The linear wave equation on \( N \)-dimensional spatial domains

Hans Zwart\(^1\), and Mikael Kurula\(^2\)

Abstract— We study the wave equation on a bounded Lipschitz set, characterizing all homogeneous boundary conditions for which this partial differential equation generates a contraction semigroup in the energy space \( L^2(\Omega)^{n+1} \). The proof uses boundary triplet techniques.

MSC 2010 — 35F15, 35L05, 93C20

Index Terms— Port-Hamiltonian system, contraction semigroup, boundary triplet

I. INTRODUCTION

Let \( \Omega \subset \mathbb{R}^n \) be a bounded set with Lipschitz-continuous boundary and let \( \Gamma_0 \) and \( \Gamma_1 \) be open subsets of \( \partial \Omega \), such that \( \Gamma_1 \cap \Gamma_0 = \emptyset \) and \( \Gamma_1 \cup \Gamma_0 = \partial \Omega \). The divergence and gradient on \( \Omega \) are defined in the distribution sense via

\[
\begin{align*}
\text{div } v &= \frac{\partial v_1}{\partial x_1} + \ldots + \frac{\partial v_n}{\partial x_n} \quad \text{and} \\
\text{grad } w &= \left( \frac{\partial w}{\partial x_1}, \ldots, \frac{\partial w}{\partial x_n} \right)^T.
\end{align*}
\]

The Laplacian is the operator \( \Delta z := \text{div} (\text{grad } z) \).

The following PDE describes a wave equation with a viscous damper on the part \( \Gamma_1 \) of \( \partial \Omega \) and a reflecting boundary condition on \( \Gamma_0 \):

\[
\begin{align*}
\frac{\partial^2 z}{\partial t^2}(\xi, t) &= (\Delta z)(\xi, t) \quad \text{on } \Omega \times \mathbb{R}^+, \\
0 &= \nu \cdot \text{grad } z(\xi, t) + k(\xi) \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\
0 &= \frac{\partial z}{\partial t}(\xi, t) \quad \text{on } \Gamma_0 \times \mathbb{R}^+
\end{align*}
\]

where \( \nu \in L^\infty(\partial \Omega; \mathbb{R}^n) \) is the outward unit normal of \( \partial \Omega \) and the non-negative real-valued function \( k \) describes the amount of damping in almost every point \( \xi \in \Gamma_1 \).

In this paper we show that the PDE (1) possesses a unique solution for all initial data in \( L^2(\Omega)^{n+1} \). However, our result is much more general. Namely, we characterize all boundary conditions for which the wave equation possesses a unique solution that is contractive with respect to the energy. In the full article [6] underlying this paper, the results are formulated for arbitrary boundary triplets, and the wave equation is merely an example.

We follow the port-Hamiltonian approach as has been done for the one-dimensional wave equation in [2], [3]. The first step is to rewrite \( \frac{\partial^2 z}{\partial t^2}(\cdot, t) = (\Delta z)(\cdot, t) \) on \( \Omega \) in the energy variables, as

\[
\frac{d}{dt} \begin{bmatrix} \dot{z}(t) \\ \text{grad } z(t) \end{bmatrix} = \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \begin{bmatrix} \dot{z}(t) \\ \text{grad } z(t) \end{bmatrix},
\]

where \( \dot{z}(t) = \frac{dz}{dt}(t) \). Note that the position can be recovered from (2) by simply integrating the first state component. Next we want to characterize those domains of the operator \( \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix} \) for which it is the infinitesimal generator of a contraction semigroup in \( L^2(\Omega)^{n+1} \). From Lemma 7.2.3 of [3] it is clear that this also characterizes existence of a contraction semigroup on the energy space, i.e., when (1) contains the physical parameters.

II. BACKGRO UND AND SETTING

The necessary background for the present article has been compiled in [4]. Here we only fix the notation very briefly and the reader is referred to [4] for more details.

We define

\[
H^{\text{div}}(\Omega) := \{ v \in L^2(\Omega)^n \mid \text{div } v \in L^2(\Omega) \},
\]

equipped with the graph norm of \( \text{div} \). This is the maximal domain for which \( \text{div} \) can be considered as operator between \( L^2 \) spaces. We will consider \( \text{grad} \) as an unbounded operator from \( L^2(\Omega) \) into \( L^2(\Omega)^n \) with domain contained in \( H^1(\Omega) \).

**Theorem 2.1:** For a bounded Lipschitz set \( \Omega \) the following hold:

1. The boundary trace mapping \( g \mapsto g|_{\partial \Omega} : C^1(\overline{\Omega}) \to C(\partial \Omega) \) has a unique continuous extension \( \gamma_{\text{H}} \) that maps \( H^1(\Omega) \) onto \( H^{1/2}(\partial \Omega) \). The space \( H^1_0(\Omega) \) equals \( \{ g \in H^1(\Omega) \mid \gamma_{\text{H}}(g) = 0 \} \).
2. The normal trace mapping \( u \mapsto \nu \cdot \gamma_{\text{H}} u : H^1(\Omega)^n \to L^2(\partial \Omega) \) has a unique continuous extension \( \gamma_{\text{H}} \) that maps \( H^1(\Omega)^n \) onto \( H^{-1/2}(\partial \Omega) \). Here the dot denotes the inner product in \( \mathbb{R}^n \), \( p \cdot q = q^\top p \) without complex conjugate. Furthermore, the space \( H^1_0(\Omega) \) equals

\[
H^1_{\text{div}}(\Omega) = \{ f \in H^1(\Omega) \mid \gamma_{\text{H}} f = 0 \}.
\]

We call \( \gamma_{\text{H}} \) the Dirichlet trace map and \( \gamma_{\text{H}} \) the normal trace map. Note that \( \gamma_{\text{H}} \) is not the Neumann trace \( \gamma_{\text{N}} \); the relation between the two is \( \gamma_{\text{N}} f = \gamma_{\text{H}} \text{grad } f \), for \( f \) smooth enough.

**Theorem 2.2:** Let \( \Omega \) be a bounded Lipschitz set in \( \mathbb{R}^n \). For all \( f \in H^1(\Omega) \) and \( g \in H^1(\Omega) \) it holds that

\[
\langle \text{div } f, g \rangle_{L^2(\Omega)} + \langle f, \text{grad } g \rangle_{L^2(\Omega)} = \langle \gamma_{\text{H}} f, \gamma_{\text{H}} g \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)}.
\]
In particular, we have the following Green’s formula:
\[
\langle \Delta h, g \rangle_{L^2(\Omega)} + \langle \text{grad} \, h, \text{grad} \, g \rangle_{L^2(\Omega)^n} = \langle \gamma_\perp \text{grad} \, h, \gamma_0 g \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)},
\]
which is valid for all \(h, g \in H^1(\Omega)\), such that \(\Delta h \in L^2(\Omega)\).

III. DUALITY OF THE DIVERGENCE AND THE GRADIENT

Since \(H^{1/2}(\partial \Omega)\) and \(H^{-1/2}(\partial \Omega)\) are each others duals with pivot space \(L^2(\partial \Omega)\), we can make the following definition: The *annihilator* in \(H^{-1/2}(\partial \Omega)\) of a subspace \(R \subset H^{1/2}(\partial \Omega)\) is the (closed) subspace
\[
R^{\perp,\ast} := \{ v \in H^{-1/2}(\partial \Omega) \mid \langle v, r \rangle = 0 \quad \forall r \in R \},
\]
Where \((v, r)\) denotes the duality pairing between \(H^{-1/2}(\partial \Omega)\) and \(H^{1/2}(\partial \Omega)\). The following result formulates an exact duality between the divergence and the gradient:

**Theorem 3.1**: Let \(\Omega\) be a bounded Lipschitz set in \(\mathbb{R}^n\) and let \(H_0^1(\Omega) \subset C \subset H^1(\Omega)\). Consider \(\text{grad} \Gamma_0\) as an unbounded operator from the dense subspace \(C \subset L^2(\Omega)\) into \(L^2(\Omega)^n\). Then its adjoint is given by \(\langle \text{grad} \Gamma_0 \rangle^* = -\text{div}_D\) with
\[
D := \{ f \in H^{1/2}(\partial \Omega) \mid \gamma_\perp f \in (\gamma_0 G)^\perp \}. \quad (4)
\]
Furthermore, the set \(D\) is a closed subspace of \(H^{1/2}(\partial \Omega)\) that contains \(H_0^{1/2}(\partial \Omega)\), i.e., \(H_0^{1/2}(\partial \Omega) \subset D \subset H^{1/2}(\partial \Omega)\).

Assume that \(G\) is closed in \(H^{1}(\Omega)\). Then \(D = H^{1/2}(\partial \Omega)\) if and only if \(G = H_0^1(\Omega)\), and \(D = H_0^{1/2}(\partial \Omega)\) if and only if \(G = H^{1}(\Omega)\).

Theorem 3.1 follows essentially from the “integration by parts formula” (3). For a given domain \(G\) of the gradient operator, (4) says that the corresponding domain \(D\) of the adjoint divergence operator is the inverse image under \(\gamma_\perp\) of the annihilator \((\gamma_0 G)^\perp\).

We proceed by specilising Theorem 3.1 to the case where the functions in the domain of the gradient operator vanish on an open subset \(\Gamma_0 \subset \partial \Omega\). Following [9, Chap. 13], we will identify \(L^2(\Gamma_0)\) with the space of functions in \(L^2(\partial \Omega)\) that vanish almost everywhere on \(\partial \Omega \setminus \Gamma_0\). Hence we have
\[
L^2(\partial \Omega) = L^2(\Gamma_0) \oplus L^2(\partial \Omega \setminus \Gamma_0),
\]
and we denote the corresponding orthogonal projection onto \(L^2(\Gamma_0)\) by \(\pi_{\Gamma_0}\). If \(\Gamma_1\) is as described in the introduction and the common boundary \(\partial \Omega \setminus (\Gamma_0 \cup \Gamma_1)\) of \(\Gamma_0\) and \(\Gamma_1\) has surface measure zero, then \(L^2(\partial \Omega \setminus \Gamma_0) = \Gamma_1\), but this seems to be unimportant in our setting.

In [9, §13.6] the following space of functions in \(H^1(\Omega)\), whose boundary trace vanish on \(\Gamma_0\), was introduced:
\[
H_0^1(\Omega) := \{ g \in H^1(\Omega) \mid (\gamma_0 g)|_{\Gamma_0} = 0 \text{ in } L^2(\Gamma_0) \}.
\]
The space \(H_0^1(\Omega)\) is closed, because it can be viewed as the kernel of the bounded operator \(\pi_0 \gamma_0 : H^1(\Omega) \to L^2(\Gamma_0)\); recall that \(\gamma_0\) is bounded from \(H^1(\Omega)\) into \(H^{1/2}(\partial \Omega)\) by

**Theorem 2.1** and that the latter is continuously embedded in \(L^2(\partial \Omega)\) by its definition.

By Theorem 3.1, \(\text{grad} \Gamma_0^* = -\text{div}_D\), where
\[
H^{1/2}(\Omega) := \{ f \in H^{1/2}(\Omega) \mid \gamma_\perp f \in \left(\gamma_0 H_0^1(\Omega)\right)^\perp \}, \quad (5)
\]
and it follows that \(H^{1/2}(\Omega)\) is closed in \(H^{1/2}(\partial \Omega)\). In particular, \(H_0^1(\Omega) = H_0^{1/2}(\Omega)\), corresponds to \(H_0^{1/2}(\Omega) = H^{1/2}(\Omega)\), and this case was used extensively in [7], [10], [11], [5]. The other extreme case is \(H^1(\Omega) = H_0^0(\Omega)\), which corresponds to \(H_0^{1/2}(\Omega) = H_0^{1/2}(\partial \Omega)\).

As a consequence of the Riesz representation theorem, there exists a unitary operator \(\Psi : H^{-1/2}(\partial \Omega) \to H^{1/2}(\partial \Omega)\), such that
\[
\langle x, z \rangle_{H^{-1/2}(\partial \Omega), H^{1/2}(\partial \Omega)} = \langle \Psi x, z \rangle_{H^{1/2}(\partial \Omega)} = \langle x, \Psi^* z \rangle_{H^{-1/2}(\partial \Omega)}
\]
for all \(x \in H^{-1/2}(\partial \Omega)\) and \(z \in H^{1/2}(\partial \Omega)\); see [8, p. 288–289] or [9, p. 57]. This \(\Psi\) is called the *duality operator* [8].

We have the following practical description of the annihilator in (5):

**Proposition 3.2**: It holds that
\[
\left(\gamma_0 H_0^1(\Omega)\right)^\perp = L^2(\Gamma_0)^{H^{-1/2}(\partial \Omega)}
\]
and
\[
\left(\gamma_0 H_0^1(\Omega)\right)^\perp \cap L^2(\partial \Omega) = L^2(\partial \Omega) \ominus \left(\gamma_0 H_0^1(\Omega)\right).
\]

IV. TOOLS FOR EXISTENCE PROOFS FOR PDES

The operator \(\mathcal{A}\) defined as
\[
\begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix}_{\mathcal{D}} \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix} \supseteq \mathcal{D} \to \begin{bmatrix} L^2(\Omega) \\ L^2(\Omega)^n \end{bmatrix}, \quad (6)
\]
with domain \(\mathcal{D} = \begin{bmatrix} H_0^1(\Omega) \\ H_0^{1/2}(\Omega) \end{bmatrix}\) is skew-adjoint by Theorem 3.1. We shall next characterize all domains \(\mathcal{D}\) in practice we characterise the boundary conditions,
\[
\begin{bmatrix} H_0^1(\Omega) \\ H_0^{1/2}(\Omega) \end{bmatrix} \subset \mathcal{D} \subset \begin{bmatrix} H^1(\Omega) \\ H^{1/2}(\partial \Omega) \end{bmatrix},
\]
which make \(A\) in (6) maximal dissipative or skew-adjoint on \(L^2(\Omega)^{n+1}\). We achieve this by associating a boundary triplet to \(A\) in (6).

The first step is to adapt the definition [1, p. 155] of a boundary triplet for a symmetric operator to the case of a skew-symmetric operator. It is based on the observation that an operator \(iA\) is skew-symmetric if and only if \(A\) is symmetric; see also [8, §5].

**Definition 4.1**: Let \(A\) be a densely defined, skew-symmetric, and closed linear operator on a Hilbert space \(\mathcal{X}\). A boundary triplet for \(A\) consists of a Hilbert space \(\mathcal{B}\) and two bounded linear operators \(B_1, B_2 : \text{dom}(A) \to \mathcal{B}\), such that
\[
\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \text{dom}(A) = \begin{bmatrix} \mathcal{B} \\ \mathcal{B} \end{bmatrix}
\]
and for all \( x, \tilde{x} \in \text{dom} (A_0^*) \) it holds that
\[
\langle A_0^*x, \tilde{x} \rangle_X + \langle x, A_0^*\tilde{x} \rangle_X = \langle B_1x, B_2\tilde{x} \rangle_B + \langle B_2x, B_1\tilde{x} \rangle_B. \tag{8}
\]
The analogue of (8) is written as follows in [1, p. 155]:
\[
\langle A^*x, \tilde{x} \rangle - \langle x, A^*\tilde{x} \rangle = \langle \Gamma_1x, \Gamma_2\tilde{x} \rangle_B + \langle \Gamma_2x, \Gamma_1\tilde{x} \rangle_B,
\]
and setting \( A_0^* = (iA)^*, \) \( B_1 = \Gamma_1, \) and \( B_2 = i\Gamma_2 \) in (8), we obtain exactly this.

**Theorem 4.2**: Let \( \Omega \) be a bounded Lipschitz set. The operator
\[
A_0 := \begin{bmatrix} 0 & -\text{div} \\ -\text{grad} & 0 \end{bmatrix}, \quad \text{dom} (A_0) := \begin{bmatrix} H^1_0(\Omega) \\ H^2(\Omega) \end{bmatrix},
\]
is closed, skew-symmetric, and densely defined on \( L^2(\Omega)^n \). Its adjoint is
\[
A_0^* := \begin{bmatrix} 0 & \text{div} \\ \text{grad} & 0 \end{bmatrix}, \quad \text{dom} (A_0^*) := \begin{bmatrix} H^1(\Omega) \\ H^2(\Omega) \end{bmatrix}. \tag{9}
\]
Setting \( B_0 := [\gamma_0 \ 0] \) and \( B_\perp := [0 \ \gamma_\perp] \), we obtain that \( (H^{1/2}(\partial\Omega); B_0, \Psi B_\perp) \) is a boundary triplet for \( A_0^* \).

One can now prove the following \( n \)-dimensional analogue of [3, Thm 7.2.4]:

**Theorem 4.3**: Let \( \mathcal{H} \) be a Hilbert space and let \( W_B = [W_1 \ W_2] : [H^{1/2}(\partial\Omega) \ W^{1/2}(\partial\Omega)] \to \mathcal{H} \) be a bounded linear operator, such that
\[
\text{ran} (W_1 - W_2 \Psi^*) \subset \text{ran} (W_1 + W_2 \Psi^*). \tag{10}
\]
Then the restriction \( A := A_0^*|_{\text{dom}(A)} \) of \( A_0^* \) in (9) to \( \text{dom}(A) := \ker (W_B [B_0 \ B_\perp]) \) is a closed operator on \( L^2(\Omega)^{n+1} \) and the following conditions are equivalent:

1) \( A \) generates a contraction semigroup on \( L^2(\Omega)^{n+1} \).

2) \( A \) is dissipative: \( \text{Re} \langle Ax, x \rangle \leq 0 \) for all \( x \in \text{dom}(A) \).

3) The operator \( W_1 + W_2 \Psi^* \) is injective and the following operator inequality holds in \( \mathcal{H} \):
\[
W_1 \Psi W_2^* + W_2 \Psi W_1^* \geq 0. \tag{11}
\]
The inequality (11) can equivalently be written as follows, with \( J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \):
\[
[ W_1 \ W_2 \Psi^* ] J \begin{bmatrix} W_1 \ W_2 \Psi^* \end{bmatrix}^* \geq 0.
\]
This inequality in fact means that \( A^* \) is dissipative, and this in turn implies that the range inclusion (10) is a maximality condition. Indeed, if (10) holds, then Theorem 4.3 essentially says that \( A \) is dissipative if and only if \( A^* \) is dissipative. If \( W_1 + W_2 \Psi^* \) is invertible, then \( W_B \) is automatically surjective and (10) holds, but this can be the case only for “minimal” choices of \( \mathcal{H} \).

We finish this section with our main result.

**Theorem 4.4**: Make the assumptions and use the notation in Theorem 4.2. Let \( V_B = [V_1 \ V_2] \) be a bounded everywhere defined operator from \( L^2(\partial\Omega)^2 \) into some Hilbert space \( \mathcal{H} \) and define
\[
A := \{ a \in \text{dom}(A_0^*) \mid B_\perp a \in L^2(\partial\Omega) \}
\]
\[
\wedge \ [V_1 \ V_2] \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} a = 0. \tag{12}
\]
Then the following two conditions are together sufficient for the closure \( A \) of the operator \( A_0^*|_{\mathcal{H}} \) to generate a contraction semigroup on \( L^2(\Omega)^{n+1} \):

1) The kernel of \( V_B \) is a dissipative relation in \( L^2(\partial\Omega) \), i.e., \( \text{Re} \langle u, v \rangle_{L^2(\partial\Omega)} \leq 0 \) for all \( u, v \in L^2(\partial\Omega) \) such that \( V_1u + V_2v = 0 \).

2) The following operator inequality holds in \( \mathcal{H} \):
\[
V_1V_2^* + V_2V_1^* \geq 0. \tag{13}
\]
The operator \( A \) generates a unitary group if \( \text{Re} \langle u, v \rangle_{L^2(\partial\Omega)} = 0 \) for all \( [u] \in \ker (V_B) \) and \( V_1V_2^* + V_2V_1^* = 0 \).

Condition 2 is also necessary for \( A \) to generate a contraction semigroup (unitary group).

The strength in the preceding result, as compared to Theorem 4.3, lies in the fact that we only need to investigate the kernel of \( [V_1 \ V_2] \) which is a relation in \( L^2(\partial\Omega) \). If we decided to use Theorem 4.3, then we would need to study a significantly less practical subspace of \( [H^{1/2}(\partial\Omega) \ W^{1/2}(\partial\Omega)] \).

**Corollary 4.5**: Under the following additional assumptions, condition 1 in Theorem 4.4 becomes necessary too:

1) The operator \( V_2 \) is injective with a closed range.

2) Denoting the orthogonal projection in \( \mathcal{H} \) onto \( \ker (V_2) \) by \( P \), the intersection \( \ker ((I - P)V_1) \cap H^{1/2}(\partial\Omega) \) is dense in \( \ker ((I - P)V_1) \).

V. APPLICATION TO THE WAVE EQUATION

In this final section, we apply Theorem 4.4 to our example in the introduction:
\[
\frac{\partial^2 z}{\partial t^2}(\xi, t) = (\Delta z)(\xi, t) \quad \text{on} \quad \Omega \times \mathbb{R}_+, \tag{14}
\]
\[
= 0 = \nu \cdot \text{grad} \ z(\xi, t) + k(\xi) \frac{\partial z}{\partial t}(\xi, t) \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+.
\]
We want to show that the operator associated to this PDE generates a contraction semigroup on the energy space \( L^2(\Omega)^{n+1} \). For that we write the wave equation in the form (2); hence we have that our state vector is \( z(t) = \begin{bmatrix} z(t) \\ \text{grad} \ z(t) \end{bmatrix} \). Furthermore, the system operator \( A \) is \( A_0^* \), from equation (9), restricted to some domain. This domain is determined by the boundary conditions in (14).

We assume that the parts \( \Gamma_0 \) and \( \Gamma_1 \) of \( \partial\Omega \) are such that \( \Gamma_0 \cap \Gamma_1 = \emptyset \), \( \overline{\Gamma_0} \cup \overline{\Gamma_1} = \partial\Omega \), and that \( \Gamma_0 \) and \( \Gamma_1 \) have a common boundary of surface measure zero. These assumptions are not restrictive; the last assumption is satisfied, e.g., if \( \Gamma_0 \) and \( \Gamma_1 \) themselves have Lipschitz-continuous boundaries.

In order to apply Theorem 4.4, we first have to reformulate the boundary conditions of (1) as the kernel of
for some bounded operators \( V_1 \) and \( V_2 \). As range space of \( V_1 \) and \( V_2 \) we take \( H := L^2(\Gamma_0) \). Recall that \( \pi_0 \) is the orthogonal projection in \( L^2(\partial \Omega) \) onto \( L^2(\Gamma_0) \), and we denote the corresponding projection onto \( L^2(\Gamma_1) \) by \( \pi_1 \). Now we define:

\[
[V_1 \ V_2] := \begin{bmatrix} \pi_1 M_k & \pi_1 \\ \pi_0 & 0 \end{bmatrix},
\]

(15)

where \( M_k \) is the bounded operator of multiplication by \( k \) in \( L^2(\partial \Omega) \). (The function \( k \in L^2(\Gamma_1; \mathbb{R}) \), \( k(\cdot) \geq 0 \), is extended by zero on \( \Gamma_0 \).)

Next we check if the kernel of \([V_1 \ V_2] \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix}\) corresponds to our boundary conditions. Since the state is \( \gamma(t) = \begin{bmatrix} \z(t) \\ \text{grad } z(t) \end{bmatrix} \), we have that

\[
[V_1 \ V_2] \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} x = \begin{bmatrix} \pi_1 M_k & \pi_1 \\ \pi_0 & 0 \end{bmatrix} \begin{bmatrix} \gamma_0 \hat{z} \\ \gamma_\perp \text{grad } z \end{bmatrix},
\]

and we see that \( x = \begin{bmatrix} \hat{z} \\ \text{grad } z \end{bmatrix} \), with \( \gamma_\perp \text{grad } z \in L^2(\partial \Omega) \), lies in \( \ker \left( [V_1 \ V_2] \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} \right) \) if and only if \( \pi_0 \gamma_0 = 0 \) and

\[
\pi_1 M_k \gamma_0 \hat{z} + \pi_1 \gamma_\perp \text{grad } z = 0,
\]

(16)

which indeed agrees with the boundary conditions in (14).

We show that \( \ker \left( [V_1 \ V_2] \right) \) is a dissipative relation in \( L^2(\partial \Omega) \) as follows. It holds that \( \begin{bmatrix} u \\ v \end{bmatrix} \in \ker \left( [V_1 \ V_2] \right) \) if and only if \( \pi_1 u = -M_k \pi_1 u \) and \( \pi_0 u = 0 \). For any such \( \begin{bmatrix} u \\ v \end{bmatrix} \), we have

\[
\text{Re} \left\langle u, v \right\rangle_{L^2(\partial \Omega)} = \text{Re} \left\langle \pi_0 u, \pi_0 v \right\rangle_{L^2(\Gamma_0)} + \text{Re} \left\langle \pi_1 u, \pi_1 v \right\rangle_{L^2(\Gamma_1)} = -\text{Re} \left\langle \pi_1 u, M_k \pi_1 u \right\rangle_{L^2(\Gamma_1)} \leq 0.
\]

We still need to verify that \( V_1 V_2^* + V_2 V_1^* \geq 0 \). For all \( p \in L^2(\Gamma_1) \) and \( q \in L^2(\Gamma_0) \) it holds that

\[
2\text{Re} \left\langle V_1 V_2^* p, q \right\rangle_{L^2(\Gamma_1)} = 2\text{Re} \left\langle \begin{bmatrix} M_k \pi_1 \\ \pi_0 \end{bmatrix} \begin{bmatrix} I_1 \\ 0 \end{bmatrix} p, q \right\rangle_{L^2(\Gamma_1)} \geq 0,
\]

where \( I_1 : L^2(\Gamma_1) \to L^2(\partial \Omega) \) is the injection operator; hence \( \pi_0 I_1 = 0 \).

By Theorem 4.4, we conclude that the closure \( \bar{A} \) of the operator \( A \) defined in (12), with \( [V_1 \ V_2] \) given by (15), generates a contraction semigroup on \( L^2(\Omega)^{n+1} \).

Using the results of [6], this operator closure can be directly characterised as \( A = A_3 \big|_{\text{dom}(A)} \), where

\[
\text{dom}(A) = \ker \left( \begin{bmatrix} \Pi_1 M_k & \Pi_1 \\ \pi_0 & 0 \end{bmatrix} \begin{bmatrix} B_0 \\ B_\perp \end{bmatrix} \right),
\]

with \( \Pi_1 \) the orthogonal projection in \( H^{-1/2}(\partial \Omega) \) onto \( H^{-1/2}(\partial \Omega) \cap L^2(\Gamma_1) \). Here \( \Pi_1 \) needs to be chosen differently from above, since \( \text{ran} (\Pi_1) \not\subset L^2(\Gamma_1) \); take for instance \( \Pi_1 \) as given in Theorem 4.3. One could also prove that \( A \) generates a contraction semigroup on \( L^2(\Omega)^{n+1} \) using this representation and Theorem 4.3, but that leads to more complicated calculations than those above.

By Proposition 3.2, we can also write \( \text{dom}(A) \) as

\[
\text{dom}(A) = \left\{ \begin{bmatrix} g \\ f \end{bmatrix} \in H^1_\text{div}(\Omega) \bigg| H^1_\text{div}(\Omega) \right\}.
\]

REFERENCES


