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Memristors: The Rise and the Fall(?)

Dimitri Jeltsema*

***Abstract*— In May 2008, HP Lab engineers announced their physical realization of the 'missing' fourth basic circuit element in electronics: the memristor. Not often a technological discovery attracted so much attention from the media. Apart from the wildest possible speculations on future applications in new non-volatile memory devices with human brain synthesizing properties and suggestions to rewrite the existing textbooks on circuit theory, the discovery met with much skepticism as well. In this talk, we review a collection of scientific statements, arguments, and counter examples that critically address the existence of memristors.**

I. Chua's Memristor

Since electronics was developed, engineers designed, analyzed, and synthesized circuits using combinations of three basic two-terminal elements: resistors, inductors, and capacitors. From a mathematical perspective, the behavior of each of these elements, whether linear or nonlinear, is described by relationships between two of the four electrical variables: voltage, current, charge, and magnetic flux(-linkage). A resistor is described by the relationship of current and voltage, a capacitor by that of voltage and charge, and an inductor by that of current and flux. But what about the relationship between charge and flux? As Professor Leon O. Chua (the inventor of the well-known chaotic Chua circuit) from the University of California, Berkeley, pointed out in his 1971 paper [2], a fourth element should be added to complete the symmetry; see Figure 1. He coined this 'missing' element the *memristor*. More specifically, if q denotes the charge and ϕ denotes the flux, then a two-terminal *charge-controlled* memristor is defined by the constitutive relationship

$$\phi = \hat{\phi}(q). \tag{1}$$

Since magnetic flux is the time integral of voltage V (Faraday's law), and charge is the time integral of current I, or equivalently, $V = \mathrm{d}\phi/\mathrm{d}t$ and $I = \mathrm{d}q/\mathrm{d}t$, we obtain, after differentiating (1) with respect to time, the more familiar expression

$$V = M(q)I, \tag{2}$$

where $M(q) := \mathrm{d}\hat{\phi}(q)/\mathrm{d}q$ is called the incremental or small-signal *memristance*. At first glance (2) shows that a two-terminal charge-controlled memristor behaves like a resistor described by Ohm's law. The difference, however, is that its resistance $M(q)$ is not a constant, but varies with the instantaneous value of the charge. Recalling that charge follows from the time integral of current, it thus records the past values of the current and hence motivates the name memory resistor, or memristor for short. It follows from (2) that the SI unit of memristance is the ohm [Ω], the same as that of resistance. Figure 2 shows the typical responses of a

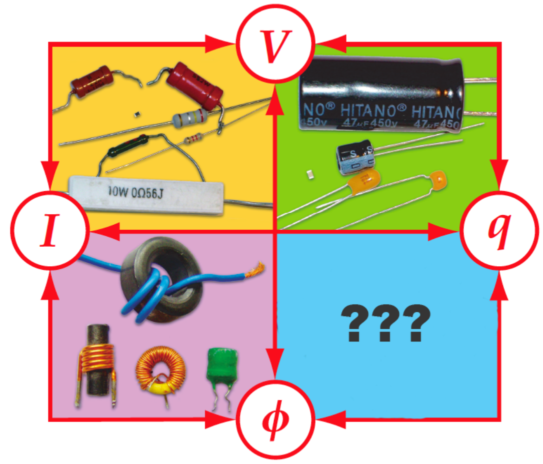

Fig. 1. A resistor is described by the relationship of current and voltage, a capacitor by that of voltage and charge, and an inductor by that of current and flux. There should exist an element that defines the relationship between charge and flux as to complete the symmetry.

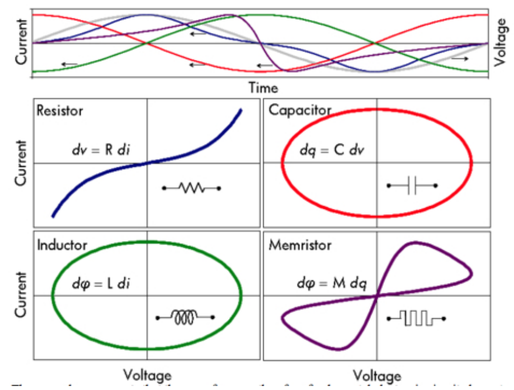

Fig. 2. The upper figure shows an applied voltage sine wave (gray) versus time with the corresponding current for a resistor (blue), capacitor (red), inductor (green), and memristor (purple). The lower figures show the current-voltage characteristics for the four basic circuit elements, with the characteristic pinched hysteresis loop of the memristor in the bottom right.

resistor, inductor, capacitor, and a memristor as a result of a sinusoidal voltage excitation.

Similarly, a two-terminal *flux-controlled* memristor (memductor) is defined by

$$q = \hat{q}(\phi), \tag{3}$$

Differentiation with respect to time yields

$$I = W(\phi)V, \tag{4}$$

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where $W(\phi) := \mathrm{d}\hat{q}(\phi)/\mathrm{d}\phi$ is called the incremental *memductance*. Clearly, the corresponding SI unit of memductance is the mho [℧] or Siemens [S], the same as that of conductance.

A. Pure two-terminal memristors

The charge-controlled and flux-controlled memristor representations can be combined into a state equation of the form

$$\begin{aligned} \dot{w} &= u, \\ y &= g(w)u, \end{aligned} \tag{5}$$

where $w \in \mathbb{R}$ denotes the internal state of the memristor (either charge or magnetic flux linkage), and $u \in \mathbb{R}$ and $y \in \mathbb{R}$ denote the input (current or voltage) and output (voltage or current), respectively. Any device that satisfies the equations (5) is in [2] referred to as a pure two-terminal (or one-port) memristor.

B. Linear versus nonlinear

Observe that (2) and (4) are just charge- and flux-modulated versions of Ohm's law, respectively. It is important to realize that for the special cases that the constitutive relations are linear, that is, when the incremental memristance M or the incremental memductance W is constant, a memristor or memductor becomes an ordinary resistor or conductor. Hence, memristors and memductors are only relevant in nonlinear circuits, which may account in part for their neglect in linear network and systems theory. Furthermore, it is directly noticed from (2) (resp. (4)) that $V \equiv 0$ (resp. $I \equiv 0$) whenever $I \equiv 0$ (resp. $V \equiv 0$), regardless of q (resp. ϕ) which incorporates the memory effect. This characteristic feature is the so-called *no energy discharge property* [5], which is related to the fact that, unlike an inductor or a capacitor, a memristor does not store energy.

C. A curious kind of pipe

In order to gain some intuition for what distinguishes a memristor from a resistor, as well as from an inductor or a capacitor, let us briefly consider the common analogy of an electrical resistor and a pipe that carries a fluid. The fluid can be considered analogous to charge, the pressure at the inlet of the pipe is similar to voltage, and the rate of flow of the fluid through the pipe is like current. As is the case with a resistor, the flow of fluid through the pipe is faster if the pipe is shorter or if it has a larger diameter and vice-versa.

Now, an analogy for a memristor is a peculiar kind of pipe that expands or shrinks when fluid flows through it. For example, if fluid flows through the pipe in one direction, the diameter of the pipe increases, thus enabling the fluid to flow faster. If fluid flows through the pipe in the opposite direction, the diameter of the pipe decreases, thus slowing down the flow of fluid. If the fluid pressure is turned off, the pipe retains its most recent diameter until the fluid pressure is turned back on. Unlike a bucket, which can be considered as a hydraulic capacitor, a memristive pipe does not store the fluid, but 'remembers' the amount of fluid that flowed through it. In the electrical domain this means that, like a capacitor, a memristor has a memory, but unlike a capacitor it does not store charge but just 'remembers' the last charge that passed through it.

D. Brother or distant cousin?

So a memristor is essentially a nonlinear element described by the same fundamental set of circuit variables as the passive two-terminal resistor, inductor, and capacitor. But does that give it the right to be just as fundamental as the latter familiar three circuit elements? This, of course, depends on how we (prefer to) look at it. From a linear perspective it is senseless to complement the linear circuit elements with a linear memristor as it precisely coincides with an ordinary resistor. In the realm of impedances it is clear that linear electronics is already complete in itself; linear resistors are purely real impedances, linear inductors and capacitors are merely the positive and negative purely imaginary impedances, and an impedance is not passive if its real part is negative. There is simply no room to complement that. One the other hand, apart from the fact that linear elements can be considered as a special case (small-signal or local approximation) of nonlinear elements, a few arguments in favor of the memristor as the fourth fundamental passive circuit element can be given as follows. A fundamental property of a resistor, inductor, and capacitor, whether linear or nonlinear, is that the values of their associated incremental or small-signal resistance, inductance, and capacitance, respectively, do not change with the frequency of an infinitesimally small sinusoidal variation about any fixed point of operation. The same property holds true for a memristor. Furthermore, from Figure 2 it is nearly obvious that there does not exist a combination of two-terminal *passive* resistors, inductors, and/or capacitors that duplicates the properties of a memristor (although including *active* elements like op-amps can do so). These features make the memristor just as fundamental as the existing three elements.

II. About HP Lab's device

In 2008, researchers at the Hewlett-Packard (HP) laboratories claimed to have found an analytical physical model for a genuine memristor device. HP's device is a two-terminal, two-layer semiconductor constructed from layers of titanium oxide TiO_2 (a substance we also find in toothpaste and suncream) sandwiched between two metal electrodes in a crossbar architecture. One layer of titanium oxide is doped with oxygen vacancies and the adjacent layer is undoped, leaving it in its natural state as an insulator. Under the influence of a bias voltage, oxygen vacancies move from the doped layer of titanium dioxide to the undoped layer. A high concentration of dopants results in a relatively low resistance. Likewise, if the polarity of the voltage is reversed, oxygen vacancies migrate back into the doped layer, thus turning to the region with relatively high resistance. The most typical feature of HP's device is that, after reversing the polarity of bias voltage, the current does not take the same reverse path, an effect we know as hystereses.

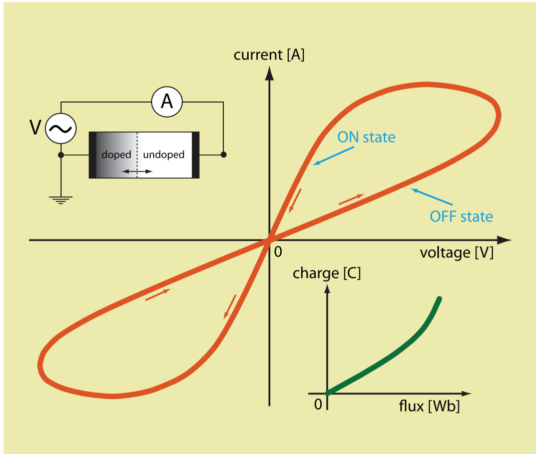

Fig. 3. Current-voltage plot demonstrating hysteretic phenomena of HP Lab's device.

An example of a typical current-voltage characteristic observed by the HP engineers is shown by the so-called Lissajous plot of Figure 3. Based on this characteristic, the HP researchers were motivated to relate their device to a genuine memristor. The originally proposed model for HP's device reads [12]

$$\begin{aligned} \dot{w} &= \mu_V \frac{R_{ON}}{D} I, \\ V &= -\left(R_{ON}\frac{w}{D} + R_{OFF}\left(1 - \frac{w}{D}\right)\right) I, \end{aligned} \tag{6}$$

where w denotes the internal state, D is the thickness of the film, μ_V represents the mobilities of the ionic defects, and R_{ON} and R_{OFF} are the electrical resistances in the ON and OFF region, respectively. At first sight (6) does not seem to coincide with the equations of a pure memristor (5) as $\dot{w} \neq I$. However, letting

$$w = \mu_V \frac{R_{ON}}{D} q,$$

and substituting the latter into (6) yields an equation set isomorphic to (5). It is straightforward to reveal that q indeed has the units of electric charge.[1] However, the first question that comes into mind: where is the (magnetic) flux linkage?

III. Criticism

In 2008, 'The missing memristor found' [12] was published in the respected science journal Nature and the memristor's supposed discovery was announced on the front pages of most major newspapers. This was indeed interesting news for physics, electrical engineering, and nanotechnology, the very field in which the devices in question were discovered. However, apart from the fact that the actually discovered device had been discovered before 2008 already, there are numerous of critical issues, not only about the practical, but also about the theoretical concepts. Below we briefly outline a few of the most prominent issues.

[1]In [12] the following units are used: R_{ON} and R_{OFF} are measured in [Ω], D in [nm], μ_V in [$cm^2V^{-1}s^{-1}$], and hence w in [m].

A. Originality of HP's discovery

The first criticism received by HP Lab's discovery is that memristors, or the memristance phenomenon in particular, already existed. Indeed, a variety of physical devices, including thermistors, discharge tubes, Josephson junctions, and even ionic systems, like the Hodgkin-Huxley model of a neuron, were shown to exhibit memristive effects. Furthermore, it is also known that there have been many researchers before who observed similar peculiar hysteretic current-voltage characteristics in various materials [1]. However, most of these observations were reported as anomalous or interpreted as difficult time-varying conductances, often leading to paradoxes and confusion.

B. Violation of Landauer's principle

In [6] it is noted that the device model proposed in [12] included a mistaken assumption regarding ionic conduction. Furthermore, the authors of [7] discuss various issues and problems in the realization of memristors. They claimed that the physics behind the HP memristor model conflicts with fundamentals of solid state electrochemistry as the coupling of electronic/ionic diffusion currents was not considered. Additionally, they pointed to issues concerning fundamentals of non-equilibrium thermodynamics: the dynamic state equations set up for memristors like the HP memristor imply the possibility of violating Landauer's principle of the minimum amount of energy required to change 'information' states in a system. This critique was endorsed by the authors of [10].

C. HP's device is not actually a true memristor

As pointed out by the HP researchers themselves, no one has (yet) been able to come up with a realistic physical model that satisfies the set of equations (5). Thus, a pure two-terminal memristor might be merely a concept which cannot be realized in our world. However, one can generalize the concept by giving up the idea that there is a direct functional relation between the memristance and the charge. By this means, one gets rid of the connotation between the total electrical charge that has been transported through a system up to some time t and the magnetic flux established somewhere in the system at time t. Indeed, as pointed out in [3], [5], memristors are a special case of a much broader class of dynamical systems called *memristive systems*. In contrast to the basic mathematical description of a pure memristor (5), the flux linkage in memristive systems is no longer uniquely defined by the charge, or vice-versa. In [3], a one-port memristive system is defined by

$$\begin{aligned} \dot{w} &= f(w, u), \\ y &= g(w, u)u. \end{aligned} \tag{7}$$

As for a pure two-terminal memristor (5), the internal state is denoted by w, and u and y denote the input and output, respectively. However, the internal state w does not necessarily have to be related to flux or charge. The main peculiarity which distinguishes a memristive system from an arbitrary dynamical system is the form of the output equations or readout maps. Indeed, as with (5), it is noticed from (7) that the output y is zero whenever the input u is zero, regardless of the state w which incorporates the systems memory effect, i.e., the 'no energy discharge property'.

D. Widrow's memistor

The majority of publications refer to the memristor as a mathematical model or entity that was discovered and made rigorous by Chua [2]. However, it is notable that in 1960 Bernard Widrow coined the term *memistor* (memory resistor) as a device defined by charge dependent conductance [13]. Widrow used the memistors concept to simulate electronic neurons. The main difference between Widrow's and Chua's memory resistors is that Widrow's memistor is a 3-terminal device and real while Chua's memristor is a 2-terminal theoretical device conceived ten years after Widrow's memistor.

E. Symmetry abandoned

The memristor, as defined by Chua [2], is based on a completion of the possible relationships between the four fundamental physical electrical variables: charge, magnetic flux, voltage, and current. As a capacitor relates voltage to physical electric charge and an inductor relates current to magnetic flux, the same physical electric charge and magnetic flux should be associated in the definition of a memristor in order to maintain a perfect symmetry between the four fundamental electrical variables. In Chua's original paper, the memristor is defined like that: an element that relates electrical charge with magnetic flux. Moreover, Chua originally insisted not only on the close relation between magnetism and the memristor, but on the interaction of both, the electric *and* the magnetic fields [2]:

"...an inductor has been identified to be an electromagnetic system where only the first-order magnetic field is negligible. [...] The remaining case where both first-order fields are not negligible has been dismissed as having no corresponding situation in circuit theory. We will now offer the suggestion that this missing combination is precisely that which gives rise to the characterization of a memristor."

However, after HP's discovery, the original suggested physical symmetry between the four fundamental electrical variables is broken and flux is just conveniently (re-)defined as the integral of voltage—without regard to physical meaning, see e.g., [5]. An interesting thought experiment assuming a world without magnetism is given in [11]. In such world inductors cannot exist, but memristors could still be constructed. On the same grounds as the memristor was historically predicted, an 'inductor' could then be predicted.

F. Quasi-static field perspective

It is well known that the circuit-theoretic definitions of resistance, inductance, and capacitance can be associated with electromagnetic systems operating in their quasi-static limit. From this point of view, a resistor corresponds to an electromagnetic system for which the first-order fields are negligible compared to its zero-order fields. Its low frequency behavior is then characterized by an instantaneous (memoryless) relationship between the zero-order electric and magnetic field intensities. Similarly, an inductor corresponds to an electromagnetic system for which both the zero-order electric field and the first-order magnetic field can be ignored. The behavior of an electromagnetic system for which both the zero-order magnetic field and the first-order electric field can be ignored corresponds to a capacitor. The fourth combination, in which both zero-order fields are negligible while the first-order fields are both relevant, naturally implies to correspond to a memristor type of device. Indeed, the latter situation gives rise to an instantaneous relationship between the first-order electric and magnetic field densities, $\mathbf{D}$ and $\mathbf{B}$, respectively, which in turn correspond to charge and flux. This would imply that in Maxwell's equations we have that $\mathrm{div}\mathbf{B} \neq 0$, a property that seems to be not unambiquously satisfied by any existing material yet.

G. A change of definition

In 2011, Chua changed his definition of memristors and states that *all* zero-crossing pinched hysteresis curves define memristors—regardless of any underlying charge-flux relationship [4]. However, as shown in [8], there actually exist many dynamical systems that do not satisfy either the equations of a pure memristor (5), or the equations of a memristive system (7), and yet also produce the same type of zero-crossing hysteresis curves claimed as a fingerprint for a memristor. Moreover, in algebraic geometry, the lemniscate of Gerono is a plane algebraic curve of degree four and genus zero and is a lemniscate curve shaped like an ∞ symbol. This establishes that zero-crossing hysteresis serves as insufficient evidence for a memristor or a memristive system.

References

[1] S. Chandra. On the Discovery of a Polarity-Dependent Memory Switch and/or Memristor (Memory Resistor). IETE Technical Review 27, 179-180, 2010.
[2] L.O. Chua. Memristor—the missing circuit element. IEEE Trans. on Circ. Theory, CT-18(2):507–519, September 1971.
[3] L.O. Chua and S.M. Kang. Memristive devices and systems. Proc. of the IEEE, vol. 64, no. 2, Feb. 1976.
[4] L.O. Chua. Resistance switching memories are memristors, Applied Physics A, vol 102, no. 4, Jan 2011.
[5] M. Di Ventra, Y. Pershin and L.O. Chua. Circuit elements with memory: memristors, memcapacitors and meminductors. Proc. of the IEEE, 97(10), 1717–1724.
[6] P. Meuffels and H. Schroeder. Comment on "Exponential ionic drift: fast switching and low volatility of thin-film memristors" by D.B. Strukov and R.S. Williams in Appl. Phys. A (2009) 94: 515-519", Applied Physics A 105: 65, 2011.
[7] P. Meuffels and R. Soni. Fundamental Issues and Problems in the Realization of Memristors, arXiv:1207.7319, 2012.
[8] B. Mouttet. Memresistors and non-memristive zero crossing hysteresis curves. arXiv:1201.2626v2, Jan. 2012
[9] Y. Pershin and M. Di Ventra. Memory effects in complex materials and nanoscale systems. Advances in Physics, 60(2), 145–227, 2011.
[10] M. Di Ventra and Y. Pershin. On the physical properties of memristive, memcapacitive and meminductive systems. Nanotechnology 24 (25): 255201, 2013.
[11] S. Vongehr. Missing the Memristor. Advanced Science Letters 17, 285-290, 2012.
[12] D.B. Strukov, G.S. Snider, D.R. Stewart, R.S. Wiliams. The missing memristor found. Nature, vol. 453, May 2008.
[13] B. Widrow. An adaptive adaline neuron using chemical memistors. Stanford Electronics Laboratories Technical Report 1553-2, Oct. 1960.