

# Nondecreasing Lyapunov functions

Michael Defoort, Mohamed Djemai and Stephan Trenn

**Abstract**— We propose the notion of nondecreasing Lyapunov functions which can be used to prove stability or other properties of the system in question. This notion is in particular useful in studying switched or hybrid systems. We illustrate the concept by a general construction of such a nondecreasing Lyapunov function for a class of planar hybrid systems. It is noted that this class encompasses switched systems for which no piecewise-quadratic (classical) Lyapunov function exists.

## I. INTRODUCTION

Switched Systems are systems involving both continuous and discrete dynamics. They consist of a finite number of subsystems and a discrete rule that dictates switching between these subsystems [10]. They have been widely studied during the last two decades (see for instance [4], [9], [12]) because they can describe a wide range of physical and engineering systems.

The design of Lyapunov functions is of high interest in systems theory. Indeed, it finds direct applications in stability analysis of nonlinear systems. This is achieved by looking at the sign of time derivative of the Lyapunov function along the solutions of the system. Furthermore, other challenges are implicitly linked such as performance analysis and controller design for instance. Hence, methods to design Lyapunov functions for general nonlinear systems are of great theoretical and practical interest.

The simple test of looking at the time derivative of the Lyapunov function is not available anymore for switched or hybrid systems as differentiability (or even continuity) of the Lyapunov functions along solutions does not hold in general. A general approach to the above problem is based on the construction of a common Lyapunov function for the family of subsystems corresponding to the considered switched system [11], [13]. However, even for the case of general linear switched systems, there is no constructive procedure to show the existence of a common Lyapunov function. Some of the available methods are rather involved due to the fact, that one still aims to construct a decreasing Lyapunov function. Additional assumptions on the switching law (dwell time for instance) have been added to analyze the stability using multiple Lyapunov functions [3], [14].

M. Defoort and M. Djemai are with Univ. Lille Nord de France, F-59000 Lille, France, UVHC, LAMIH, F-59313 Valenciennes, France, CNRS, UMR 8201, F-59313 Valenciennes, France; S. Trenn is with the Department of Mathematics, Univ. of Kaiserslautern, 67663 Kaiserslautern, Germany

michael.defoort@univ-valenciennes.fr,  
mohamed.djemai@univ-valenciennes.fr,  
trenn@mathematik.uni-kl.de

This work was prepared while the third author was a visiting professor at the University of Valenciennes.

However, in this case, the subsystems are required to be asymptotically stable.

In [2], it is shown that stabilizability of switched linear systems does not imply the existence of a convex Lyapunov function (including norms and quadratic functions). The search for a piecewise quadratic Lyapunov function is formulated as a convex optimization problem in terms of linear matrix inequalities [7]. This approach enables to derive a Lyapunov function for a class of switched linear systems where the subsystems are not required to be asymptotically stable. However, the search of a decreasing function needs hard restrictions on the switching signals. In [3], it is shown that this is not a mandatory condition for the design of Lyapunov function candidates. For conewise linear systems a construction for piecewise-quadratic Lyapunov functions is presented in [6], [5]; we will compare this approach with ours in Section IV.

In this paper, the notion of nondecreasing<sup>1</sup> Lyapunov functions is introduced for a general class of nonlinear switched systems. Using this notion, it will be easier in general to study stability, analyze performance and design controllers. Furthermore, for a rather general class of switched linear systems a constructive procedure for deriving a nondecreasing Lyapunov function is given.

Similar ideas were proposed for the discrete time case in [1], [8] but due the discrete nature of time the methods are quite different to the ones used for the continuous time case.

The outline of this paper is as follows. Section II derives the framework of nondecreasing Lyapunov functions for switched or hybrid nonlinear systems. Then in Section III, a class of piecewise linear systems is studied and a general construction for a nondecreasing Lyapunov function is given. It should be highlighted that the proposed scheme is a powerful tool since it provides a solution even if there does not exist a piecewise-quadratic Lyapunov function in the sense of [7]; this is illustrated with a specific example in Section IV.

We conclude this introduction with some remarks on notation. For a piecewise continuous function  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  we denote the limit of  $f(s)$  as  $s \in \mathbb{R}$  approaches  $t \in \mathbb{R}$  from the left by  $f(t^-)$  and from the right by  $f(t^+)$ . For simplicity we will assume that  $f(t) = f(t^+)$  for all  $t \in \mathbb{R}$ , i.e. any piecewise continuous function is assumed to be right continuous. A continuous function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is called a  $\mathcal{K}$ -function, if  $\alpha(0) = 0$  and  $\alpha$  is strictly increasing.

<sup>1</sup>Here “nondecreasing” is not used in the usual sense, because we do not assume that the function is nowhere decreasing; it would be more precise to use “not necessarily decreasing” or “non-monotonically decreasing” which are, however, a bit clumsy.

With  $\mathcal{K}_\infty$ -functions we denote all  $\mathcal{K}$ -functions which grow unboundedly. A continuous function  $\gamma : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is called an  $\mathcal{L}$ -function if it is strictly decreasing with  $\gamma(t) \rightarrow 0$  as  $t \rightarrow \infty$ . A continuous function  $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is called a  $\mathcal{KL}$ -function if  $\beta(\cdot, t)$  is a  $\mathcal{K}$ -function for each fixed  $t \geq 0$  and  $\beta(r, \cdot)$  is an  $\mathcal{L}$ -function for each fixed  $r > 0$ .

## II. NONDECREASING LYAPUNOV FUNCTIONS

We consider hybrid systems of the following form

$$\begin{aligned} \dot{x}(t) &= f_{q(t)}(x(t)), \quad \forall t \geq 0 \text{ with } q(t) = q(t^-), \\ x(t) &= g_{q(t^-), q(t)}(x(t^-)), \quad q(t) \neq q(t^-), \\ x(0^-) &= x_0 \in \mathbb{R}^n \\ q(t) &= h(q(t^-), x(t^-), \sigma(t)) \quad \forall t \geq 0, \\ q(0^-) &= q_0 \in \mathcal{Q}. \end{aligned} \quad (1)$$

Here  $\mathcal{Q}$  is the set of discrete states,  $f_q : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the vector field of mode  $q \in \mathcal{Q}$ ,  $g_{q^-, q} : \mathbb{R}^n \rightarrow \mathbb{R}^n$  is the jump map whenever the discrete state changes from  $q^- \in \mathcal{Q}$  to  $q \in \mathcal{Q}$ ,  $\sigma \in \Sigma \subseteq \mathcal{P}^{[0, \infty)}$  is an external right continuous switching signal (i.e. state independent) with values in the set  $\mathcal{P}$  and where  $\Sigma$  denotes the allowed class of these switching signals (e.g. no accumulation of switching times, some dwell time condition or some condition on the mode sequences), and  $h : \mathcal{Q} \times \mathbb{R}^n \times \mathcal{P} \rightarrow \mathcal{Q}$  is the discrete state map. This system class is quite general and encompasses purely time-dependent switched systems (i.e.  $h(q, x, \sigma) = \sigma$ ) as well as pure state-dependent switched systems (where  $h(q, x, \sigma)$  only depends on  $x$ ); in the latter case we will (with some abuse of notation) write  $\sigma(x)$  to denote the state-dependent switching signal. We make the following assumption on the solvability of (1)

**Assumption 1:** For any switching signal  $\sigma \in \Sigma$  and any initial values  $x_0 \in \mathbb{R}^n$ ,  $q_0 \in \mathcal{Q}$  there exists a solution  $(x, q) : \mathbb{R} \rightarrow \mathbb{R}^n \times \mathcal{Q}$  of (1), in particular,

- (i)  $q$  is piecewise constant, right-continuous and  $q(t-)$  exists for all  $t \geq 0$ , i.e. the set of discontinuities of  $q$  on  $[0, \infty)$  has no accumulation points,
- (ii)  $x$  is continuously differentiable on each interval  $[t_1, t_2) \subseteq \mathbb{R}_{\geq 0}$  on which  $q$  is constant.

We recall the definition of a classical Lyapunov function (in the framework of  $\mathcal{KL}$ -functions).

*Definition 2.1 (c.f. [2]):* Consider a dynamical system (1) satisfying Assumption 1 which produces trajectories  $x(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ . We call  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  a *Lyapunov function* for (1) if, and only if,

- (i) There exist  $\mathcal{K}_\infty$ -functions  $\alpha_1$  and  $\alpha_2$  such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|) \quad \forall x \in \mathbb{R}^n.$$

- (ii) There exists a  $\mathcal{KL}$ -function  $\beta$  such that for all trajectories  $x(\cdot)$  of (1)

$$V(x(t)) \leq \beta(V(x(t_0)), t - t_0) \quad \forall t \geq t_0. \quad (2)$$

- (iii) The  $\mathcal{KL}$ -function  $\beta$  from above additionally fulfills

$$\beta(v, 0) = v.$$

It is straightforward to see (even for the general system class given by (1)) that a Lyapunov function in the above sense ensures asymptotic stability uniformly in the initial values  $(x_0, q_0)$  as well as in the external switching signal  $\sigma$ . A key feature which can also be explored to verify the existence of a Lyapunov function (see discussion below) is the fact that the above conditions ensures that  $V$  decreases along solutions:

$$V(x(t_2)) \leq \beta(V(x(t_1), t_2 - t_1) < \beta(V(x(t_1), 0) = V(x(t_1)))$$

for any  $t_2 > t_1$ .

However, in order to prove convergence of the trajectories to zero it is in fact not necessary to assume that  $V$  decreases along solution as long as (2) holds. It is therefore a natural generalization to consider Lyapunov functions which are not necessarily decreasing along solutions:

*Definition 2.2:* We call  $\bar{V} : \mathbb{R}^n \rightarrow \mathbb{R}$  a *nondecreasing Lyapunov function* for the dynamical system (1) if, and only if,

- (i) There exist  $\mathcal{K}_\infty$ -functions  $\bar{\alpha}_1$  and  $\bar{\alpha}_2$  such that

$$\bar{\alpha}_1(\|x\|) \leq \bar{V}(x) \leq \bar{\alpha}_2(\|x\|) \quad \forall x \in \mathbb{R}^n. \quad (3)$$

- (ii) There exists a  $\mathcal{KL}$ -function  $\bar{\beta}$  such that for all trajectories  $x(\cdot)$  of (1)

$$\bar{V}(x(t)) \leq \bar{\beta}(\bar{V}(x(t_0)), t - t_0) \quad \forall t \geq t_0. \quad (4)$$

From (4) together with (3) it immediately follows that each trajectory of (1) converges to zero. Furthermore, (4) ensures that each trajectory remains within the compact sublevel set of  $\bar{V}$  with the level  $\bar{\beta}(\bar{V}(x(t_0)), 0)$ . The latter can be made arbitrarily small for sufficiently small  $x(t_0)$ , hence we have also shown stability. Altogether, we have asymptotic stability for (1) satisfying Assumption 1 when a nondecreasing Lyapunov function in the sense of Definition 2.2 exists.

*Remark 2.3:* If  $\bar{V}$  is some norm on  $\mathbb{R}^n$  then the definition of a nondecreasing Lyapunov function is actually identical to the usual asymptotic stability definition (in the framework of  $\mathcal{KL}$ -functions). However,  $\bar{V}$  is not necessarily a norm and even if it is a norm there is more flexibility as one is not restricted to a given norm (e.g. the Euclidian norm usually used in the stability definition).

The popularity for the usage of Lyapunov functions to show stability of nonlinear systems is the possibility to show existence of a Lyapunov function without knowing the solutions. This is achieved by looking at the derivative of  $V$  along solutions which can be expressed by  $\nabla V(x)f(x)$  when the nonlinear system is given by  $\dot{x} = f(x)$ . This simple test is, however, not available anymore for switched or hybrid systems as differentiability (or even continuity) of the Lyapunov functions along solutions does not hold in general. There are numerous ways to deal with this problem. Some of these methods are rather involved due to the fact, that one still aims to construct a decreasing (common) Lyapunov function. As we have pointed out above this is an unnecessary restriction of Lyapunov function candidates. We therefore conjecture that it is in general much easier to find a nondecreasing Lyapunov function than it is to find a

usual Lyapunov function. One way towards this construction could be the following result:

*Lemma 2.4:* Consider the switched system (1) satisfying Assumption 1. Assume there is a function  $\widehat{V}$  satisfying the following:

(i) There exist  $\mathcal{K}_\infty$ -functions  $\widehat{\alpha}_1$  and  $\widehat{\alpha}_2$  such that

$$\widehat{\alpha}_1(\|x\|) \leq \widehat{V}(x) \leq \widehat{\alpha}_2(\|x\|) \quad \forall x \in \mathbb{R}^n.$$

(ii) For every trajectory  $x(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$  there exist a function  $\widehat{x}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$  and a  $\mathcal{K}$ -function  $\widehat{\alpha}_3$  such that:

$$\|x(t)\| \leq \widehat{\alpha}_3(\|\widehat{x}(t)\|) \quad \forall t \geq 0.$$

(iii) Furthermore, there exists a  $\mathcal{KL}$ -function  $\widehat{\beta}$  such that for  $\widehat{x}$  as above

$$\widehat{V}(\widehat{x}(t)) \leq \widehat{\beta}(\widehat{V}(\widehat{x}(t_0)), t - t_0) \quad \forall t \geq t_0.$$

Then  $\widehat{V}$  is a nondecreasing Lyapunov function for (1). In particular, finding a function  $\widehat{V}$  and  $\widehat{x}$  with the above properties ensures that the switched system (1) is asymptotically stable.

*Proof:* It suffices to construct  $\bar{\beta}$  such that (2) holds.

$$\begin{aligned} \widehat{V}(x(t)) &\leq \widehat{\alpha}_2(\|x(t)\|) \leq \widehat{\alpha}_2(\widehat{\alpha}_3(\|\widehat{x}(t)\|)) \\ &\leq \widehat{\alpha}_2(\widehat{\alpha}_3(\widehat{\alpha}_1^{-1}(\widehat{V}(\widehat{x}(t)))) \\ &\leq \widehat{\alpha}_2(\widehat{\alpha}_3(\widehat{\alpha}_1^{-1}(\widehat{\beta}(\widehat{V}(\widehat{x}(t_0)), t - t_0)))) \\ &=: \bar{\beta}(\widehat{V}(x(t_0)), t - t_0) \end{aligned}$$

This shows that  $\widehat{V}$  in addition to (3) also satisfies (4), i.e.  $\widehat{V}$  is a nondecreasing Lyapunov function; in particular, (1) is asymptotically stable. ■

### III. CONSTRUCTION OF A NONDECREASING LYAPUNOV FUNCTION FOR A GENERIC EXAMPLE CLASS

We consider the switched system

$$\dot{x} = A_{\sigma(x)}x \quad (5)$$

satisfying the following assumptions:

- (i) The state space is two dimensional, i.e.  $x(t) \in \mathbb{R}^2$ .
- (ii) The switching signal  $\sigma : \mathbb{R}^2 \setminus \{0\} \rightarrow \{1, 2, \dots, N\}$  is such that  $\mathcal{C}_i := \sigma^{-1}(i)$ ,  $i = 1, 2, \dots, N$ , are (disjoint) cones and  $\mathbb{R}^2 \setminus \{0\} = \bigcup_i \mathcal{C}_i$ . With only a slight restriction of generality, we can assume that there are pairwise distinct vectors  $c_1, c_2, \dots, c_{N-1}, c_N \in \mathbb{R}^n \setminus \{0\}$  and  $c_{N+1} := c_1$  arranged anti-clockwise such that

$$\mathcal{C}_i = \{ \lambda c_i + \mu c_{i+1} \mid \lambda > 0, \mu \geq 0 \},$$

i.e. the cone  $\mathcal{C}_i$  is bounded by the lines spanned by  $c_i \in \mathcal{C}_i$  and  $c_{i+1} \notin \mathcal{C}_i$ . In the following we will call  $c_i$  the *right* and  $c_{i+1}$  the *left border* of the cone  $\mathcal{C}_i$ . Furthermore we can assume that  $\langle c_i, c_{i+1} \rangle \geq 0$  (i.e. the angle between  $c_{i+1}$  and  $c_i$  is less or equal than  $90^\circ$ ), because if this is not the case for some  $i$  one could split the cone  $\mathcal{C}_i$  in the middle to reduce the angle between

the right and left border. This will introduce one new mode (but with the same dynamics as the mode  $i$ ).

- (iii) In each mode the dynamics flow from the right border of  $\mathcal{C}_i$  to the left border, i.e. for all  $i = 1, 2, \dots, N$  there exists  $\varepsilon > 0$  such that

$$c_i + \varepsilon A_i c_i \in \mathcal{C}_i \setminus \{c_i\} \quad \wedge \quad c_{i+1} - \varepsilon A_i c_{i+1} \in \mathcal{C}_i,$$

and let  $\Delta_i > 0$  denote the time a trajectory of mode  $i$  needs to reach the left border from the right border, i.e.

$$x_i(\Delta_i) = \gamma_i c_{i+1} \quad \text{for some } \gamma_i > 0, \quad (6)$$

where  $x_i$  is the solution of  $\dot{x}_i = A_i x_i$ ,  $x_i(0) = c_i$ . Due to linearity, for any  $\eta > 0$  it holds that  $x_i(\Delta_i) = \eta \gamma_i c_{i+1}$  if  $x_i(0) = \eta c_i$ , i.e.  $\Delta_i$  and  $\gamma_i$  do not depend on where the trajectory starts on the left border.

- (iv) There exists a norm  $\|\cdot\|$  on  $\mathbb{R}^2$  such that  $\|x_i(\Delta_i)\| < \|c_i\|$  where  $x_i$  is a solution of  $\dot{x}_i = A_i x_i$ ,  $x_i(0) = c_i$ . Without restriction of generality (just scale  $c_i$  accordingly) we assume  $\|c_i\| = 1$ . Under this rescaling we have  $\gamma_i < 1$  in (6).

- (v) For each sector  $\mathcal{C}_i$  the two “flow arrows” starting at  $c_i$  and ending at  $c_{i+1}$  intersect within  $\mathcal{C}_i$ , i.e. there exist  $\nu_i, \kappa_i > 0$  such that

$$c_i + \nu_i A_i c_i = c_{i+1} - \kappa_i A_i c_{i+1}.$$

In summary we consider (state-dependent) switched linear systems whose trajectory move anti-clockwise. The strongest and maybe most difficult to verify condition is assumption (iv). However, the desired inequality only has to hold for  $N$  pairs of points and might be easy to check in many practical examples. In particular, it is *not* assumed that  $t \mapsto \|x_i(t)\|$  is decreasing on the whole of the interval  $[0, \Delta_i]$ , i.e.  $\|\cdot\|$  is not a Lyapunov function in general. Nevertheless it is not difficult to show that under these assumptions the switched system is asymptotically stable by looking at the discretized system  $z_{i+1} = x_i(\Delta_i)$  where  $x_i$  solves  $\dot{x}_i = A_i x_i$ ,  $x_i(0) = z_i$  which is contracting in the norm  $\|\cdot\|$ . However, it is not easy to come up with a Lyapunov function for this switched system; in some cases, however, the approach in [7] might give a solution. One specific example, which cannot be handled by the approach in [7] is discussed in the next section.

A construction of a nondecreasing Lyapunov function under the above assumptions is however quite simple (for an illustration see Figure 1):

**Step 1.** For each cone  $\mathcal{C}_i$  consider the straight lines  $\bar{L}_i$  and  $\bar{R}_i$  given by

$$\begin{aligned} \bar{R}_i &:= \{ c_i + \nu A_i c_i \mid \nu > 0 \} \\ \bar{L}_i &:= \{ c_{i+1} - \kappa A_i c_{i+1} \mid \kappa > 0 \} \end{aligned}$$

**Step 2.** If the intersections of the lines  $\bar{L}_i$  and  $\bar{R}_i$  is outside the triangle given by  $0, c_i, c_{i+1}$  (Case A) then let  $\nu_i > 0$  and  $\kappa_i > 0$  be the corresponding parameters of the intersection and let

$$\begin{aligned} R_i &:= \{ c_i + \nu A_i c_i \mid \nu \in [0, \nu_i] \} \\ L_i &:= \{ c_{i+1} - \kappa A_i c_{i+1} \mid \kappa \in (0, \kappa_i] \}. \end{aligned}$$

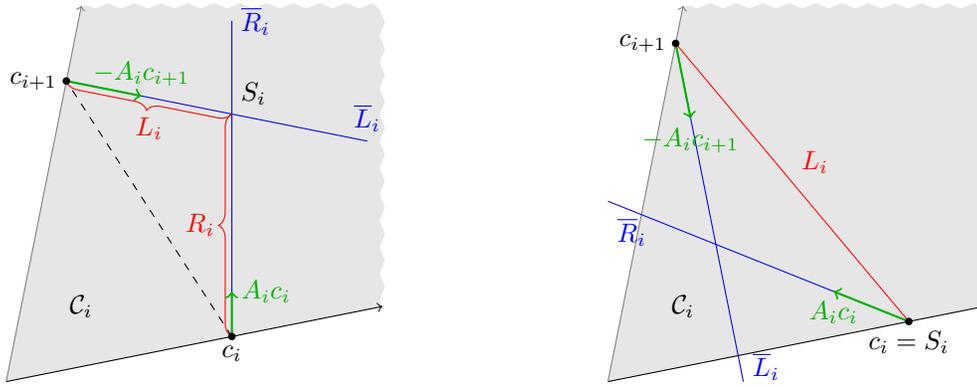


Fig. 1: Illustration of construction of  $\hat{V}$ , Case A left and Case B right.

Otherwise (Case B) let  $L_i$  be the line between  $c_i$  and  $c_{i+1}$ , i.e.

$$L_i := \{ \kappa c_i + (1 - \kappa)c_{i+1} \mid \kappa \in [0, 1) \}$$

and set  $R_i := \emptyset$ . Denote with  $S_i$  the right-most point of  $L_i$ , i.e. in the first case  $S_i$  is the intersection of  $\bar{R}_i$  and  $\bar{L}_i$  and in the second case  $S_i$  is just  $c_i$ .

**Step 3.** Consider the closed (piecewise-linear) curve  $C := R_1 \cup L_1 \cup R_2 \cup L_2 \cup \dots \cup R_N \cup L_N$  in  $\mathbb{R}^2$ . It is easily seen that there exists a homogenous, piecewise-linear function  $\hat{V}$  such that  $\hat{V}^{-1}(\{1\}) = C$ .

**Step 4.** Construct  $\hat{x}$  as follows (see also Figure 2): Let  $t_0 = 0$  and  $t_i = t_{i-1} + \Delta_i$ , for  $i \in \mathbb{N}$ , where for  $i > N$  we set  $\Delta_i := \Delta_{(i \bmod N)+1}$  and similar for  $\gamma_i$  and  $c_i$  in the following. Furthermore let  $\hat{x}_i$  be defined inductively as follows:  $\hat{x}_0 = x(0)$  and  $\hat{x}_{i+1} := \gamma_i \|\hat{x}_i\| c_{i+1}$ . Now  $\hat{x}$  is defined on each interval  $[t_i, t_{i+1})$  such that its graph coincides with the line through  $\hat{S}_i := \|\hat{x}_i\| S_i$  and  $\hat{x}_{i+1}$ . The specific time parametrization is irrelevant as long as  $\hat{x}(t)$  moves monotonically from  $\hat{S}_i$  to  $\hat{x}_{i+1}$  as  $t$  increases from  $t_i$  to  $t_{i+1}^-$ . Note that, for regions  $C_i$  where Case A from Step 2 holds  $\hat{x}$  has jumps at the switching time  $t_i$ .

**Theorem 3.1:** Consider the switched system (5) satisfying the assumptions (i)-(v). Then the above constructed function  $\hat{V}$  is a nondecreasing Lyapunov function for the switched system (5).

*Proof:* We are utilizing Lemma 2.4 to show that  $\hat{V}$  is a nondecreasing Lyapunov function. By construction,  $\hat{V}$  is a homogenous function with compact level sets containing the origin in its interior, hence  $\hat{V}$  is positive definite in the sense of (i) in Lemma 2.4. To show (ii) of Lemma 2.4 we first study the qualitative behavior of  $x(t)$  for  $t \in [t_i, t_{i+1})$ . For simplicity we assume that  $x(t_i) = c_i$ . By linearity the curvature of  $x(t)$  has constant sign, hence the curve  $x(t)$  remains within the region enclosed by the origin,  $c_i$ ,  $S_i$  and  $c_{i+1}$ , see Figure 2.

In fact, the curve even remains within the smaller region enclosed by the origin,  $c_i$ ,  $S_i$  and  $x(t_{i+1}^-) = \gamma_i c_{i+1}$ . Hence it is possible to construct  $\hat{x}(t)$  as in Step 4 above and to choose a  $\mathcal{K}$ -function  $\hat{\alpha}_3$  such that property (ii) of Lemma 2.4 holds. Furthermore, by construction of  $\hat{V}$  and  $\hat{x}$  it follows that  $\hat{V}(\hat{x}(t))$  is strictly decreasing on  $(t_i, t_{i+1})$  (just observe how  $\hat{x}$  intersects with the level curves of  $\hat{V}$ ). For Case A we further observe that the jump at the beginning is along the level curve of  $\hat{V}$ , hence  $\hat{V}(\hat{x}(t_i^-)) = \hat{V}(\hat{x}(t_i^+))$ . Since we only have finitely many regions and homogenous  $\hat{V}$  and  $\hat{x}$  is possible to find a  $\mathcal{KL}$ -function  $\hat{\beta}$  such that property (iii) of Lemma 2.4 holds. Therefore we have shown that the assumptions of Lemma 2.4 hold and  $\hat{V}$  is a nondecreasing Lyapunov function for (5). ■

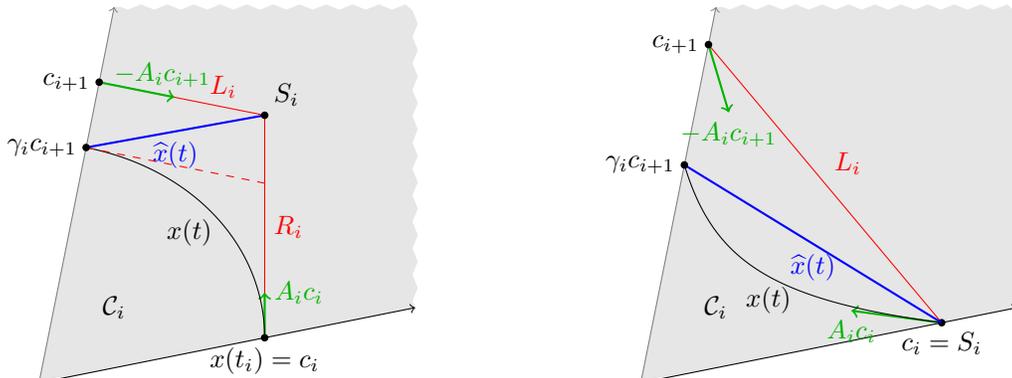


Fig. 2: Illustration of construction of  $\hat{x}$  and the proof idea of Theorem 3.1, Case A left and Case B right.

IV. A SPECIFIC EXAMPLE

We will illustrate the construction of a nondecreasing Lyapunov function from the previous section by the following specific example:

$$A_1 = A_3 = \begin{bmatrix} 1 & -5 \\ 0.2 & 1 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & -0.2 \\ 5 & 1 \end{bmatrix} \quad (7a)$$

and

$$\begin{aligned} \mathcal{C}_1 = -\mathcal{C}_3 &= \{ x \in \mathbb{R}^2 \mid x_1 > 0, x_2 \geq 0 \}, \\ \mathcal{C}_2 = -\mathcal{C}_4 &= \{ x \in \mathbb{R}^2 \mid x_1 \leq 0, x_2 > 0 \}. \end{aligned} \quad (7b)$$

We will show that this example satisfies the assumptions of (5) (with any standard  $p$ -norm) and the trajectories converge to zero as is shown in Figure 3.

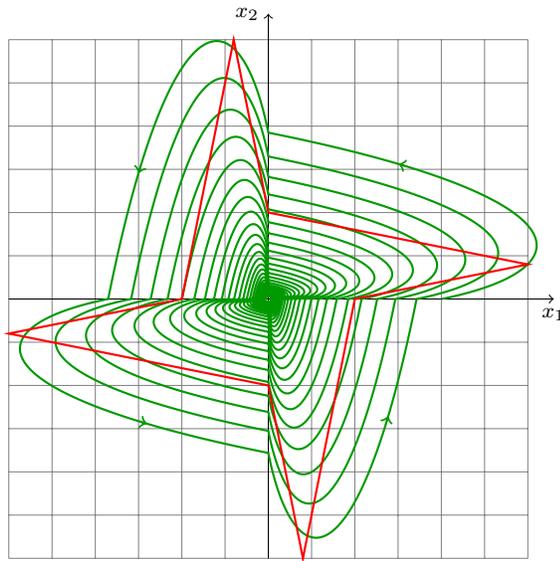


Fig. 3: Converging trajectory (green) of the switched system (5) given by (7) together with a level curve (red) of a nondecreasing Lyapunov function  $\hat{V}$ .

But first we would like to highlight that for this example it is impossible to construct a piecewise-quadratic Lyapunov function:

*Lemma 4.1:* Consider the switched system (5) given by (7). Then there does not exist a piecewise-quadratic Lyapunov function in the sense of [7].

*Proof:* We consider the Lyapunov function candidate given by

$$V(x) = V_i(x) \text{ for } x \in \mathcal{C}_i$$

where  $V_i(x) = x^\top P_i x$  for some symmetric  $P_i = \begin{bmatrix} a_i & b_i \\ b_i & c_i \end{bmatrix} \in \mathbb{R}^{2 \times 2}$  satisfying

$$x^\top P_i x > 0 \text{ and } x^\top (A_i^\top P_i + P_i A_i) x < 0 \quad \forall x \in \mathcal{C}_i.$$

First note that the unit vectors  $e_1, e_2$  or its negations are elements of the closure of  $\mathcal{C}_i$  for each  $i \in \{1, 2, 3, 4\}$ . Hence  $x^\top P_i x > 0$  for all  $x \in \mathcal{C}_i$  implies  $e_1^\top P e_1 = a_i \geq 0$  and

$e_2^\top P e_2 = c_i \geq 0$ . In the following it suffices to consider  $i \in \{1, 2\}$  because of symmetry. From

$$\begin{aligned} A_1^\top P_1 + P_1 A_1 &= \begin{bmatrix} 2a_1 + \frac{2b_1}{5} & 2b_1 - 5a_1 + \frac{c_1}{5} \\ 2b_1 - 5a_1 + \frac{c_1}{5} & 2c_1 - 10b_1 \end{bmatrix} \\ A_2^\top P_2 + P_2 A_2 &= \begin{bmatrix} 2a_2 + 10b_2 & 2b_2 - \frac{a_2}{5} + 5c_2 \\ 2b_2 - \frac{a_2}{5} + 5c_2 & 2c_2 - \frac{2b_2}{5} \end{bmatrix} \end{aligned}$$

and the same argument as above it follows that  $x^\top (A_i^\top P_i + P_i A_i) x < 0$  for  $x \in \mathcal{C}_i$  implies

$$\begin{aligned} 2a_1 + \frac{2b_1}{5} < 0, \quad 2c_1 - 10b_1 \leq 0, \\ 2a_2 + 10b_2 \leq 0, \quad 2c_2 - \frac{2b_2}{5} < 0. \end{aligned}$$

Simple rearrangements yield the following contradictions:

$$\begin{aligned} 0 \leq \frac{c_1}{5} \leq b_1 < -5a_1 \leq 0, \\ 0 \leq 5c_2 < b_2 \leq -\frac{a_2}{5} \leq 0. \end{aligned}$$

Hence a piecewise-quadratic Lyapunov function does not exist for this example (note that we did not even assume continuity of  $V$  to show the contradiction). ■

*Remark 4.2:* Note that we showed that the switched system (5) given by (7) does not have a piecewise-quadratic Lyapunov function whose pieces are given by the four cones  $\mathcal{C}_1, \dots, \mathcal{C}_4$ . In fact, it is possible to construct a piecewise-quadratic Lyapunov function if we allow more cones and use the procedure described in [6], [5], whose authors reported to us that their construction uses 108 cones for our specific example.

We now derive explicitly  $\hat{V}$  for the switched linear system (5) given by (7). In accordance with property (ii) of (5) we choose  $c_1 = -c_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $c_2 = -c_4 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ . Then assumption (iii) is satisfied for  $\Delta_i = \pi/2$  and  $\gamma := \gamma_i = e^{\pi/2/5} \approx 0.962$  for all  $i \in \{1, 2, 3, 4\}$ . In particular, any  $p$ -norm will satisfy (iv). Finally, assumption (v) holds for  $\nu = 2$  and  $\kappa_i = 3/5$  for all  $i \in \{1, 2, 3, 4\}$ . We can now construct  $\hat{V}$  and  $\hat{x}$  according to Steps 1-4 above. Because of symmetry it suffices to consider the first cone  $\mathcal{C}_1$  only. The lines  $\bar{R}_1$  and  $\bar{L}_1$  are given by

$$\bar{R}_1 = \left\{ \begin{pmatrix} \nu + 1 \\ \nu/5 \end{pmatrix} \mid \nu \geq 0 \right\}, \bar{L}_1 = \left\{ \begin{pmatrix} 5\kappa \\ 1 - \kappa \end{pmatrix} \mid \kappa \geq 0 \right\}.$$

The intersection of  $\bar{R}_1$  and  $\bar{L}_1$  is the point  $\begin{pmatrix} 3 \\ 2/5 \end{pmatrix}$  which lays outside the triangle whose corners are  $c_1, c_2$  and the origin; hence we are in Case A of the construction of  $\hat{V}$ , c.f. Figure 2. The nondecreasing Lyapunov function on  $\mathcal{C}_1$  is now defined to be constant on the lines between  $c_1$  and  $S_1 := \begin{pmatrix} 3 \\ 2/5 \end{pmatrix}$  as well as between  $S_1$  and  $c_2$ , c.f. Figure 3, i.e. for  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathcal{C}_1$

$$\hat{V}(x) = \begin{cases} x_1 - 5x_2, & \text{if } x_2 < \frac{2}{15}x_1, \\ x_1/5 + x_2, & \text{if } x_2 \geq \frac{2}{15}x_1 \end{cases}$$

and analogously for  $x \in \mathcal{C}_2 \cup \mathcal{C}_3 \cup \mathcal{C}_4$ . Finally, for  $x(0) = c_1$  the auxiliary trajectory  $\hat{x}(\cdot)$  on the time interval  $[0, \pi/2)$  is given by

$$\hat{x}(t) = \begin{cases} c_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, & \text{if } t = 0, \\ \begin{pmatrix} 3 \\ 2/15 \end{pmatrix} - \frac{t}{\pi/2} \begin{pmatrix} 3 \\ \gamma - 2/15 \end{pmatrix}, & \text{if } t \in (0, \pi/2) \end{cases}$$

and analogously for  $t > \pi/2$ .

With this choice, the following comparison functions  $\hat{\alpha}_1$  and  $\hat{\alpha}_2$  satisfy assumption (i) from Lemma 2.4 (where we take  $\|\cdot\| = \|\cdot\|_\infty$  to be the infinity norm):

$$\hat{\alpha}_1 \equiv \frac{1}{3}, \quad \hat{\alpha}_2 \equiv \frac{6}{5}$$

Considering the trajectory  $x(\cdot)$  with initial value  $x(0) = c_1$  we can conclude the following inequalities (c.f. Figure 3):

$$\forall t \in [0, \pi/2): \quad \|x(t)\| \leq 2 \quad \text{and} \quad \|\hat{x}(t)\| \geq \frac{2}{5}.$$

From these it follows in general that  $\hat{\alpha}_3 \equiv 5$  satisfies assumption (ii) of Lemma 2.4. Note that with an adequate time parametrization for  $\hat{x}$  and a more precise analysis one could actually choose  $\hat{\alpha}_3 \equiv 1$ . Finally, we define

$$\hat{\beta}_0(\eta, s) := \eta\gamma^k \left(1 - \frac{1-\gamma}{\pi/2}\right) (s - k\pi/2)$$

where  $k \in \mathbb{N}$  is such that  $s \in [k\pi/2, (k+1)\pi/2)$ . Then  $\hat{\beta}_0$  satisfies assumption (iii) of Lemma 2.4 under the assumption that  $x(t_0) = \eta c_1$  for some  $\eta > 0$ . For arbitrary initial values it suffices to scale  $\hat{\beta}_0$  accordingly, i.e.

$$\hat{\beta}(\eta, s) := \hat{\beta}_0(\eta, s)/\gamma$$

is one possible comparison function which makes assumption (iii) of Lemma 2.4 valid.

## V. CONCLUSION

We have presented the concept of a nondecreasing Lyapunov function which is suitable to investigate asymptotic stability of switched and hybrid systems. We proposed a construction of a piecewise-linear nondecreasing Lyapunov function for a class of switched systems and illustrated the usefulness with a specific example for which no piecewise-quadratic Lyapunov function exists. The aforementioned construction is rather simple and we claim that for a wide

variety of hybrid systems it is much simpler to construct a nondecreasing Lyapunov function than finding a usual (common) Lyapunov function.

## REFERENCES

- [1] Amir Ali Ahmadi and Pablo A. Parrilo. Non-monotonic Lyapunov functions for stability of discrete time nonlinear and switched systems. In *Proc. 47th IEEE Conf. Decis. Control, Cancun, Mexico*, pages 614–621. IEEE, 2008.
- [2] Franco Blanchini and Carlo Savorgnan. Stabilizability of switched linear systems does not imply the existence of convex Lyapunov functions. *Automatica*, 44(4):1166 – 1170, 2008.
- [3] Michael S. Branicky. Multiple Lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Trans. Autom. Control*, 43(4):475–482, 1998.
- [4] Raymond A. DeCarlo, Michael S. Branicky, Stefan Pettersson, and Bengt Lennartson. Perspectives and results on the stability and stabilizability of hybrid systems. *Proc. of the IEEE*, 88(7):1069–1082, 2000.
- [5] Raffaele Iervolino, Francesco Vasca, and Luigi Iannelli. Piecewise quadratic stability of uncertain conewise linear systems through cone-copositive programming. submitted for publication.
- [6] Raffaele Iervolino, Francesco Vasca, and Luigi Iannelli. A cone-copositive approach for the stability of piecewise linear differential inclusions. *Proc. 50th IEEE Conf. Decis. Control and European Control Conference ECC 2011, Orlando, USA*, 2011.
- [7] Mikael Johansson and Anders Rantzer. Computation of piecewise quadratic Lyapunov functions for hybrid systems. *IEEE Trans. Autom. Control*, 43(4):555–559, 1998.
- [8] Mircea Lazar. Flexible control Lyapunov functions. In *Proc. American Control Conf. 2009*, pages 102–107. IEEE, 2009.
- [9] Daniel Liberzon. *Switching in Systems and Control. Systems and Control: Foundations and Applications*. Birkhäuser, Boston, 2003.
- [10] Hai Lin and Panos J. Antsaklis. Stability and stabilizability of switched linear systems: A survey of recent results. *IEEE Trans. Autom. Control*, 54(2):308–322, 2009.
- [11] Kumpati S. Narendra and Jeyendran Balakrishnan. A common Lyapunov function for stable LTI systems with commuting  $A$ -matrices. *IEEE Trans. Autom. Control*, 29(12):2469–2471, December 1994.
- [12] Zhendong Sun and Shuzhi Sam Ge. *Switched linear systems. Communications and Control Engineering*. Springer-Verlag, London, 2005.
- [13] L. Vu and D. Liberzon. Common Lyapunov functions for families of commuting nonlinear systems. *Syst. Control Lett.*, 2005.
- [14] X. Zhao, L. Zhang, P. Shi, and M. Liu. Stability of switched positive linear systems with average dwell time switching. *Automatica*, 2012.