21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Disturbance decoupling for continuous piecewise linear bimodal systems

A.R.F. Everts and M.K. Camlibel

Abstract— **In this paper we tackle the disturbance decoupling problem for continuous bimodal piecewise linear systems. After establishing necessary and sufficient geometric conditions for such a system to be disturbance decoupled, we study state feedback and dynamic feedback controllers, both mode-dependent and mode-independent. For these feedback schemes, we provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. Also, we provide subspace algorithms in order to verify the presented conditions.**

I. Introduction

The disturbance decoupling problem amounts to eliminating, by means of feedback, the effect of the disturbance from the output of a given input/state/output dynamical system. Apart from its practical relevance, the study of the disturbance decoupling problem had a quite strong impact on feedback control theory by paving the road to the development of geometric control theory [1], [2], [11]. Both in the context of linear and (smooth) nonlinear systems, geometric control theory constitutes a rich collection of powerful tools employed in solving numerous control problems (see e.g. [3], [5], [6], [9], [10]).

In the context of hybrid dynamical systems, the results on disturbance decoupling are limited to state-independent linear switching systems [7], [8], [12], [13]. In this paper, we focus on a particular class of hybrid systems exhibiting state-dependent switchings, namely piecewise linear bimodal systems. The main goal of the paper is to provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem for this class of systems.

The main difference, in the context of disturbance decoupling, between the state-independent and state-dependent switchings stems from the different nature of the set of reachable states by the disturbances for these two cases. In the case of linear state-independent switching systems, the set of states that can be reached from the origin by the disturbances constitute a subspace of the whole state space. In [7], [12], this leads to the solution of the disturbance decoupling problem by following the footsteps of the classical results for the linear systems. However, the same set of states is, in general, neither a subspace nor even a convex set for the case of state-dependent switchings. As such, the ideas employed in the context of linear state-independent switching systems cannot be indiscriminately applied to linear state-dependent switching systems.

This research was funded by NWO (The Netherlands Organisation for Scientific Research) project CODAVI (613.001.108).

Both authors are with the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, 9747 AG Groningen, The Netherlands `a.r.f.everts@rug.nl`, `m.k.camlibel@rug.nl`

To overcome this obstacle, we first investigate under which conditions a given bimodal system is disturbance decoupled. It turns out that one can still provide easily verifiable geometric necessary and sufficient conditions for disturbance decoupling (see Theorem 1), even though the set of reachable states does not, in general, enjoy nice geometric properties such as being convex. Based on these geometric necessary and sufficient conditions, we study the disturbance decoupling problem for both state feedback controllers and dynamic feedback controllers. For both feedback schemes, we consider mode-independent and mode-dependent controllers, and provide necessary and sufficient conditions for the solvability of the disturbance decoupling problem. These conditions amount to checking certain subspace inclusions very much analogous to linear systems and linear state-independent switching systems. To verify these conditions, we also propose subspace algorithms.

The organization of the paper is as follows. Section II quickly reviews basic notation and notions of geometric control theory of linear systems. In Section III, we introduce the class of continuous piecewise linear bimodal systems as well as the disturbance decoupling problem for this class of systems. This is followed by a complete characterization of the disturbance decoupled (open-loop) bimodal systems. Based on this characterization, we first turn our attention to the disturbance decoupling problem by state feedback in Section IV. Subsequently, we discuss the disturbance decoupling problem by dynamic feedback in Section V. In order to verify the conditions presented in these sections, we provide subspace algorithms in Section VI. Finally, the paper closes with conclusions in Section VII.

II. Preliminaries and notation

We quickly recall some notational conventions as well as terminology of the geometric approach to linear systems (see e.g. [9] for more details).

Consider matrices $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times p}$ and $C \in \mathbb{R}^{q\times m}$. With $\langle A \mid \operatorname{im} B\rangle$ we denote the subspace $\operatorname{im} B + \operatorname{im} AB + \cdots + \operatorname{im} A^{m-1}B$, that is the smallest A-invariant subspace containing $\operatorname{im} B$.

A subspace $\mathcal{V} \subseteq \mathbb{R}^m$ is called controlled invariant with respect to A and B, or (A,B)-invariant in short, if there exists a matrix F such that $(A+BF)\mathcal{V} \subseteq \mathcal{V}$, or equivalently, if $A\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. The subspace $\mathcal{V}^*(\ker C, A, B)$ is defined to be the largest (A,B)-invariant subspace contained in $\ker C$.

A subspace $\mathcal{S} \subseteq \mathbb{R}^m$ is called conditioned invariant with respect to C and A, or (C,A)-invariant in short, if there exist a matrix G such that $(A + GC)\mathcal{S} \subseteq \mathcal{S}$, or equivalently, if

$A(\mathcal{S} \cap \ker C) \subseteq \mathcal{S}$. The subspace $\mathcal{S}^*(\operatorname{im} B, C, A)$ is defined to be the smallest (C, A)-invariant subspace containing $\operatorname{im} B$.

A pair of subspaces $(\mathcal{S}, \mathcal{V})$ is called a (C, A, B)-pair if $\mathcal{S}$ is (C, A)-invariant, $\mathcal{V}$ is (A, B)-invariant and $\mathcal{S} \subseteq \mathcal{V}$. If $(\mathcal{S}, \mathcal{V})$ is a (C, A, B)-pair, then there is a linear map $N : \mathbb{R}^q \to \mathbb{R}^p$ such that $(A + BNC)\mathcal{S} \subseteq \mathcal{V}$ (see e.g. [9, Lemma 6.3]).

III. Disturbance decoupling

In this paper, we consider bimodal systems of the form

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Ed(t) & \text{if } c^T x(t) \leqslant 0, \\ A_2 x(t) + Ed(t) & \text{if } c^T x(t) \geqslant 0, \end{cases} \tag{1a}$$

$$z(t) = Hx(t), \tag{1b}$$

where $x \in \mathbb{R}^{\mathrm{x}}$ is the state, $d \in \mathbb{R}^{\mathrm{d}}$ is the unknown disturbance, $z \in \mathbb{R}^{\mathrm{z}}$ is the output, and the matrices A_1, A_2, E, H and the vector c are of appropriate sizes. Throughout the paper we assume that the right-hand side of (1a) is continuous in x. In other words, the implication

$$c^T x = 0 \quad \Rightarrow \quad A_1 x = A_2 x$$

holds. Equivalently, we have

$$A_1 - A_2 = hc^T \tag{2}$$

for a vector $h \in \mathbb{R}^{\mathrm{x}}$. As such, the right-hand side of (1a) is Lipschitz continuous in the x variable. Therefore, for each x_0 and locally integrable disturbance d there exists a unique absolutely continuous function $x^{x_0,d}(t)$ satisfying $x(0) = x_0$ and (1a) for almost all t. We denote the corresponding output by $z^{x_0,d}(t)$.

We say that the system (1) is *disturbance decoupled* if

$$z^{x_0,d_1}(t) = z^{x_0,d_2}(t)$$

for all $x_0 \in \mathbb{R}^{\mathrm{x}}$, locally integrable functions d_1 and d_2, and $t \geqslant 0$.

For $x_0 \in \mathbb{R}^{\mathrm{x}}$ and $T \geqslant 0$, define the set

$$R(x_0, T) := \{\, x^{x_0,d}(T) \mid d \text{ is locally integrable} \,\}.$$

It follows immediately that system (1) is disturbance decoupled if and only if for every $x_0 \in \mathbb{R}^{\mathrm{x}}$ and $T \geqslant 0$ the difference between any two elements in $R(x_0, T)$ is in the kernel of H. Equivalently,

$$\bigcup_{T \geqslant 0} \bigcup_{x_0 \in \mathbb{R}^{\mathrm{x}}} \big(R(x_0, T) - R(x_0, T)\big) \subseteq \ker H. \tag{3}$$

Neither the set $R(x_0, T)$ nor $R(x_0, T) - R(x_0, T)$ is necessarily convex in general. As such, condition (3) is rather hard to check. However, the following theorem provides equivalent geometric conditions that are easier to verify.

Theorem 1 *The system (1) is disturbance decoupled if and only if*

$$\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle \subseteq \ker H. \tag{4}$$

In order to give a proof of Theorem 1, we need the following three auxiliary lemmas.

Lemma 2 *Let A_1 and A_2 be two square matrices such that $A_1 - A_2 = hc^T$. Then $c^T(sI - A_1)^{-1}E$ is identically zero if and only if so is $c^T(sI - A_2)^{-1}E$.*

Proof. We use the well-known identity

$$(sI - A_1)^{-1} - (sI - A_2)^{-1} = (sI - A_1)^{-1}(A_1 - A_2)(sI - A_2)^{-1}.$$

By pre-multiplying both sides by c^T and post-multiplying by E and using $A_1 - A_2 = hc^T$ we get

$$c^T(sI - A_1)^{-1}E - c^T(sI - A_2)^{-1}E = c^T(sI - A_1)^{-1}hc^T(sI - A_2)^{-1}E.$$

Hence, if $c^T(sI - A_2)^{-1}E$ is identically zero, then so is $c^T(sI - A_1)^{-1}E$. By symmetry, the converse also holds. ■

Lemma 3 *The subspace $\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$ is the smallest subspace containing $\operatorname{im} E$ that is invariant under both A_1 and A_2. Furthermore, if $c^T(sI - A_1)^{-1}E$ is not identically zero, then*

$$\langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle = \langle A_1 \mid \operatorname{im} \begin{bmatrix} h & E \end{bmatrix} \rangle. \tag{5}$$

Proof. Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E \rangle + \langle A_2 \mid \operatorname{im} E \rangle$ and $\mathcal{U} = \langle A_1 \mid \operatorname{im} \begin{bmatrix} h & E \end{bmatrix} \rangle$. The subspace $\mathcal{U}$ contains $\operatorname{im} h$ and is invariant under A_1, hence it is also invariant under $A_2 = A_1 - hc^T$. Since $\mathcal{U}$ contains $\operatorname{im} E$, we have $\langle A_i \mid \operatorname{im} E \rangle \subseteq \mathcal{U}$ for $i = 1, 2$. Hence, the inclusion $\mathcal{V} \subseteq \mathcal{U}$ follows.

Suppose that $c^T(sI - A_1)^{-1}E$ is not identically zero, and let p be the smallest non-negative integer such that $c^T A_1^p E \neq 0$. For any element $y \in \mathcal{V}^\perp$ we have

$$y^T A_1^k E = y^T A_2^k E = 0, \quad \forall k \geqslant 0.$$

In particular, we obtain

$$0 = y^T A_2^{p+1} E = y^T (A_1 - hc^T)^{p+1} E = y^T hc^T A_1^p E,$$

by using $c^T A_1^k E = 0$ for $0 \leqslant k \leqslant p - 1$. Since $c^T A_1^p E$ is nonzero, this implies that $y^T h = 0$. Hence, we get $h \in (\mathcal{V}^\perp)^\perp = \mathcal{V}$. Then, we have $A_1(v_1 + v_2) = A_1 v_1 + A_2 v_2 + hc^T v_2 \in \mathcal{V}$ for all $v_1 \in \langle A_1 \mid \operatorname{im} E \rangle$ and $v_2 \in \langle A_2 \mid \operatorname{im} E \rangle$. As such, $\mathcal{V}$ is A_1-invariant. Furthermore, we have $\mathcal{U} \subseteq \mathcal{V}$ since $\mathcal{V}$ contains both $\operatorname{im} h$ and $\operatorname{im} E$. Hence, (5) holds. Since $\mathcal{U}$ is invariant under both A_1 and A_2, so is the subspace $\mathcal{V}$.

In the case that $c^T(sI - A_1)^{-1}E$ is identically zero, we have $c^T A_1^k E = 0$ for all integers $k \geqslant 0$. We claim that $A_1^k E = A_2^k E$ for all $k \geqslant 0$. To prove this claim, we employ induction on k. It clearly holds for $k = 0$. Suppose that $A_1^k E = A_2^k E$ holds for $k = 0, 1, \ldots, \ell$, then

$$A_1^{\ell+1} E = A_1 A_1^\ell E = (A_2 + hc^T) A_1^\ell E = A_2 A_1^\ell E = A_2^{\ell+1} E.$$

Hence, we have $\langle A_1 \mid \operatorname{im} E \rangle = \langle A_2 \mid \operatorname{im} E \rangle$. Consequently, also in this case $\mathcal{V}$ is invariant under both A_1 and A_2.

Since any subspace that contains $\operatorname{im} E$ and is invariant under A_1 and A_2 must contain both $\langle A_1 \mid \operatorname{im} E \rangle$ and $\langle A_2 \mid \operatorname{im} E \rangle$, we see that $\mathcal{V}$ is the smallest of such subspaces. ■

Lemma 4 *If $c^T(sI - A_1)^{-1}E$ is identically zero, then for all $x_0 \in \mathbb{R}^n$ and integrable disturbances d_1 and d_2 we have $c^T x^{x_0,d_1}(t) = c^T x^{x_0,d_2}(t)$ for all $t \geqslant 0$.*

Proof. Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle$. It follows from Lemma 2 that $c^T A_1^k E = c^T A_2^k E = 0$ for $k \geqslant 0$. Hence, we get $\mathcal{V} \subseteq \ker c^T$. By Lemma 3, $\mathcal{V}$ is invariant under both A_1 and A_2. Let $v_1, v_2, \dots, v_\ell$ be a basis for $\mathcal{V}$, and extend this to a basis $v_1, v_2, \dots, v_{\mathrm{x}}$ for $\mathbb{R}^{\mathrm{x}}$. Let $S = \begin{bmatrix} v_1 & v_2 & \dots & v_{\mathrm{x}}\end{bmatrix}$, then the basis transformation $\xi = S^{-1}x$ results in the system

$$\dot{\xi}(t) = \begin{cases} \tilde{A}_1\xi + \tilde{E}d & \text{if } \tilde{c}^T\xi \leqslant 0,\\ \tilde{A}_2\xi + \tilde{E}d & \text{if } \tilde{c}^T\xi \geqslant 0.\end{cases}$$

Write $\xi = (\xi_1^T, \xi_2^T)^T$, where ξ_1 contains the first ℓ entries of ξ. Since $\operatorname{im} E \subseteq \mathcal{V}$ and $\mathcal{V} \subseteq \ker c^T$, we see that the matrices $\tilde{A}_1$, $\tilde{A}_2$, $\tilde{E}$ and $\tilde{c}^T$ are of the form

$$\tilde{A}_1 = \begin{pmatrix} A_{11}^1 & A_{12}^1 \\ 0 & A_{22}^1\end{pmatrix}, \qquad \tilde{A}_2 = \begin{pmatrix} A_{11}^2 & A_{12}^2 \\ 0 & A_{22}^2\end{pmatrix}$$

$$\tilde{E} = \begin{pmatrix} E_1 \\ 0\end{pmatrix}, \qquad \tilde{c}^T = \begin{pmatrix} 0 & c_2^T\end{pmatrix}.$$

In particular, ξ_2 satisfies

$$\dot{\xi}_2 = \begin{cases} A_{22}^1\xi_2 & \text{if } c_2^T\xi_2 \leqslant 0\\ A_{22}^2\xi_2 & \text{if } c_2^T\xi_2 \geqslant 0.\end{cases}$$

Note that ξ_2 does not depend on the disturbance d. Therefore, the value of $c^T x = \tilde{c}^T\xi = c_2^T\xi_2$ does not depend on the disturbance, thus we can conclude that $c^T x^{x_0,d_1}(t) = c^T x^{x_0,d_2}(t)$ for all $t \geqslant 0$, initial conditions x_0 and integrable disturbances d_1, d_2. ■

Now, we are in a position to give a proof of Theorem 1.

Proof of Theorem 1. *Necessity:* Suppose that the system (1) is disturbance decoupled. Let x_0 be such that $c^T x_0 < 0$ and take $d_1(t) = d \in \mathbb{R}^{\mathrm{d}}$ a constant vector, and $d_2(t) = 0$. Let $x_i(t) = x^{x_0,d_i}(t)$ for $i = 1, 2$ denote the state trajectories of the system (1) corresponding to the initial state x_0 and disturbances d_i, and let $z_i(t) = Hx_i(t)$. Since x_1 and x_2 are continuous, there exists an $\varepsilon > 0$ such that $c^T x_i(t) < 0$ for all $t \in (0, \varepsilon)$ and $i = 1, 2$. This means that for $t \in (0, \varepsilon)$ we have

$$\dot{x}_i(t) = A_1 x_i(t) + Ed_i(t), \qquad i = 1, 2.$$

Since the system (1) is disturbance decoupled, the outputs satisfy $z_1(t) = z_2(t)$ for $t \geqslant 0$. Therefore, we have

$$Hx_1(t) = Hx_2(t), \qquad t \geqslant 0.$$

By differentiating both sides k times, we get

$$HA_1^k x_1(t) + HA_1^{k-1}Ed = HA_1^k x_2(t), \quad t \in (0, \varepsilon), \quad k \geqslant 1.$$

Taking the limit as $t \downarrow 0$ and using $x_1(0) = x_2(0)$ gives us

$$HA_1^k Ed = 0, \qquad k \geqslant 0.$$

Since this must hold for all vectors $d \in \mathbb{R}^q$, we can conclude that $HA_1^k E = 0$ for all $k \geqslant 0$. By choosing x_0 such that $c^T x_0 > 0$ and employing similar arguments, we obtain $HA_2^k E = 0$ for all $k \geqslant 0$. Consequently, (4) holds.

Sufficiency: Let $\mathcal{V} = \langle A_1 \mid \operatorname{im} E\rangle + \langle A_2 \mid \operatorname{im} E\rangle$. In view of (3), it suffices to show that $R(x_0, T) - R(x_0, T) \subseteq \mathcal{V}$, or equivalently $\mathcal{V}^\perp \subseteq (R(x_0, T) - R(x_0, T))^\perp$ for all x_0 and $T \geqslant 0$.

Let x_0 be an initial state and d_1, d_2 two disturbances. Also, let $x_i(t) = x^{x_0,d_i}(t)$ for $i = 1, 2$ denote the two corresponding trajectories of the system (1). Let y be an element of $\mathcal{V}^\perp = \langle A_1 \mid \operatorname{im} E\rangle^\perp \cap \langle A_2 \mid \operatorname{im} E\rangle^\perp$. Then $y^T A_1^k E = 0$ and $y^T A_2^k E = 0$ for all $k \geqslant 0$. In the case that $c^T(sI - A_1)^{-1}E$ is not identically zero, we know from Lemma 3 that $\operatorname{im} h \subseteq \mathcal{V}$. As such, we have $y^T h = 0$. Together with $y^T E = 0$, this yields

$$\begin{aligned} y^T \dot{x}_i(t) &= \begin{cases} y^T A_1 x_i(t) & \text{if } c^T x_i(t) \leqslant 0\\ y^T A_2 x_i(t) & \text{if } c^T x_i(t) \geqslant 0\end{cases}\\ &= y^T A_1 x_i(t),\end{aligned}$$

for $t \geqslant 0$ and $i = 1, 2$. In the case that $c^T(sI - A_1)^{-1}E$ is identically zero, it follows from Lemma 4 that $c^T x_1(t) = c^T x_2(t)$ for all $t \geqslant 0$. Hence, we have $hc^T(x_1(t) - x_2(t)) = 0$, which implies that

$$\begin{aligned} y^T(\dot{x}_1(t) - \dot{x}_2(t)) &= \begin{cases} y^T A_1(x_1(t) - x_2(t)), & c^T x_1(t) \leqslant 0\\ y^T A_2(x_1(t) - x_2(t)), & c^T x_1(t) \geqslant 0\end{cases}\\ &= y^T A_1(x_1(t) - x_2(t)),\end{aligned}$$

for $t \geqslant 0$. In conclusion, in both cases we have

$$y^T(\dot{x}_1(t) - \dot{x}_2(t)) = y^T A_1(x_1(t) - x_2(t)), \tag{6}$$

for all $y \in \mathcal{V}^\perp$ and for almost all $t \geqslant 0$.

Let λ be an eigenvalue of A_1^T. Suppose that $y \in \mathcal{V}^\perp$ satisfies $(A_1^T - \lambda I)^k y = 0$ for some integer $k \geqslant 1$. Let $y_1, y_2, \dots, y_k$ be a Jordan chain for the eigenvalue λ satisfying

$$y_j = (A_1^T - \lambda I)^{k-j} y \quad \text{for} \quad 1 \leqslant j \leqslant k.$$

Since $y_k = y \in \mathcal{V}^\perp$ and $\mathcal{V}^\perp$ is A_1^T-invariant, we get $y_j \in \mathcal{V}^\perp$ for all $j = 1, 2, \dots, k$. We will prove by induction on j that

$$y_j^T(x_1(t) - x_2(t)) = 0 \tag{7}$$

for all $j = 1, 2, \dots, k$ and all $t \geqslant 0$. For $j = 1$, we have $A_1^T y_1 = \lambda y_1$. Hence, it follow from (6) that

$$\frac{d}{dt}[y_1^T(x_1(t) - x_2(t))] = \lambda y_1^T(x_1(t) - x_2(t)),$$

for almost all $t \geqslant 0$. This results in

$$y_1^T(x_1(t) - x_2(t)) = e^{\lambda t} y_1^T(x_1(0) - x_2(0)) = 0.$$

Now, assume that (7) holds for $j = 1, 2, \dots, \ell$ for some integer ℓ with $1 \leqslant \ell < k$. By using (6) and $A_1^T y_{\ell+1} = \lambda y_{\ell+1} + y_\ell$, we obtain

$$\begin{aligned} \frac{d}{dt}[y_{\ell+1}^T(x_1(t) - x_2(t))] &= y_{\ell+1}^T A_1(x_1(t) - x_2(t))\\ &= (\lambda y_{\ell+1} + y_\ell)^T(x_1(t) - x_2(t))\\ &= \lambda y_{\ell+1}^T(x_1(t) - x_2(t)),\end{aligned}$$

for almost all $t \geqslant 0$. Therefore, we have

$$y_{\ell+1}^T(x_1(t) - x_2(t)) = e^{\lambda t} y_{\ell+1}^T(x_1(0) - x_2(0)) = 0.$$

This completes the proof of (7). Clearly, (7) implies that $y_j \in (R(x_0,T) - R(x_0,T))^\perp$ for all j, x_0 and $T \geqslant 0$.

Let $\mathcal{M}$ be the subspace $\mathcal{M} = \mathcal{V}^\perp \oplus i\mathcal{V}^\perp \subseteq \mathbb{C}^n$. Consider A_1^T as a linear map from $\mathbb{C}^n$ to $\mathbb{C}^n$. Since $\mathcal{V}^\perp$ is A_1^T-invariant, so is $\mathcal{M}$. Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the distinct eigenvalues of A_1^T and define the corresponding root subspaces R_{λ_i} for $i = 1, 2, \ldots, r$ as

$$\mathcal{R}_{\lambda_i}(A_1^T) := \ker(A_1^T - \lambda_i I)^{p_i},$$

where p_i is the geometric multiplicity of the eigenvalue λ_i. By [4, Thm. 2.1.5], we can decompose $\mathcal{M}$ as follows:

$$\mathcal{M} = \bigoplus_{i=1}^{r} (\mathcal{M} \cap \mathcal{R}_{\lambda_i}(A_1^T)).$$

For fixed x_0 and $T \geqslant 0$, we can consider $R(x_0,T) - R(x_0,T)$ as a subset of $\mathbb{C}^n$, it now follows from the preceding argument on Jordan chains that $\mathcal{M} \subseteq (R(x_0,T) - R(x_0,T))^\perp$. Hence, $\mathcal{V}^\perp \subseteq (R(x_0,T) - R(x_0,T))^\perp$ for all x_0 and $T \geqslant 0$. ■

For later use in the next two sections, and to relate our result to similar results for switched linear systems, we state the following corollary, which follows from combining Theorem 1 with Lemma 3.

Corollary 5 *The system (1) is disturbance decoupled if and only if there exists a subspace $\mathcal{V} \subseteq \mathbb{R}^{\mathrm{x}}$ that is invariant under both A_1 and A_2 such that* $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$.

Remark 6 In [12], the disturbance decoupling problem for switched linear systems is studied. The results presented in [12, Thm. 3.7 and 3.9] provide sufficient conditions for the disturbance decoupling of piecewise linear systems. Applied to the bimodal system (1), these conditions boil down to the conditions in Corollary 5, but with the extra condition that the subspace $\mathcal{V}$ and the matrices A_1 and A_2 should satisfy $\operatorname{im}(A_1 - A_2) = \operatorname{im} hc^T \subseteq \mathcal{V}$. This condition implies that $h \in \mathcal{V}$ which is not necessary in the case that $c^T(sI - A_1)^{-1}E$ is identically zero.

IV. Disturbance decoupling by state feedback

The next question we will address is under what conditions a bimodal system can be rendered disturbance decoupled by means of static state feedback. To do so, we consider the bimodal system

$$\dot{x}(t) = \begin{cases} A_1 x(t) + Bu(t) + Ed(t) & \text{if } c^T x(t) \leqslant 0 \\ A_2 x(t) + Bu(t) + Ed(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{8a}$$

$$z(t) = Hx(t) \tag{8b}$$

where $x \in \mathbb{R}^{\mathrm{x}}$ is the state, $u \in \mathbb{R}^{\mathrm{u}}$ is the input, $z \in \mathbb{R}^{\mathrm{z}}$ is the output, $d \in \mathbb{R}^{\mathrm{d}}$ is the disturbance, and the matrices A_1, A_2, B, E, H and the vector c are of suitable sizes. We assume that A_1 and A_2 satisfy $A_1 - A_2 = hc^T$ for a vector $h \in \mathbb{R}^{\mathrm{x}}$ and that B has full column rank.

In this section we will provide necessary and sufficient conditions for the existence of a static state feedback law that renders the closed-loop system disturbance decoupled. We consider two forms of static feedback: mode-dependent and mode-independent.

A. Mode-dependent state feedback

Consider a mode-dependent static feedback law of the form

$$u(t) = \begin{cases} F_1 x(t) & \text{if } c^T x(t) \leqslant 0 \\ F_2 x(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{9}$$

for two matrices $F_1, F_2 \in \mathbb{R}^{\mathrm{u}\times\mathrm{x}}$ with the property that $c^T x = 0$ implies $F_1 x = F_2 x$. Equivalently, $\ker c^T \subseteq \ker(F_1 - F_2)$. This implies that there exists $f \in \mathbb{R}^{\mathrm{u}}$ such that $F_1 - F_2 = fc^T$. In other words, we consider mode-dependent and continuous (in x) state feedback. Clearly, such a feedback results in the (continuous) closed-loop system

$$\dot{x}(t) = \begin{cases} (A_1 + BF_1)x(t) + Ed(t) & \text{if } c^T x(t) \leqslant 0 \\ (A_2 + BF_2)x(t) + Ed(t) & \text{if } c^T x(t) \geqslant 0 \end{cases} \tag{10a}$$

$$z(t) = Hx(t). \tag{10b}$$

In view of Corollary 5, we see that the closed-loop system (10) is disturbance decoupled if and only if we can find a subspace $\mathcal{V}$ and matrices F_1 and F_2 such that $\mathcal{V}$ is invariant under both $A_1 + BF_1$ and $A_2 + BF_2$, $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$ and $\ker c^T \subseteq \ker(F_1 - F_2)$.

In order to check whether such a subspace exists or not, we need to introduce some nomenclature. Define the set of subspaces

$$V_{\mathrm{md}}(H, \{A_1, A_2\}, B) := \{\, \mathcal{V} \subseteq \ker H \mid \exists F_1, F_2 \text{ s.t. } (A_j + BF_j)\mathcal{V} \subseteq \mathcal{V},\ j = 1, 2 \,\}. \tag{11}$$

Let $\mathcal{V}$ and $\mathcal{W}$ be two subspace in $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Then $\mathcal{V}$ and $\mathcal{W}$ are both (A_1, B)-invariant, hence $\mathcal{V} + \mathcal{W}$ is (A_1, B)-invariant as well. Similarly, we see that $\mathcal{V} + \mathcal{W}$ is (A_2, B)-invariant too. Therefore, $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ is closed under subspace addition. Let $\mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$ be the largest of the subspaces in $V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. If the context is clear, we will denote it by $\mathcal{V}^*_{\mathrm{md}}$.

Note that we dropped the continuity condition in (11). However, for any subspace $\mathcal{V}$ in $V_{\mathrm{md}}(\ker H, \{A_1, A_2\}, B)$, we can find matrices F_1 and F_2 such that the feedback (9) is continuous in x and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$, as shown in the following lemma.

Lemma 7 *If $\mathcal{V}$ is (A_1, B)-invariant and (A_2, B)-invariant, and $A_1 - A_2 = hc^T$, then there exist matrices $F_1, F_2 \in \mathbb{R}^{\mathrm{u}\times\mathrm{x}}$ such that $F_1 - F_2 = fc^T$ for some $f \in \mathbb{R}^{\mathrm{u}}$ and $(A_i + BF_i)\mathcal{V} \subseteq \mathcal{V}$ for $i = 1, 2$.*

Proof. Since $\mathcal{V}$ is (A_1, B)-invariant, there exists a matrix F_1 such that $(A_1 + BF_1)\mathcal{V} \subseteq \mathcal{V}$. Since $\mathcal{V}$ is (A_2, B)-invariant as well, it is also (hc^T, B)-invariant, so $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

This implies that $h \in \mathcal{V} + \operatorname{im} B$ or $\mathcal{V} \subseteq \ker c^T$. In the former case, there exists an $f \in \mathbb{R}^{\mathrm{u}}$ such that $h + Bf \in \mathcal{V}$. In the latter case, let f be any vector in $\mathbb{R}^{\mathrm{u}}$. In both cases $(h+Bf)c^T\mathcal{V} \subseteq \mathcal{V}$. Let $F_2 = F_1 - fc^T$, then $A_2 + BF_2 = A_1 + BF_1 - (h+Bf)c^T$, and hence $(A_2+BF_2)\mathcal{V} \subseteq \mathcal{V}$. ■

The following theorem shows how we can use the subspace $\mathcal{V}^*_{\text{md}}(H, \{A_1, A_2\}, B)$ to determine whether there exists a mode-dependent state feedback controller that renders the system (8) disturbance decoupled.

Theorem 8 *There exists a mode-dependent static state feedback of the form* (9) *that renders the closed-loop system (10) disturbance decoupled if and only if*

$$\operatorname{im} E \subseteq \mathcal{V}^*_{\text{md}}(H, \{A_1, A_2\}, B).$$

Proof. *Sufficiency:* Since $\mathcal{V}^*_{\text{md}}$ is (A_1, B)-invariant and (A_2, B)-invariant, by Lemma 7 there exist F_1 and F_2 such that $F_1 - F_2 = fc^T$ for some $f \in \mathbb{R}^{\mathrm{u}}$ and $(A_i + BF_i)\mathcal{V}^*_{\text{md}} \subseteq \mathcal{V}^*_{\text{md}}$ for $i = 1, 2$. From the hypothesis, we have $\operatorname{im} E \subseteq \mathcal{V}^*_{\text{md}} \subseteq \ker H$. Then, it follows from Corollary 5 that mode-dependent static feedback given by (9) renders the closed-loop system (10) disturbance decoupled.

Necessity: Suppose that F_1 and F_2 are such that the input (9) renders the closed-loop system (10) disturbance decoupled. It follows from Corollary 5 that there exists a subspace $\mathcal{V}$ that is invariant under both $A_1 + BF_1$ and $A_2 + BF_2$, and such that $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$. Therefore, $\mathcal{V} \in V_{\text{md}}(H, \{A_1, A_2\}, B)$. Hence, $\operatorname{im} E \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\text{md}}$. ■

In Section VI we will provide an algorithm to compute $\mathcal{V}^*_{\text{md}}(H, \{A_1, A_2\}, B)$. Once the condition $\operatorname{im} E \subseteq \mathcal{V}^*_{\text{md}}(H, \{A_1, A_2\}, B)$ is satisfied, one can construct the feedback matrices F_1 and F_2 by following the steps in the proof of Lemma 7.

B. Mode-independent state feedback

Consider the static state feedback law $u = Fx$ for a matrix $F \in \mathbb{R}^{\mathrm{u}\times\mathrm{x}}$. This can be seen as a special case of the mode-dependent state feedback, with $F_1 = F_2$. Such a feedback law results in the closed-loop system

$$\dot{x}(t) = \begin{cases} (A_1 + BF)x(t) + Ed(t) & \text{if } c^T x(t) \leqslant 0 \\ (A_2 + BF)x(t) + Ed(t) & \text{if } c^T x(t) \geqslant 0. \end{cases} \tag{12}$$

By Corollary 5, we see that the closed-loop system is disturbance decoupled if and only if we can find a subspace $\mathcal{V}$ and a feedback matrix F such that $\mathcal{V}$ is invariant under both $A_1 + BF$ and $A_2 + BF$, and $\operatorname{im} E \subseteq \mathcal{V} \subseteq \ker H$. Similar to the mode-dependent case, we define

$$V_{\text{mi}}(H, \{A_1, A_2\}, B) := \{\, \mathcal{V} \subseteq \ker H \mid \exists F \text{ s.t. } (A_j + BF)\mathcal{V} \subseteq \mathcal{V} \text{ for } j = 1, 2 \,\}. \tag{13}$$

A subspace $\mathcal{V}$ is an element of $V_{\text{mi}}(\ker H, \{A_1, A_2\}, B)$ if and only if $\mathcal{V} \subseteq \ker H$, $\mathcal{V}$ is (A_1, B)-invariant and invariant under hc^T. Hence, the set of subspaces $V_{\text{mi}}(H, \{A_1, A_2\}, B)$ is closed under subspace addition, and we let $\mathcal{V}^*_{\text{mi}}(H, \{A_1, A_2\}, B)$ denote its largest subspace. If the context is clear, we will denote it by $\mathcal{V}^*_{\text{mi}}$. In Section VI we will provide an algorithm to compute $\mathcal{V}^*_{\text{mi}}$.

The following theorem can be proven by using similar arguments employed in the proof of Theorem 8.

Theorem 9 *There exists a matrix $F \in \mathbb{R}^{\mathrm{u}\times\mathrm{x}}$ such that the state feedback $u(t) = Fx(t)$ renders the closed-loop system (8) disturbance decoupled if and only if*

$$\operatorname{im} E \subseteq \mathcal{V}^*_{\text{mi}}(H, \{A_1, A_2\}, B).$$

V. Disturbance decoupling by dynamic feedback

In this section, we address the disturbance decoupling problem by dynamic feedback. Consider the bimodal system (8) together with the output

$$y(t) = Cx(t), \tag{14}$$

where $y \in \mathbb{R}^{\mathrm{y}}$. The main goal of this section is to investigate under which conditions there exists a dynamic controller from y to u rendering the closed-loop system disturbance decoupled. Similar to the state feedback problem, we distinguish two cases: mode-dependent and mode-independent controllers.

A. Mode-dependent dynamic feedback

Consider the mode-dependent dynamic feedback controller given by

$$\dot{w}(t) = \begin{cases} Kw(t) + L_1 y(t) & \text{if } c^T x \leqslant 0 \\ Kw(t) + L_2 y(t) & \text{if } c^T x \geqslant 0 \end{cases} \tag{15a}$$

$$u(t) = \begin{cases} Mw(t) + N_1 y(t) & \text{if } c^T x \leqslant 0 \\ Mw(t) + N_2 y(t) & \text{if } c^T x \geqslant 0 \end{cases} \tag{15b}$$

where $w \in \mathbb{R}^{\mathrm{w}}$, $u \in \mathbb{R}^{\mathrm{u}}$, $y \in \mathbb{R}^{\mathrm{y}}$, and the matrices K, L_1, L_2, M, N_1 and N_2 are of suitable sizes. Interconnecting this controller with the system given by (8) and (14) results in the closed-loop system

$$\begin{pmatrix} \dot{x}(t) \\ \dot{w}(t) \end{pmatrix} = \begin{cases} A_{\text{e},1} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + E_{\text{e}} d(t) & \text{if } c_{\text{e}}^T \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \leqslant 0 \\ A_{\text{e},2} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + E_{\text{e}} d(t) & \text{if } c_{\text{e}}^T \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \geqslant 0 \end{cases} \tag{16a}$$

$$z(t) = H_{\text{e}} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \tag{16b}$$

where

$$A_{\text{e},i} = \begin{pmatrix} A_i + BN_iC & BM \\ L_iC & K \end{pmatrix}, \qquad i = 1, 2, \tag{16c}$$

$$E_{\text{e}} = \begin{pmatrix} E \\ 0 \end{pmatrix}, \quad H_{\text{e}} = \begin{pmatrix} H & 0 \end{pmatrix}, \quad c_{\text{e}}^T = \begin{pmatrix} c^T & 0 \end{pmatrix}. \tag{16d}$$

We only consider mode-dependent feedback controllers that render the closed-loop system continuous (both in x and w).

This amounts to imposing the following conditions on the matrices L_1, L_2, N_1 and N_2:

$$\ker c^T \subseteq \ker(L_1 - L_2)C, \quad \ker c^T \subseteq \ker(N_1 - N_2)C. \quad (17)$$

Hence, there are vectors $\ell \in \mathbb{R}^{\mathtt{w}}$ and $n \in \mathbb{R}^{\mathtt{u}}$ such that

$$(L_1 - L_2)C = \ell c^T, \quad (N_1 - N_2)C = nc^T \qquad (18)$$

In this way, we have $\ker c_{\mathrm{e}}^T \subseteq \ker(A_{\mathrm{e},1} - A_{\mathrm{e},2})$.

The objective of this section is to find such a mode-dependent dynamic controller that renders the closed-loop system disturbance decoupled. By employing (C, A_1, B)-pairs, the following theorem provides necessary and sufficient conditions for the existence of such a controller.

Theorem 10 *There exists a mode-dependent dynamic controller of the form* (15) *satisfying the continuity condition* (17) *such that the closed-loop system* (16) *is disturbance decoupled if and only if there exist subspaces $\mathcal{S}$ and $\mathcal{V}$ such that $(\mathcal{S}, \mathcal{V})$ is a (C, A_1, B)-pair,* $\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H$ *and* $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

Proof. *Necessity:* Assume that there exists such a controller given by K, L_1, L_2, M, N_1, N_2. Let $\mathbb{R}^{\mathtt{w}}$ denote the state space of the controller. The (extended) state space of the interconnected system is then given by $\mathbb{R}^{\mathtt{x}} \times \mathbb{R}^{\mathtt{w}}$. By Corollary 5, there exists a subspace $\mathcal{V}_{\mathrm{e}} \subseteq \mathbb{R}^{\mathtt{x}} \times \mathbb{R}^{\mathtt{w}}$ that is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$, satisfying $\operatorname{im} E_{\mathrm{e}} \subseteq \mathcal{V}_{\mathrm{e}} \subseteq \ker H_{\mathrm{e}}$. For this subspace $\mathcal{V}_{\mathrm{e}}$, we define the following two subspaces of $\mathbb{R}^{\mathtt{x}}$:

$$p(\mathcal{V}_{\mathrm{e}}) := \{x \in \mathbb{R}^{\mathtt{x}} \mid \exists w \in \mathbb{R}^{\mathtt{w}} \text{ such that } \begin{pmatrix} x \\ w \end{pmatrix} \in \mathcal{V}_{\mathrm{e}}\}, \quad (19a)$$

$$i(\mathcal{V}_{\mathrm{e}}) := \{x \in \mathbb{R}^{\mathtt{x}} \mid \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathcal{V}_{\mathrm{e}}\}, \quad (19b)$$

which can be seen as the projection of $\mathcal{V}_{\mathrm{e}}$ on $\mathbb{R}^{\mathtt{x}}$ and the intersection of $\mathcal{V}_{\mathrm{e}}$ and $\mathbb{R}^{\mathtt{x}} \times \{0\}$ respectively. Let $\mathcal{S} = i(\mathcal{V}_{\mathrm{e}})$ and $\mathcal{V} = p(\mathcal{V}_{\mathrm{e}})$. Since $\mathcal{V}_{\mathrm{e}}$ is $A_{\mathrm{e},1}$-invariant, $(\mathcal{S}, \mathcal{V})$ is a (C, A_1, B)-pair (see e.g. [9, Theorem 6.2]).

For any $x \in \operatorname{im} E$, we have that $(x^T, 0^T)^T \in \operatorname{im} E_{\mathrm{e}} \subseteq \mathcal{V}_{\mathrm{e}}$. Therefore, we get $x \in i(\mathcal{V}_{\mathrm{e}}) = \mathcal{S}$, hence we have $\operatorname{im} E \subseteq \mathcal{S}$. For $x \in \mathcal{V} = p(\mathcal{V}_{\mathrm{e}})$, there exists a $w \in \mathbb{R}^{\mathtt{x}}$ such that $(x^T, w^T)^T \in \mathcal{V}_{\mathrm{e}} \subseteq \ker H_{\mathrm{e}}$. Then, we get $Hx = H_{\mathrm{e}}(x^T, w^T)^T = 0$ and hence $\mathcal{V} \subseteq \ker H$.

Since L_1, L_2, N_1 and N_2 satisfy (17), there are vectors ℓ and n such that (18) holds. Consequently, we have

$$A_{\mathrm{e},1} - A_{\mathrm{e},2} = \begin{pmatrix} (h + Bn)c^T & 0 \\ \ell c^T & 0 \end{pmatrix}. \qquad (20)$$

Let $x \in \mathcal{V}$. Then, a $w \in \mathbb{R}^w$ such that $(x^T, w^T)^T \in \mathcal{V}_{\mathrm{e}}$. Since $\mathcal{V}_{\mathrm{e}}$ is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$, we have

$$(A_{\mathrm{e},1} - A_{\mathrm{e},2}) \begin{pmatrix} x \\ w \end{pmatrix} = \begin{pmatrix} (h + Bn)c^T x \\ \ell c^T x \end{pmatrix} \in \mathcal{V}_{\mathrm{e}}.$$

Consequently, we obtain $(h + Bn)c^T x \in \mathcal{V}$ and hence $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$.

Sufficiency: Let $(\mathcal{S}, \mathcal{V})$ be a such a (C, A_1, B)-pair. Then there exist F and G such that

$$(A_1 + BF)\mathcal{V} \subseteq \mathcal{V}, \qquad (A_1 + GC)\mathcal{S} \subseteq \mathcal{S}.$$

Furthermore, we can find (see e.g. [9, Lemma 6.3]) a linear mapping N_1 such that

$$(A_1 + BN_1C)\mathcal{S} \subseteq \mathcal{V}.$$

Since $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$, we have $\mathcal{V} \subseteq \ker c^T$ or $h \in \mathcal{V} + \operatorname{im} B$. If the latter holds, then there exists an $n \in \mathbb{R}^{\mathtt{u}}$ such that $h + Bn \in \mathcal{V}$. Then choose $\ell \in \mathcal{V}$ such that $h + Bn - \ell \in \mathcal{S}$. If we have $\mathcal{V} \subseteq \ker c^T$, then we can choose $n \in \mathbb{R}^{\mathtt{u}}$ and $\ell \in \mathcal{V}$ arbitrarily. In both cases, we can find n and ℓ such that $(h + Bn - \ell)c^T\mathcal{V} \subseteq \mathcal{S}$.

Let $L_1 = BN_1 - G$ and define

$$K = A_1 + BF + GC - BN_1C, \qquad L_2 = L_1 - \ell c^T,$$
$$M = F - N_1C, \qquad N_2 = N_1 - nc^T.$$

and let K, L_1, L_2, M, N_1, N_2 define a controller of the form (15), with $\mathtt{w} = \mathtt{x}$. Note that $(L_1 - L_2)C = \ell c^T$ and $(N_1 - N_2)C = nc^T$, so L_1, L_2, N_1 and N_2 satisfy the continuity condition (17). The system matrices of the corresponding closed-loop system (16) are then given by

$$A_{\mathrm{e},i} = \begin{pmatrix} A_i + BN_iC & B(F - N_1C) \\ L_iC & A_1 + BF + GC - BN_1C \end{pmatrix},$$

for $i = 1, 2$.

Let $\mathcal{V}_{\mathrm{e}}$ be the subspace of $\mathbb{R}^{\mathtt{x}} \times \mathbb{R}^{\mathtt{w}}$ given by

$$\mathcal{V}_{\mathrm{e}} = \{ \begin{pmatrix} s \\ 0 \end{pmatrix} + \begin{pmatrix} v \\ v \end{pmatrix} \in \mathbb{R}^{\mathtt{x}} \times \mathbb{R}^{\mathtt{w}} \mid s \in \mathcal{S},\, v \in \mathcal{V} \}.$$

Straightforward calculations show that $\mathcal{V}_{\mathrm{e}}$ is invariant under both $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$, and satisfies $\operatorname{im} E_{\mathrm{e}} \subseteq \mathcal{V}_{\mathrm{e}} \subseteq \ker H_{\mathrm{e}}$. It follows from Corollary 5 that the closed-loop system is disturbance decoupled. ■

The conditions presented in Theorem 10 are existential in nature. Next, we articulate these conditions and provide easily verifiable conditions based on subspace algorithms. Recall that $\mathcal{S}^*(E, A_1, C)$ is the smallest (C, A_1)-invariant subspace containing $\operatorname{im} E$.

Theorem 11 *There exists a mode-dependent dynamic controller of the form* (15) *satisfying the continuity condition* (17) *that renders the closed-loop system* (16) *disturbance decoupled if and only if*

$$\mathcal{S}^*(E, A_1, C) \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Proof. *Necessity*: If there exists such a controller, then by Theorem 10 there are subspaces $\mathcal{S}$ and $\mathcal{V}$ such that $(\mathcal{S}, \mathcal{V})$ is a (C, A_1, B)-pair, $\operatorname{im} E \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \ker H$ and $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$. We clearly have $\mathcal{S}^*(E, A_1, C) \subseteq \mathcal{S}$. The subspace $\mathcal{V}$ is (A_1, B)-invariant. Since $hc^T\mathcal{V} \subseteq \mathcal{V} + \operatorname{im} B$, it is also (A_2, B)-invariant. Therefore, we have $\mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B)$. Hence, we can conclude that

$$\mathcal{S}^*(E, A_1, C) \subseteq \mathcal{S} \subseteq \mathcal{V} \subseteq \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Sufficiency: Let $(\mathcal{S},\mathcal{V})$ be the (C,A_1,B)-pair given by $\mathcal{S}=\mathcal{S}^*(E,A_1,C)$ and $\mathcal{V}=\mathcal{V}^*_{\text{md}}(H,\{A_1,A_2\},B)$. Then we have $\operatorname{im}E\subseteq\mathcal{S}\subseteq\mathcal{V}\subseteq\ker H$. Since $\mathcal{V}$ is both (A_1,B)-invariant and (A_2,B)-invariant, we have $A_i\mathcal{V}\subseteq\mathcal{V}+\operatorname{im}B$ for $i=1,2$. As such, we obtain $hc^T\mathcal{V}=(A_1-A_2)\mathcal{V}\subseteq\mathcal{V}+\operatorname{im}B$. Then, it follows from Theorem 10 that the closed-loop system is disturbance decoupled. ■

B. Mode-independent dynamic feedback

As a special case, we consider in this section the linear time-invariant mode-independent feedback controller

$$\dot{w}(t)=Kw(t)+Ly(t) \tag{21a}$$
$$u(t)=Mw(t)+Ny(t), \tag{21b}$$

where $w\in\mathbb{R}^{\mathrm{w}}$, $u\in\mathbb{R}^{\mathrm{u}}$, $y\in\mathbb{R}^{\mathrm{y}}$, and all involved matrices are of suitable sizes. By interconnecting this controller with system given by (8) and (14), we obtain the closed-loop system (16) with the system matrices $A_{\mathrm{e},1}$ and $A_{\mathrm{e},2}$ now given by

$$A_{\mathrm{e},i}=\begin{pmatrix}A_i+BNC & BM\\ LC & K\end{pmatrix}\ \text{for } i=1,2, \tag{22}$$

We can adapt Theorem 10 for mode-dependent dynamic controllers to obtain a similar, but more restrictive, result for mode-independent dynamic controllers.

Theorem 12 *There exists a mode-independent dynamic controller of the form* (21) *that renders the system given by* (8) *and* (14) *disturbance decoupled if and only if there exist subspaces* $\mathcal{S}$ *and* $\mathcal{V}$ *such that* $(\mathcal{S},\mathcal{V})$ *is a* (C,A_1,B)*-pair,* $\operatorname{im}E\subseteq\mathcal{S}\subseteq\mathcal{V}\subseteq\ker H$ *and* $hc^T\mathcal{V}\subseteq\mathcal{S}$.

Proof. A proof of the statement follows from the proof of Theorem 10 by taking $n=0$ and $\ell=0$. ■

Just like for the mode-dependent case, we would like to define some minimal $\mathcal{S}^*$ and maximal $\mathcal{V}^*$ such that $(\mathcal{S}^*,\mathcal{V}^*)$ is a (C,A_1,B)-pair that satisfies the conditions of Theorem 12 exactly when the system can be rendered disturbance decoupled by means of a mode-dependent dynamic feedback controller. For this reason, we define the set of subspaces

$$S_{\text{mi}}(E,\{A_1,A_2\},C):=\{\mathcal{S}\subseteq\mathbb{R}^{\mathrm{x}}\ \mid\ \operatorname{im}E\subseteq\mathcal{S},\ \text{and } \exists G \text{ s.t. } (A_j+GC)\mathcal{S}\subseteq\mathcal{S} \text{ for } j=1,2\}. \tag{23}$$

Similar to the fact that set V_{mi} (defined in (13)) has a maximal element with respect to subspace addition, the set S_{mi} has a minimal element. Let $\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)$ denote the smallest subspace in S_{mi}. In Section VI we present an algorithm to compute $\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)$.

The existence of a controller of the form (21) that renders the closed-loop system disturbance decoupled does not imply that $(\mathcal{S}^*_{\text{mi}},\mathcal{V}^*_{\text{mi}})$ is a (C,A_1,B)-pair satisfying the conditions of Theorem 12 as $hc^T\mathcal{V}^*_{\text{mi}}\subseteq\mathcal{S}^*_{\text{mi}}$ is not necessarily satisfied. However, the following assertion holds.

Theorem 13 *There exists a controller of the form* (21) *that renders the system given by* (8) *and* (14) *disturbance decoupled if and only if one of the following two conditions holds*

1. $\mathcal{S}^*_{\text{mi}}(\begin{bmatrix}E & h\end{bmatrix},\{A_1,A_2\},C)\subseteq\mathcal{V}^*_{\text{mi}}(H,\{A_1,A_2\},B)$,
2. $\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)\ \subseteq\mathcal{V}^*_{\text{mi}}(\begin{bmatrix}H^T & c\end{bmatrix}^T,\{A_1,A_2\},B)$.

Proof. *Sufficiency:* If the first condition holds, then let $\mathcal{S}=\mathcal{S}^*_{\text{mi}}(\begin{bmatrix}E & h\end{bmatrix},\{A_1,A_2\},C)$ and $\mathcal{V}=\mathcal{V}^*_{\text{mi}}(H,\{A_1,A_2\},B)$. Then, we have $h\in\mathcal{S}$ which implies that $hc^T\mathcal{V}\subseteq\mathcal{S}$.

If the second condition holds, let $\mathcal{S}$ and $\mathcal{V}$ be the subspaces $\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)$ and $\mathcal{V}^*_{\text{mi}}(\begin{bmatrix}H^T & c\end{bmatrix}^T,\{A_1,A_2\},B)$ respectively. Then, we have $\mathcal{V}\subseteq\ker c^T$ which implies that $hc^T\mathcal{V}\subseteq\mathcal{S}$. Also, in both cases we have that $(\mathcal{S},\mathcal{V})$ is a (C,A_1,B)-pair that satisfies $\operatorname{im}E\subseteq\mathcal{S}\subseteq\mathcal{V}\subseteq\ker H$. Therefore, it follows from Theorem 12 that there exists a controller of the form (21) such that the closed-loop system is disturbance decoupled.

Necessity: Suppose there exists such a controller. By Theorem 12, there exist subspaces $\mathcal{S}$ and $\mathcal{V}$ such that $(\mathcal{S},\mathcal{V})$ is a (C,A_1,B)-pair, $\operatorname{im}E\subseteq\mathcal{S}\subseteq\mathcal{V}\subseteq\ker H$, and $hc^T\mathcal{V}\subseteq\mathcal{S}$. The last condition implies that $(\mathcal{S},\mathcal{V})$ is also a (C,A_2,B)-pair. Hence, we have $\mathcal{S}\in S_{\text{mi}}(E,\{A_1,A_2\},\ker H)$ and $\mathcal{V}\in V_{\text{mi}}(H,\{A_1,A_2\},B)$. Furthermore, the last condition also implies that $h\in\mathcal{S}$ or $\mathcal{V}\subseteq\ker c^T$.

If $h\in\mathcal{S}$, then $S\in S_{\text{mi}}(\begin{bmatrix}E & h\end{bmatrix},\{A_1,A_2\},C)$, so

$$\mathcal{S}^*_{\text{mi}}(\begin{bmatrix}E & h\end{bmatrix},\{A_1,A_2\},C)\subseteq \mathcal{S}\subseteq\mathcal{V}\subseteq\mathcal{V}^*_{\text{mi}}(H,\{A_1,A_2\},B).$$

If $\mathcal{V}\subseteq\ker c^T$, then $\mathcal{V}\in V_{\text{mi}}(\begin{bmatrix}H^T & c\end{bmatrix}^T,\{A_1,A_2\},B)$. Hence, we get

$$\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)\subseteq \mathcal{S}\subseteq\mathcal{V}\subseteq\mathcal{V}^*_{\text{mi}}(\begin{bmatrix}H^T & c\end{bmatrix}^T,\{A_1,A_2\},B).$$ ■

VI. SUBSPACE ALGORITHMS

We will first propose algorithms for computing $\mathcal{V}^*_{\text{md}}(H,\{A_1,A_2\},B)$ and $\mathcal{V}^*_{\text{mi}}(H,\{A_1,A_2\},B)$. Both algorithms are similar to the invariant subspace algorithm for computing $\mathcal{V}^*(H,A_1,B)$ for linear systems (see e.g. [9]), and to the subspace algorithms proposed in [12] for switched linear systems. Then we will provide an algorithm for computing $\mathcal{S}^*_{\text{mi}}(E,\{A_1,A_2\},C)$.

A. Algorithm for $\mathcal{V}^*_{\text{md}}(H,\{A_1,A_2\},B)$

For computing $\mathcal{V}^*_{\text{md}}(H,\{A_1,A_2\},B)$, we propose the following algorithm. We first define

$$\mathcal{V}_0=\ker H, \tag{24a}$$

Then, for $i\geqslant 0$, we define

$$\mathcal{V}_{i+1}=\mathcal{V}_i\cap A_1^{-1}(\mathcal{V}_i+\operatorname{im}B), \tag{24b}$$

if $h\in\mathcal{V}_i+\operatorname{im}B$, and otherwise

$$\mathcal{V}_{i+1}=\mathcal{V}_i\cap A_1^{-1}(\mathcal{V}_i+\operatorname{im}B)\cap\ker c^T. \tag{24c}$$

It is clear that we have $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for all $i \geqslant 0$ and hence there is a $k \leqslant \mathrm{x}$ such that $\mathcal{V}_k = \mathcal{V}_{k+1}$. Moreover, it follows from the definition of $\mathcal{V}_i$ that we have $\mathcal{V}_{k+2} = \mathcal{V}_{k+1}$. Therefore, we get $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geqslant k$.

Theorem 14 *Let* $\mathcal{V}_i$ *be defined as in algorithm* (24). *Then for* $q = \min\{k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leqslant \mathrm{x}$ *we have*

$$\mathcal{V}_q = \mathcal{V}^*_{\mathrm{md}}(H, \{A_1, A_2\}, B).$$

Proof. As the subspaces $\mathcal{V}_i$ are nested, we have $\mathcal{V}_q \subseteq \mathcal{V}_0 = \ker H$. Since $\mathcal{V}_q$ satisfies $\mathcal{V}_q = \mathcal{V}_{q+1}$, it follows that $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \mathrm{im}\, B)$ if $h \in \mathcal{V}_q + \mathrm{im}\, B$, and $\mathcal{V}_q = \mathcal{V}_q \cap A_1^{-1}(\mathcal{V}_q + \mathrm{im}\, B) \cap \ker c^T$ otherwise. In both cases $\mathcal{V}_q$ is (A_1, B)-invariant. Furthermore, we have $h \in \mathcal{V}_q + \mathrm{im}\, B$ or $\mathcal{V}_q \subseteq \ker c^T$, which implies that $hc^T\mathcal{V}_q \subseteq \mathcal{V}_q + \mathrm{im}\, B$. Hence, $\mathcal{V}_q$ is (A_2, B)-invariant as well. Therefore, we see that $\mathcal{V}_q \in V_{\mathrm{md}}(H, \{A_1, A_2\}, B)$, and hence $\mathcal{V}_q \subseteq \mathcal{V}^*_{\mathrm{md}}$.

To prove that we have $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_q$ as well, we use mathematical induction on i. Firstly, we have that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_0 = \ker H$. Secondly, assume that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_i$ for some $i \geqslant 0$. Since $\mathcal{V}^*_{\mathrm{md}}$ is both (A_1, B)-invariant and (A_2, B)-invariant, it is (hc^T, B)-invariant as well. Therefore, we have

$$hc^T\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}^*_{\mathrm{md}} + \mathrm{im}\, B \subseteq \mathcal{V}_i + \mathrm{im}\, B.$$

Hence, we get $h \in \mathcal{V}_i + \mathrm{im}\, B$ or $\mathcal{V}^*_{\mathrm{md}} \subseteq \ker c^T$. In both cases, it holds that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_{i+1}$. Therefore, we see that $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_k$ for all $k \geqslant 0$. In particular, we have $\mathcal{V}^*_{\mathrm{md}} \subseteq \mathcal{V}_q$. ∎

B. Algorithm for $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$

To compute $\mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B)$, we propose the following algorithm. We define

$$\mathcal{V}_0 = \ker H \tag{25a}$$

and

$$\mathcal{V}_{i+1} = \mathcal{V}_i \cap A_1^{-1}(\mathcal{V}_i + \mathrm{im}\, B) \cap (A_1 - A_2)^{-1}(\mathcal{V}_i) \tag{25b}$$

for $i \geqslant 0$. Again, it is clear that $\mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ for $i \geqslant 0$ and that there is a $k \leqslant \mathrm{x}$ such that $\mathcal{V}_i = \mathcal{V}_k$ for all $i \geqslant k$.

Theorem 15 *Let* $\mathcal{V}_i$ *be defined as in algorithm* (25). *Then for* $q = \min\{\, k \in \mathbb{N} \mid \mathcal{V}_k = \mathcal{V}_{k+1}\} \leqslant \mathrm{x}$ *we have*

$$\mathcal{V}_q = \mathcal{V}^*_{\mathrm{mi}}(H, \{A_1, A_2\}, B).$$

The proof is very similar to that of Theorem 14 and is hence omitted.

C. Algorithm for $\mathcal{S}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$

By making use of the well-known duality between controlled invariance and conditioned invariance (see e.g. [9, Theorem 5.6]), we adapt the algorithm (25) for computing $\mathcal{V}^*_{\mathrm{mi}}$ to obtain the following algorithm. We define

$$\mathcal{S}_0 = \mathrm{im}\, E, \tag{26a}$$

and

$$\mathcal{S}_{i+1} = \mathrm{im}\, E + A_1(\mathcal{S}_i \cap \ker C) + hc^T\mathcal{S}_i \tag{26b}$$

for $i \geqslant 0$. It is easy to see that $\mathcal{S}_i \subseteq \mathcal{S}_{i+1}$ for $i \geqslant 0$. Since $\mathcal{S}_i \in \mathbb{R}^{\mathrm{x}}$ for all $i \geqslant 0$, there is a k such that $\mathcal{S}_k = \mathcal{S}_{k+1}$. Furthermore, it follows from the definition of $\mathcal{S}_i$ that we have $\mathcal{S}_i = \mathcal{S}_k$ for all $i \geqslant k$. The next theorem shows that this algorithm indeed gives us $S^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C)$. We omit the proof, since it follows from similar arguments as employed in the proof of Theorem 14.

Theorem 16 *Let* $\mathcal{S}_i$ *be defined as in algorithm* (26). *Then for* $q = \min\{k \in \mathbb{N} \mid \mathcal{S}_k = \mathcal{S}_{k+1}\} \leqslant \mathrm{x}$ *we have*

$$\mathcal{S}_q = \mathcal{S}^*_{\mathrm{mi}}(E, \{A_1, A_2\}, C).$$

VII. CONCLUSIONS

In this paper, we studied the disturbance decoupling problem for continuous piecewise linear bimodal systems. The main contributions of the paper include necessary and sufficient conditions for such systems to be disturbance decoupled as well as a complete characterization of solvability of the disturbance decoupling problem with mode-independent and mode-dependent feedback controllers. Also, we provided algorithms in order to verify the presented conditions.

Future research possibilities include the extension of the presented results to general piecewise affine dynamical systems.

REFERENCES

[1] G. Basile and G. Marro. Controlled and conditioned invariant subspaces in linear system theory. *J. Optim. Th. & Appl.*, 3:306–315, 1969.

[2] G. Basile and G. Marro. On the observability of linear, time-invariant systems with unknown inputs. *J. Optim. Th. & Appl.*, 3:410–415, 1969.

[3] G. Basile and G. Marro. *Controlled and Conditioned Invariants in Linear System Theory*. Prentice Hall, 1992.

[4] I. Gohberg, P. Lancaster, and L. Rodman. *Invariant Subspaces of Matrices with Applications*, volume 51 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), 2006.

[5] A. Isidori. *Nonlinear Control Systems*. Communications and Control Engineering. Springer, 1995.

[6] H. Nijmeijer and A.J. van der Schaft. *Nonlinear Dynamical Control Systems*. Springer, 1990.

[7] N. Otsuka. Disturbance decoupling with quadratic stability for switched linear systems. *Systems & Control Letters*, 59(6):349–352, 2010.

[8] N. Otsuka. Disturbance decoupling via dynamic output feedback for switched linear systems. In *Proceedings of the IFAC World Congress*, 2011.

[9] H.L. Trentelman, A.A. Stoorvogel, and M.L.J. Hautus. *Control Theory for Linear Systems*. Springer, London, 2001.

[10] W.M. Wonham. *Linear Multivariable Control: a Geometric Approach*. Applications of Mathematics. Springer-Verlag, 1985.

[11] W.M. Wonham and A.S. Morse. Decoupling and pole assignment in linear multivariable systems: A geometric approach. *SIAM J. Contr. & Opt.*, 8:1–18, 1970.

[12] E. Yurtseven, W.P.M.H. Heemels, and M.K. Camlibel. Disturbance decoupling of switched linear systems. *Systems Control & Letters*, 61(1):69–78, 2012.

[13] E. Zattoni and G. Marro. Measurable disturbance rejection with quadratic stability in continuous-time linear switching systems. In *Proceedings of the European Control Conference*, 2013.