive state*

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***Abstract*— Controllability of componentwise nonnegative discrete-time linear systems is considered. The key difference here from the well-established positive systems theory is that we permit the case where the input takes negative values, provided that the state remains nonnegative. Such a framework is very natural, moreover necessary, in situations such as population ecology to describe the control actions of harvesting or culling. The present contribution summarises recently published material by the authors and considers a novel application in low-gain PI control.**

I. Introduction

Controllability is a fundamental concept in control theory and the formulation used presently dates back to Kalman [1]. For finite-dimensional, linear, time-invariant, continuous time systems the notions of reachability, controllability and null controllability are all equivalent. When the A operator in (1) is invertible then the same is true for discrete-time systems. As is well-known, part of their appeal lies in the interplay between analytic and algebraic concepts. For instance, the existence of a control steering the system to a desired state is equivalent to the reachability matrix having full rank.

In many physically motivated systems the state and input variables cannot take negative values, for example, drug ingestion and metabolism. Componentwise nonnegativity is a property that, somewhat problematically, is not *a priori* respected by controllability in its most general form. The need to understand controllability for such systems, and of course other similar concepts, motivated the development of positive systems theory and there now exist several textbooks on the subject (for example, [2]–[5]). Naturally, controllability, that is *positive input controllability*, in such a framework is more limited than the general case, but the situation is well understood ([6], [7] and the references therein). A key feature of positive systems theory is the notion that both the state and the input variables must be nonnegative.

Here we present recently published results [8] pertaining to the controllability of the discrete-time linear system

$$x(t+1) = Ax(t) + Bu(t)\,, \quad x(0) = x^0\,, \quad t \in \mathbb{N}_0\,, \quad (1)$$

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where A, B and x^0 are componentwise nonnegative under the constraint that just the state must remain nonnegative. We denote such a framework in [8] as *positive state controllability* and our primary example of where this framework is necessary is population ecology. Here matrix models are often used (see, for example, Caswell [9] or Cushing [10]) with the nonnegative state x denoting a stage- or age-structured population, and the control u denoting a conservation strategy or a form of pest control or harvesting. There are many papers (including, for example, [11]-[14]) where the model (1) is suitable for describing the addition or removal of individuals from a population and for a full description of these actions we require that u can take negative values.

We present a selection of results from [8], without proof, that demonstrate that under a certain assumption, the problem of positive state controllability is equivalent to positive input controllability of a related positive system. We refer the reader to [8] for proofs of these results. We present two examples from population ecology that seek to a) highlight the possible uses of the theory and b) demonstrate the seemingly non-trivial 'middle ground' between positive state and positive input controllability of positive linear systems. Novel to this contribution, we also consider the implications of positive state control in describing which nonnegative reference vectors are candidate asymptotic outputs of the system

$$\left.\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \quad x(0) = x^0 \\ y(t) &= Cx(t)\,, \end{aligned}\right\} \; t \in \mathbb{N}_0\,, \quad (2)$$

where A is Schur (that is, stable). The motivation for such a result is in PI control of MIMO positive state systems, for regulating the output of (2) to a desired reference. PI control has recently been suggested as a possible tool for population management [15] in the case of SISO systems. Proofs of these later results are in preparation [16].

II. Positive state controllability

A. Definitions

For $n \in \mathbb{N}$, $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$ and $e_i \in \mathbb{R}^n$ is the i^{th} standard basis vector. For vectors x and matrices X, $x \geq 0$ (also $0 \leq x$) and $X \geq 0$ (also $0 \leq X$) denotes componentwise nonnegativity. The superscript ^{T} denotes matrix transposition. For nonnegative vectors

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$x_1, x_2, \ldots, x_k \in \mathbb{R}^n_+$ we let $\langle x_1, x_2, \ldots, x_k \rangle_+ \subseteq \mathbb{R}^n_+$ denote their nonnegative linear span. For nonnegative matrices $X_1, X_2, \ldots, X_k \in \mathbb{R}^{n\times n}_+$ we let $\langle X_1, X_2, \ldots, X_k \rangle_+ \subseteq \mathbb{R}^n_+$ denote the nonnegative linear span of their columns.

We are interested in the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ generating the controlled system (1) where $A, B \geq 0$ and the state x is nonnegative. In order to formalise reachability and null controllability, both with nonnonegative state, we introduce the following definitions.

Definition 2.1: Given the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $A, B \geq 0$, we say that $x_T \in \mathbb{R}^n_+$ is positive state reachable in finite time if there exists a control sequence that steers the state x of (A, B) from 0 to x_T in N steps and additionally maintains nonnegativity of x. The collection of all such $x_T \in \mathbb{R}^n_+$ is called the positive state reachable set (in finite time). We say that (A, B) is positive state reachable in finite time if this set is $\mathbb{R}^n_+$.

We note that the positive state reachable set is a convex cone of the linear space $\mathbb{R}^n$ over $\mathbb{R}$, and not a linear subspace.

Definition 2.2: Given the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $A, B \geq 0$, we say that $x_0 \in \mathbb{R}^n_+$ is positive state null controllable in finite time if there exists a control sequence that steers the state x of (A, B) from x_0 to 0 in N steps and additionally maintains nonnegativity of x. The collection of all such $x_0 \in \mathbb{R}^n_+$ is called the positive state null controllable set (in finite time). We say that (A, B) is positive state null controllable in finite time if this set is $\mathbb{R}^n_+$.

As the underlying system (1) is linear, the natural notion of positive state controllability is (more or less) the combination of positive state reachability and positive state null controllability, and is addressed in [8].

B. Results

Theorem 2.6 provides a recipe (under a certain assumption) for describing the positive state reachable set and set of positive state null controllable states by relating nonnegative state trajectories with possibly nonpositive inputs of the pair (A, B) to nonnegative state trajectories with nonnegative inputs of a related system. Our key assumption is the following:

(A) Given the pair $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $A, B \geq 0$ there exists $F \in \mathbb{R}^{m\times n}$ such that with $\tilde{A} := A - BF$ both $\tilde{A} \geq 0$ and if $v \in \mathbb{R}^n_+, w \in \mathbb{R}^m$ satisfy $\tilde{A}v + Bw \geq 0$ then $w \geq 0$.

The idea of assumption **(A)** is that, roughly speaking, the quantity that can be removed from the state $x(t)$ by the input $u(t)$, while keeping the state $x(t + 1)$ nonnegative, is proportional to $x(t)$ itself; "you cannot take away what is not there". Such a notion is one of state feedback, and note is *not* the case for positive input control. There adding a nonnegative quantity to something already nonnegative can only make things larger. By decomposing A into $\tilde{A} + BF$, then negative controls u in $Ax + Bu$ can be absorbed as $\tilde{A}x + B(Fx + u)$. Lemma 2.3 provides a constructive characterisation of assumption **(A)** and demonstrates that **(A)** holds if, and only if, **(A)** holds for a specified F that can be computed.

Lemma 2.3: Assumption **(A)** holds for $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ with $A, B \geq 0$ if, and only if, there exist m rows of B such that the $m \times m$ submatrix, denoted $\underline{B}$, formed by taking these m rows from B is a positive monomial matrix and

$$A - B\underline{B}^{-1}\underline{A} \geq 0\,. \tag{3}$$

Here $\underline{A}$ is formed of the m rows of A that appear in $\underline{B}$. Consequently, **(A)** holds if, and only if, it holds with $F = \underline{B}^{-1}\underline{A}$ so that $\tilde{A} := A - BF \geq 0$.

Remark 2.4: (i) We comment here that **(A)** holds for *any* $A \geq 0$ in the single input case $B = b = c_i e_i$, $c_i > 0$ and the corresponding multiple input version case when B is a combination of e_i, that is, $B = [c_{i_1}e_{i_1}, \ldots, c_{i_m}e_{i_m}]$ for positive c_{i_k}. These two cases are arguably the most important for applications.

(ii) Lemma 2.3 provides an algorithm for checking assumption **(A)**. First, we see that B containing an $m\times m$ monomial submatrix is necessary for **(A)**. Second, there are then only finitely many $\underline{B}$ (formed from the monomial rows of B) to check whether **(A)** holds by verifying whether $A - B\underline{B}^{-1}\underline{A} \geq 0$.

The following corollary interprets Lemma 2.3 in the single input case.

Corollary 2.5: Let $A \geq 0$ with i^{th} row denoted by r_i and $B = b$ be given by

$$b = \sum_{k=1}^{n} c_{i_k} e_{i_k} \quad \text{with} \quad c_{i_k} > 0\,.$$

Assumption **(A)** holds for (A, b) if, and only if, there exists $i_k \in \{i_1, \ldots, i_n\}$ such that

$$r_{i_j} - \frac{c_{i_j} r_{i_k}}{c_{i_k}} \geq 0, \quad \forall\, i_j \in \{i_1, \ldots, i_n\}\,, \tag{4}$$

and in this case $F = f^T = \frac{r_{i_k}}{c_{i_k}}$, where i_k is as in (4).

Theorem 2.6: Let the pair (A, B) satisfy **(A)** and denote $\tilde{A} := A - BF$. The state trajectories of (A, B) from initial state $x_0 \in \mathbb{R}^n_+$ with nonnegative state are precisely the state trajectories of $(\tilde{A}, B)$ from initial state $x_0 \in \mathbb{R}^n_+$ with nonnegative control.

With the above characterisation we obtain the following corollaries by appealing to existing positive input results, for instance [6, pp. 41–42, Proposition 1, Proposition 2].

Corollary 2.7: Let the pair (A, B) satisfy **(A)**. The positive state reachable set of the pair (A, B) in finite time is precisely

$$\bigcup_{N\in\mathbb{N}} \langle B, \tilde{A}B, \ldots, \tilde{A}^{N-1}B\rangle_+ \,, \tag{5}$$

where $\tilde{A}$ is as in **(A)**. The positive state reachable set of the pair (A, B) in infinite time is precisely

$$\overline{\bigcup_{N\in\mathbb{N}} \langle B, \tilde{A}B, \ldots, \tilde{A}^{N-1}B\rangle_+} \,. \tag{6}$$

Remark 2.8: (i) For standard reachability of single-input systems (A, B) the Cayley-Hamilton Theorem implies that every reachable state is reachable in at most n steps, where n is the dimension of A. It is known ([6, p. 42]) that this is not the case for positive systems, and by Corollary 2.7 we see that the same is true for positive state reachability. Example 4.2 contains a pair (A, B) that is positive state reachable, but only in infinite time.

(ii) Clearly, classical positive input reachability of the nonnegative pair (A, B) pair implies positive state reachability of (A, B). This is apparent from Corollary 2.7 as for each $N \in \mathbb{N}$

$$\langle B, AB, \ldots, A^{N-1}B\rangle_+ \subseteq \langle B, \tilde{A}B, \ldots, \tilde{A}^{N-1}B\rangle_+ \,.$$

Example 4.5 contains a pair (A, B) where the above inclusion is strict.

Corollary 2.9: Let the pair (A, B) satisfy **(A)**. The positive state null controllable set of the pair (A, B) in finite time is precisely

$$\mathbb{R}^n_+ \cap \ker \tilde{A}^n. \tag{7}$$

The set of positive null controllable states in infinite time is precisely

$$\mathbb{R}^n_+ \cap (\ker \tilde{A}^n + E(\tilde{A})), \tag{8}$$

where $E(\tilde{A})$ is the sum of (generalised) eigenspaces corresponding to the stable eigenvalues of $\tilde{A}$ (that is, the eigenvalues λ of $\tilde{A}$ with $|\lambda| < 1$).

Remark 2.10: It is important to note that given $A, B \geq 0$, when $A = \tilde{A} + BF$ with $\tilde{A}, F \geq 0$ then, even without assumption **(A)**, every state trajectory of $(\tilde{A}, B)$ with nonnegative control *is* a state trajectory of (A, B) with nonnegative state (this is one of the implications in Theorem 2.6). Consequently, the positive input reachable set of the pair $(\tilde{A}, B)$ is a *subset* of the positive state reachable set of the pair (A, B). Although not giving the complete picture of positive state control, this connection can be used as an intermediate stage between positive state control and positive input control and is sometimes sufficient to fully describe positive state control, as illustrated in Example 4.3.

III. Positive outputs with positive state

Here we consider the outputs y of the input-state-output system (2) specified by $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{m\times n}$, with $A, B, C \geq 0$. Assuming that $r(A) < 1$ (that is, A is Schur) then it is well-known that

$$\lim_{t\to\infty} y(t) = G(1) \lim_{t\to\infty} u(t)\,,$$

where G is the transfer function of the triple (A, B, C). If $G(1)$ is invertible then for the output to aymptotically track a prescribed reference r then obviously $u \equiv G^{-1}(1)r$ is a suitable input (or limit of an input). Since it is not clear that $G^{-1}(1)r \geq 0$ for general $r \in \mathbb{R}^m_+$, we thus seek to describe which nonnegative references r can be tracked asymptotically whilst preserving nonnegativity of the state? These results are motivated by a desire to describe PI control for MIMO positive state systems, with suggested applications in population management [15]. a manuscript containing proofs of these results is in preparation [16]. We introduce some notation.

Definition 3.1: For $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$, we say that $r \in \mathbb{R}^p$ is *trackable* if there exists a convergent input u such that the output y of (2) has limit r. Supposing further that $A, B, C \geq 0$, we say that $r \in \mathbb{R}^p_+$ is *trackable with positive state* if r is trackable and moreover the state $x(t)$ of (2) is componentwise nonnegative for every $t \in \mathbb{N}_0$. We call the set of such r the set of trackable outputs of (A, B, C) with positive state.

Lemma 3.2: Suppose that $(A, B, C) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m} \times \mathbb{R}^{m\times n}$ with $A, B, C \geq 0$ and $r(A) < 1$. Then for each $F \in \mathbb{R}^{n\times m}_+$ such that $\tilde{A} := A - BF \geq 0$, it follows that $G_{C\tilde{A}B}(1) = C(I - \tilde{A})^{-1}B \geq 0$ and the set of trackable outputs of (A, B, C) with positive state contains $\langle G_{C\tilde{A}B}(1)\rangle_+$.

When the (A, B) component of (2) satisfy **(A)** then we can say more.

Lemma 3.3: Using the notation and assumptions of Lemma 3.2, if additionally (A, B) satisfy assumption **(A)** then the set of trackable outputs of (A, B, C) with positive state is precisely equal to $\langle G_{C\tilde{A}B}(1)\rangle_+$.

The next result provides a recipe for enlarging the guaranteed set of possible trackable outputs with positive state, particularly in the case that assumption **(A)** fails.

Lemma 3.4: Using the notation and assumptions of Lemma 3.2, for each $F \in \mathbb{R}^{n\times m}$, $F \geq 0$ such that $\tilde{A} := A - BF \geq 0$ it follows that $\langle G_{CAB}(1)\rangle_+ \subseteq \langle G_{C\tilde{A}B}(1)\rangle_+$.

Remark 3.5: (i) Lemma 3.3 demonstrates that, under assumption **(A)**, the largest possible set for tracking with positive state is $\langle G_{C\tilde{A}B}(1)\rangle_+$, where $\tilde{A}$ is as in assumption **(A)**.

(ii) A straightforward adjustment to the proof of Lemma 3.4 demonstrates that the sets $\langle G_{CAB}(1)\rangle_+$ have a monotonically decreasing nested structure with respect to the partial ordering of componentwise nonnegativity on A, in that

$$0 \le A_1 \le A_2 \;\Rightarrow\; \langle G_{CA_2B}(1)\rangle_+ \subseteq \langle G_{CA_1B}(1)\rangle_+ \,.$$

Therefore, the largest possible trackable set with positive state over all nonnegative A is equal to $\langle CB\rangle_+$ and is attained when $A = 0$.

IV. Examples

Example 4.1: Checking assumption **(A)***:* Consider the systems

$$(a)\quad A_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$(b)\quad A_2 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 1 & 1 & 3 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$(c)\quad A_3 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Assumption **(A)** holds in (a), fails in (b) and holds in (c). We proceed to prove these claims. For (a), if we take $f_1^T := \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}$ so that

$$\tilde{A}_1 := A_1 - bf_1^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 4 \\ 0 & 0 & 0 \end{bmatrix}, \tag{9}$$

then whenever $v \in \mathbb{R}^n_+, w \in \mathbb{R}$ are such that $\tilde{A}_1 v + bw \ge 0$, by inspection of the third component we see that $w \ge 0$, which by definition is **(A)**. Alternatively, with $i_1 = 1$ and $i_2 = 3$ we compute

$$r_1 - r_3 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \ge 0 \,.$$

Thus Corollary 2.5 applies with $i_k = i_2 = 3$, so that $f_1 = \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$, as in (9). However, repeating this process in (b) gives

$$r_1 - r_3 = \begin{bmatrix} 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \end{bmatrix} \not\ge 0 \,,$$

and also that $r_3 - r_1 \not\ge 0$. We conclude from Corollary 2.5 that **(A)** does not hold for (b). For (c) we note that B contains two 2×2 positive monomial submatrices

$$\underline{B}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \underline{B}_2 = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

formed from rows one and two, and rows two and three of B respectively. Taking the corresponding submatrices from A_3 gives

$$\underline{A}_1 = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \end{bmatrix} \quad \text{and} \quad \underline{A}_2 = \begin{bmatrix} 0 & 3 & 4 \\ 1 & 1 & 2 \end{bmatrix},$$

from which we compute

$$A_3 - B\underline{B}_1^{-1}\underline{A}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix} \ge 0 \,,$$

$$\text{and} \quad A_3 - B\underline{B}_2^{-1}\underline{A}_2 = \begin{bmatrix} 0 & -1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \not\ge 0 \,.$$

We conclude that **(A)** holds for (c) with $F = \left[\begin{smallmatrix} 2 & 1 & 2 \\ 0 & 3 & 4 \end{smallmatrix}\right]$.

Example 4.2: Reachability in infinite time: Consider the 3×3 nonnegative matrix and control vector

$$A = \begin{bmatrix} a_1 & 1 & 0 \\ 0 & a_1 & 1 \\ 0 & 0 & a_2 \end{bmatrix}, \quad b = e_3 \,,$$

with $1 > a_1 > 0$ and $a_2 \ge 0$. By Corollary 2.5 it follows that **(A)** applies to the pair (A, b) with $F = f^T$ the third row of A. A calculation shows that $\tilde{A}b = e_2$ and for $k \ge 2$

$$\tilde{A}^k b = \begin{bmatrix} (k-1)a_1^{k-2} & a_1^{k-1} & 0 \end{bmatrix}^T . \tag{10}$$

The positive state reachable set in $k+1$ steps is all nonnegative linear combinations of these vectors, which here is strictly increasing with increasing k and note *does not* include e_1 for finite k. However, by noting that for $k \ge 2$

$$\frac{1}{(k-1)a_1^{k-2}} \cdot \tilde{A}^k b = \begin{bmatrix} 1 & \frac{a_1}{k-2} & 0 \end{bmatrix}^T \to e_1 \,,$$

as $k \to \infty$, it follows that the pair (A, b) is positive state reachable in infinite time.

Example 4.3: Positive state reachability without **(A)***:* Consider the pair $A = \left[\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ with

$$\langle B, AB\rangle_+ = \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 3 & 2 \\ 3 & 1 \end{bmatrix} \right\rangle_+ . \tag{11}$$

An induction argument shows that the positive input reachable space is $\langle e_1, e_1 + e_2\rangle_+$; the area R in Figure 1. The

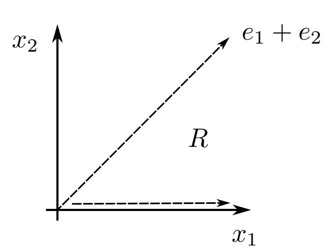


Fig. 1. Positive input reachable set (depicted R) of the pair (A, B) in (11).

pair (A, B) in this example do not satisfy assumption **(A)** (as B contains no 2×2 monomial submatrices). However, taking $F = B$ gives

$$\tilde{A} := A - BF = A - B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \geq 0 \,,$$

and so the positive state reachable set contains

$$\langle B, \tilde{A}B \rangle_+ = \left\langle \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\rangle_+ = \mathbb{R}^2_+ \,.$$

We conclude that, although assumption **(A)** does not hold, the pair (A, B) is positive state reachable in finite time. Note that in this instance the choice $F = B$ is in no sense unique; the same conclusions are reached for the pair (A, B) with $F = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$ as here $\tilde{A} = A - BF = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 2 \end{smallmatrix}\right] \geq 0$.

Notwithstanding the above, it is true that if $F_1 \geq F_2$ then $\tilde{A}_1 := A - BF_1 \leq A - BF_2 =: \tilde{A}_2$ and thus for each $k \in \mathbb{N}$

$$\langle B, \tilde{A}_2 B, \ldots, \tilde{A}_2^k B \rangle_+ \subseteq \langle B, \tilde{A}_1 B, \ldots, \tilde{A}_1^k B \rangle_+ \,.$$

Consequently, to describe positive state control for a pair (A, B) when assumption **(A)** fails, the above suggests considering $\tilde{A} := A - BF$, where F is chosen as (componentwise) large as possible so that $\tilde{A} \geq 0$. Such a process can simplify calculations considerably. For example, consider the pair $A = \left[\begin{smallmatrix} 2 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 4 \end{smallmatrix}\right]$ and $B = \left[\begin{smallmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ with

$$\langle B, AB, A^2 B \rangle_+ = \left\langle \begin{bmatrix} 1 & 1 & 4 & 2 & 19 & 6 \\ 1 & 0 & 5 & 1 & 25 & 6 \\ 1 & 0 & 6 & 1 & 33 & 7 \end{bmatrix} \right\rangle_+ \,. \tag{12}$$

Choosing $F = B^T$ gives $\tilde{A} := A - BF = \left[\begin{smallmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{smallmatrix}\right] \geq 0$ so that

$$\langle B, \tilde{A}B, \tilde{A}^2 B \rangle_+ = \left\langle \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 1 & 0 & 3 & 0 & 9 & 0 \end{bmatrix} \right\rangle_+ \,. \tag{13}$$

We see by inspection of columns two and three in the matrices of the right hand sides of (12) and (13) that in this example the difference between positive state control and positive input control is that in the former the directions e_1 and $2e_2 + 3e_3$ have been ‘decoupled’.

Example 4.4: Positive state null controllability: From Corollary 2.9 it follows that for a pair (A, B) satisfying **(A)**, e_i is positive state null controllable in k steps if, and only if, $\tilde{A}^k$ has i^{th} column zero.

Example 4.5: Positive state controllability for Leslie matrices: We recall that an $n \times n$ Leslie [17] matrix has the following structure

$$A = \begin{bmatrix} f_1 & f_2 & \cdots & \cdots & f_n \\ s_1 & 0 & \cdots & & 0 \\ 0 & s_2 & 0 & & \vdots \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & 0 & s_{n-1} & 0 \end{bmatrix} \,, \tag{14}$$

which models a population partitioned into discrete, increasing age-stages. Correspondingly, $f_i \geq 0$ denote reproductive rates and $s_i \geq 0$ denote survival rates, the latter as proportions are each no greater than one. For ecologically meaningful models [18], we shall always assume that the Leslie matrix (14) has $s_1, \ldots, s_{n-1} > 0$, $f_1, \ldots, f_n \geq 0$ and there exists at least one $i \in \{1, 2, \ldots, n\}$ such that $f_i > 0$. Noting that the $n \times n$ positive diagonal matrix

$$T = \text{diag}\left(1, \frac{1}{s_1}, \frac{1}{s_1 s_2}, \ldots, \frac{1}{s_1 \ldots s_{n-1}}\right) \,,$$

has $T^{-1} \geq 0$ for single input positive state controllability with $b = e_i$ it is sufficient to consider the similarity transformed pair $(T^{-1}AT, T^{-1}b)$. This is because $T^{-1}AT$ has the same structure as A with ones on the subdiagonal and top row with entries $\hat{f}_i > 0$ (which we abuse notation and write as f_i) and $T^{-1}e_i = c_i e_i$, for some $c_i > 0$. Consequently, when considering controllability with positive state there is no loss of generality in assuming that a Leslie matrix has $s_j = 1$ for each j.

We consider the (transformed) pair (A, b) and single input $b = e_j$, $j \in \{1, 2, \ldots, n\}$. Assumption **(A)** is always satisfied for such a pair, with $F = f^T$ the j^{th} row of A.

When $j = 1$ so that $b = e_1$ it follows that $\tilde{A}$ is nilpotent with $\tilde{A}^n = 0$. The positive state reachable set is therefore

$$\langle b, \tilde{A}b, \ldots \tilde{A}^{n-1}b \rangle_+ = \langle e_1, e_2, \ldots, e_n \rangle_+ = \mathbb{R}^n_+ \,,$$

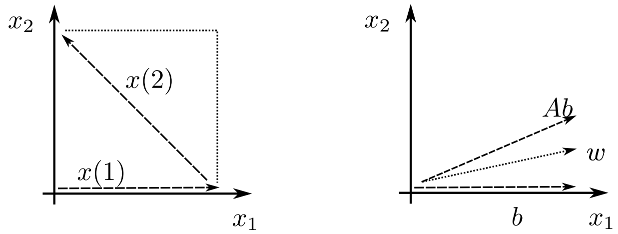
and by Corollary 2.7 the pair (A, b) is positive state reachable. Furthermore,

$$\tilde{A}^n = 0, \quad \Rightarrow \quad \mathbb{R}^n_+ \cap \ker \tilde{A}^n = \mathbb{R}^n_+ \,,$$

and so the pair (A, b) is positive state null controllable. In this very special case it follows that positive state controllability and standard controllability coincide: the unique control u that steers x between any two nonnegative states in n steps is such that the state remains in $\mathbb{R}^n_+$. Such a control can take negative values and thus is not permitted in a positive input framework.

For example, consider the pair $A = \left[\begin{smallmatrix} 2 & 2 \\ 1 & 0 \end{smallmatrix}\right]$, $b = e_1$. Trivially e_1 is reachable from zero with positive state in one step and the control $u(0) = 1$, $u(1) = -2$ steers the state from zero to e_2 in two steps. The resulting state trajectories are plotted in Figure 2(a). By taking suitable linear combinations of these inputs all of $\mathbb{R}^2_+$ can be reached with nonnegative state. If we restrict attention to (A, b) with only positive inputs, then the positive input reachable space is spanned by $b = e_1$ and $Ab = 2e_1 + e_2$, and is depicted in Figure 2(b): note that not all of $\mathbb{R}^2_+$ is reachable. Even in this very simple example there is a difference between positive state controllability and classical positive input controllability.

For $b = e_j$, $j > 1$ the situation is somewhat different. Assumption **(A)** holds with $F = f^T$ the j^{th} row of A and



(a) (b)

Fig. 2. (a) Positive state trajectories steering the state of (A, b) to e_1 and e_2 respectively. (b) The positive input reachable space of the positive system (A, b) is the area between the dashed vectors b and $Ab = 2e_1 + e_2$. The dotted line w is parallel to $\lim_{k\to\infty} A^k b$ and so here the positive input reachable set is not all of $\mathbb{R}^2_+$ but is attained in finite time.

we note that the characteristic polynomial of $\tilde{A}$ is given by

$$t^n - \sum_{k=1}^{j-1} f_k t^{n-k} ,$$

which follows easily from, for example, the expression on [5, p. 121]. Consequently, by [8, Lemma 2.12] the positive state reachable set of the pair (A, b) is achieved in finite time; indeed, in n steps. However, the pair (A, b) is not, in general, positive state reachable. In the case $n = 3$ and $b = e_2$ the positive state reachable set in finite time is contained in

$$\langle b, \tilde{A}b, \tilde{A}^2 b\rangle_+ = \left\langle \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} f_2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} f_3 + f_1 f_2 \\ 0 \\ 0 \end{bmatrix} \right\rangle_+ .$$

If $f_2 > 0$ then e_3 cannot be steered to whilst maintaining nonnegative state (in finite time), but the vectors e_1 and e_2 can (in the former case provided that $f_1 f_2 + f_3 > 0$). For positive state null controllability we see that

$$\tilde{A}^2 = \begin{bmatrix} f_1^2 & f_1 f_2 + f_3 & f_1 f_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{A}^3 = f_1 \tilde{A}_1^2 .$$

The important term here is f_1, reproduction of individuals in the first stage class. If $f_1 = 0$ then $\tilde{A}^3 = 0$ and so every state can be steered to zero with nonnegative state. However, if $f_1, f_2, f_3 > 0$ then the top row of $\tilde{A}^3$ is positive and thus there are no nontrivial positive state null controllable states in finite time!

The above conclusions are biologically sensible. When state $i > 1$ is controlled then the structure of A means that it is not possible to remove individuals from the earlier stage classes. Consequently the later stage classes e_j, $j > i$ can be steered to (these terms appear in $\tilde{A}^j b$), but also with a contribution from stages one to $i - 1$. When $i = 1$ then it follows that all stages are positive state reachable.

What is most remarkable in the $b = e_1$ case is that for Leslie matrices, the pair (A, b) is in fact fully positive state controllable; that the state can transition between any two nonnegative states (including zero) whilst remaining in $\mathbb{R}^n_+$! Leslie matrices of course have a very simple structure, but already the results presented here demonstrate the differences that arise between positive state control and positive input control.

Example 4.6: Positive outputs with positive state: Consider the input-state-ouput system (2) specified by the triple:

$$A = \begin{bmatrix} 0.7 & 1.2 \\ 0.2 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 1 & 0 \\ 0 & 0.25 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \quad (15)$$

Note that A is a Leslie matrix with $r(A) = 0.9521 < 1$ and that in the context of a population model, the first and second observations are total abundance and abundance of the second stage class, respectively. Intuitively, when the state x is nonnegative then $y_1(t) \geq y_2(t)$ for each $t \in \mathbb{N}$. By Lemma 3.2, the set of trackable outputs of (A, B, C) with positive state contains

$$\langle G(1)\rangle_+ = \left\langle \begin{bmatrix} 20 & \frac{25}{4} \\ \frac{10}{3} & \frac{5}{4} \end{bmatrix} \right\rangle_+ = \left\langle \begin{bmatrix} 48 & 15 \\ 8 & 3 \end{bmatrix} \right\rangle_+ , \quad (16)$$

which is the darker region graphed in Figure 3. However, we note that (A, B) satisfy assumption **(A)** with $F = \left[\begin{smallmatrix} 0.7 & 1.2 \\ 0.8 & 0 \end{smallmatrix}\right]$, so that $\tilde{A} := A - BF = 0$. Thus, by Corollary 3.3, it follows that the set of trackable outputs of (A, B, C) with positive state is equal to

$$\langle G_{C\tilde{A}B}(1)\rangle_+ = \langle CB\rangle_+ = \left\langle \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & \frac{1}{4} \end{bmatrix} \right\rangle_+ ,$$

which is the lighter region sketched in Figure 3, and is much 'larger', than that in (16). Indeed, we could not expect the region to be any larger, as then the inequality $y_1(t) \geq y_2(t)$ would be violated.

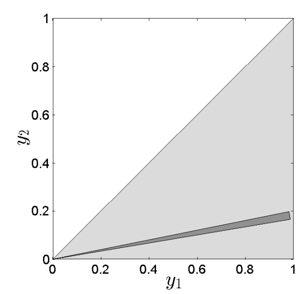


Fig. 3. Sets of trackable outputs of (A, B, C) with positive state where (A, B, C) are given by (15). The darker region is $\langle G(1)\rangle_+$, but underestimates the set of possible outputs. The lighter region is exactly the set of trackable outputs of (A, B, C) with positive state.

Example 4.7: Discrete time matrix models for the invasive weed *Cirsium vulgare* (spear thistle) in Nebraska, USA, are considered in [19], and also [20, Section 3.1]. Here time-steps correspond to years and a four stage model is used with states one to four corresponding to the seed bank, small plants, medium plants and large plants respectively (see [19]). The nominal uncontrolled system has A given by

$$A = \begin{bmatrix} 0 & 0 & \tilde{f}_1 & \tilde{f}_2 \\ s_1 & 0 & \tilde{f}_3 & \tilde{f}_4 \\ 0 & s_2 & s_3 & 0 \\ 0 & s_4 & s_5 & s_6 \end{bmatrix}, \qquad (17)$$

with

$$\begin{aligned} &s_1 = 0.0077, \quad s_2 = 0.12, \quad s_3 = 0.11\,, \quad s_4 = 0.02, \\ &s_5 = 0.27, \quad s_6 = 0.17, \quad \tilde{f}_1 = 93.1, \quad \tilde{f}_2 = 423\,, \\ &\tilde{f}_3 = 6.74, \quad \tilde{f}_4 = 30.6\,. \end{aligned} \qquad (18)$$

As with Leslie matrices, the s_i denote survival and growth parameters and the $\tilde{f}_i$ are reproductive values. We note that as $s_4 > 0$, small plants can grow into large plants in one year. Furthermore, $\tilde{f}_3, \tilde{f}_4 > 0$ means that in a given year both medium and large plants can produce seeds that germinate and grow into small plants (in addition to seeds that germinate the following year).

The uncontrolled population is unstable as the spectral radius of A is $r(A) = 1.57 > 1$. We first seek to reduce the weed population by using an additive management strategy, so that the system is of the form (1). As a management strategy we add or remove large plants so that $B = b = e_4$. When this action is performed (shortly) *before* the census or measurement (so-called pre census) then the resulting model is well described by (1). Here assumption **(A)** holds with $F = f^T$ the fourth row of A so that $\tilde{A} := A - bf^T$ is given by

$$\tilde{A} = \begin{bmatrix} 0 & 0 & \tilde{f}_1 & \tilde{f}_2 \\ s_1 & 0 & \tilde{f}_3 & \tilde{f}_4 \\ 0 & s_2 & s_3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

As $\tilde{A}^k$ has no zero columns for any $k \in \mathbb{N}$ we see that no state is null controllable in finite time. Here $r(\tilde{A}) = 1.0024 > 1$ and although $\tilde{A}$ is not primitive (or even irreducible), $r(\tilde{A})$ is a simple eigenvalue and the following limit holds

$$\lim_{k\to\infty} \frac{\tilde{A}^k}{(r(\tilde{A}))^k} x_0 = \frac{v^T x_0}{v^T w} w\,, \qquad (19)$$

where v^T and w are left and right eigenvectors of $\tilde{A}$ corresponding to $r(\tilde{A})$ respectively (which are both positive once positively scaled and satisfy $v^T w \neq 0$). When $x_0 \geq 0$ and $x_0 \neq 0$ the right hand side of (19) is positive and hence there are no non-trivial states that are positive state null controllable in infinite time. Equivalently, the negative control $u(t) = -f^T x(t)$ *does not* stabilise any nonzero initial population. Furthermore, the characterisation from Theorem 2.6 shows that this system *cannot* be stabilised by positive state control.

If instead the control action is in fact performed (shortly) *after* the census or measurement (so called post census), then a more accurate model is

$$x(t+1) = A(x(t) + bu(t)) = Ax(t) + Abu(t), \quad t \in \mathbb{N}_0\,,$$

and so we replace $b = e_4$ by $Ab = \begin{bmatrix} \tilde{f}_2 & \tilde{f}_4 & 0 & s_6 \end{bmatrix}^T$. Corollary 2.5 can be applied to check whether assumption **(A)** holds for the pair (A, Ab). Of rows one, two and four (the nonzero rows of Ab) the only possible candidate 'smallest' row of A (in the sense of (4)) is the first (as rows two and four have nonzero entries that are zero in the first row). A straightforward calculation shows that **(A)** holds with $F = f^T = [\, 0 \;0\; \tilde{f}_1/\tilde{f}_2 \;1 \,]$ if, and only if,

$$\tilde{f}_1 \tilde{f}_4 \leq \tilde{f}_2 \tilde{f}_3 \quad \text{and} \quad \tilde{f}_1 s_6 \leq \tilde{f}_2 s_5\,. \qquad (20)$$

Both of these conditions are satisfied for the parameters in (18). Therefore, if $F = f^T = [\, 0 \;0\; \tilde{f}_1/\tilde{f}_2 \;1 \,]$ then

$$\tilde{A} := A - Abf^T = \begin{bmatrix} 0 & 0 & 0 & 0 \\ s_1 & 0 & \tilde{f}_3 - \frac{\tilde{f}_1 \tilde{f}_4}{\tilde{f}_2} & 0 \\ 0 & s_2 & s_3 & 0 \\ 0 & s_4 & s_5 - \frac{\tilde{f}_1 s_6}{\tilde{f}_2} & 0 \end{bmatrix} \geq 0\,,$$

and hence positive state control for the pair (A, Ab) is precisely positive input control for the pair $(\tilde{A}, Ab)$. As the fourth column of $\tilde{A}$ is zero, we have that $x = e_4$ is null controllable (in finite time) and as $r(\tilde{A}) = 0.1153 < 1$, every state is positive state null controllable in infinite time. The above observations suggest that when control actions act on large weeds, organising these actions to take place post census is preferable to pre census. This is not biologically surprising because, loosely speaking, the fourth stage class is the most reproductive and the post census control strategy limits to a greater extent reproduction in this stage class.

Finally, suppose that the state of (1) is unknown and instead only access to some output y is available for making management decisions and we seek to regulate this output to some chosen reference. We use the low-gain PI controller

$$\left.\begin{aligned} x(t+1) &= Ax(t) + Bu(t)\,,\; x(0) = x^0 \\ y(t) &= Cx(t)\,, \\ x_c(t+1) &= x_c(t) + gK(r - y(t)) \\ &\quad - E(x_c(t) - \mathrm{sat}\, x_c(t))\,,\; x_c(0) = x_c^0 \\ u(t) &= -ky(t) + \mathrm{sat}\,(x_c(t))\,, \end{aligned}\right\} \qquad (21)$$

for $t \in \mathbb{N}_0$, where $E, K \in \mathbb{C}^{m\times m}$ and $k, g > 0$ are matrix and scalar gains respectively and sat is the diagonal saturation nonlinearity, saturating at both zero and some chosen upper limits U_i. It is well-known that PI control of MIMO systems subject to a saturating input can suffer from so-called activator saturation or integrator windup [21] and the term E in (21) is a static anti-windup component. Anti-windup controllers are well studied and we refer the reader to [22] for an overview. The system (21) is considered in [16] where it is proven that under the following assumptions

$$\tilde{A}_k := A - kBC \geq 0,\ r(\tilde{A}_k) < 1\,,\ \sigma(KG_{C\tilde{A}_kB}(1)) \subseteq \mathbb{C}_0^+\,,$$

($\mathbb{C}_0^+$ the open right half complex plane) and choice of

$$E = gKG(1)\,,$$

then there exists $g^* > 0$ such that for all $g \in (0, g^*)$, all $r \in \langle G_{C\tilde{A}_kB}(1)\rangle_+$ with $r \leq G_{C\tilde{A}_kB}(1)U$ and all $(x^0, x_c^0) \in \mathbb{R}_+^n \times \mathbb{R}_+^m$, the output y of (21) converges to r and moreover the state x remains nonnegative.

As a demonstration of the theory, we apply the above result to A given by (17)–(18), with B and C given by

$$B = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{22}$$

and $g = 0.1, k = 0.11$, $K = G^{-1}_{C\tilde{A}_kB}(1)$, seeking to control an initial population distribution with $\|x^0\|_1 = 10^5$ to

$$r = \begin{bmatrix} 209.1902 \\ 5.5994 \end{bmatrix} = G_{C\tilde{A}_kB}(1) \begin{bmatrix} 0.25 \\ 0.5 \end{bmatrix}.$$

The results are plotted in Figure 4.

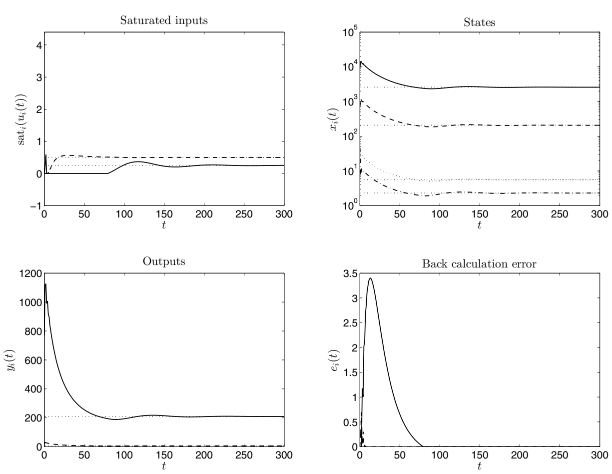


Fig. 4. Low-gain PI control (21) applied to the weed model (17), (18) and (22). The outputs converge to the chosen reference r whilst maintaining nonnegativity of the state. The backtracking error is $e = \mathrm{sat}\,(x_c) - x_c$. Note the semi-log scale on the state plot.

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