From Tensor to Coupled Matrix/Tensor Decomposition*

Laurent Sorber1 and Mikael Sørensen2 and Marc Van Barel3 and Lieven De Lathauwer4

Abstract—Decompositions of higher-order tensors are becoming more and more important in signal processing, data analysis, machine learning, scientific computing, optimization and many other fields. A new trend is the study of coupled matrix/tensor decompositions (i.e., decompositions of multiple matrices and/or tensors that are linked in one or several ways). Applications can be found in various fields and include recommender systems, advanced array processing systems, multimodal biomedical data analysis and data completion.

We give a short overview and discuss the state-of-the-art in the generalization of results for tensor decompositions to coupled matrix/tensor decompositions. We briefly discuss the remarkable uniqueness properties, which make these decompositions important tools for signal separation. Factor properties (such as orthogonality and triangularity, but also nonnegativity, exponential structure, etc.) may be imposed when useful but are not required for uniqueness per se. Also remarkable, in the exact case the decompositions may under mild conditions be computed using only tools from standard linear algebra. We touch upon the computation of exact decompositions via numerical optimization. We illustrate some of the ideas using Tensorlab, a Matlab toolbox for tensors and tensor computations that we have recently released, and of which version 2 is available in Tensorlab v2.0 [18].

I. TENSOR DECOMPOSITIONS

A canonical polyadic decomposition of a rank-\( R \) tensor \( A \in \mathbb{R}^{I_1 \times I_2 \times I_3} \) is a decomposition in a linear combination of \( R \)-rank-1 terms [1], [12]:

\[
A = \sum_{r=1}^{R} u_r^{(1)} \circ u_r^{(2)} \circ u_r^{(3)},
\]

where \( u_r^{(1)}, u_r^{(2)}, u_r^{(3)} \) are matrices of appropriate sizes.

A Tucker decomposition of a multilinear rank-\((R_1, R_2, R_3)\) tensor \( A \in \mathbb{R}^{I_1 \times I_2 \times I_3} \) is a decomposition of the form

\[
A = S \cdot_1 U_1^{(1)} \cdot_2 U_2^{(2)} \cdot_3 U_3^{(3)},
\]

where the matrices \( U_r^{(1)} \in \mathbb{R}^{I_1 \times R_1}, U_r^{(2)} \in \mathbb{R}^{I_2 \times R_2} \) and \( U_r^{(3)} \in \mathbb{R}^{I_3 \times R_3} \) have full column rank and where the tensor \( S \in \mathbb{R}^{R_1 \times R_2 \times R_3} \) [26], [27], [13], [2], [3].

A block term decomposition of a tensor \( A \in \mathbb{R}^{I_1 \times I_2 \times I_3} \) is a decomposition of \( A \) of the form

\[
A = \sum_{r=1}^{R} S_r \cdot_1 U_r^{(1)} \cdot_2 U_r^{(2)} \cdot_3 U_r^{(3)},
\]

where the matrices \( U_r^{(1)} \in \mathbb{R}^{I_1 \times R_{1r}}, U_r^{(2)} \in \mathbb{R}^{I_2 \times R_{2r}}, U_r^{(3)} \in \mathbb{R}^{I_3 \times R_{3r}} \) have full column rank and where the tensors \( S_r \in \mathbb{R}^{R_{1r} \times R_{2r} \times R_{3r}} \) have full multilinear rank [5], [6], [7].

II. COUPLED MATRIX/TENSOR DECOMPOSITIONS

Let a number of matrices and/or higher-order tensors be given. We assume that they can be decomposed as in (1), (3) or (5). We study the situation where these decompositions are coupled in the sense that each of them involves at least one factor that is shared with at least one of the other decompositions. This can be relaxed to the situation where factors only have generating variables in common.

We study the uniqueness of such coupled decompositions [21], [25], generalizing results from [14], [11], [9].

In several cases coupled decompositions can be computed exactly by means of standard linear algebra (i.e., essentially by solving (overdetermined) sets of linear equations and by computing (generalized) EVD) [22]. Here we generalize results obtained in [4], [10].

Uniqueness conditions and conditions for analytic computability can in some cases be alleviated by constraining factor matrices to be Vandermonde or orthogonal. We generalize results from [19], [20].

We discuss numerical optimization based computation [17], generalizing results from [15], [16]. Our new algorithms are available in Tensorlab v2.0 [18].

![Image](image.jpg)

*This work was supported by: (1) Research Council KU Leuven: GOA-MaNet, CoE EF/05/006 Optimization in Engineering (OPTEC), CIF1 and STSR1/08/023 (2) F.W.O.: projects G.0427.10N, G.0830.14N and G.0881.14N, (3) the Belgian Federal Science Policy Office: IUAP P7/19 (DYSCO, “Dynamical systems, control and optimization”, 2012–2017).

1Laurent Sorber is with the Stadius Center for Dynamical Systems, Signal Processing and Data Analytics of the Department of Electrical Engineering (ESAT), KU Leuven, Leuven, Belgium, with iMinds Future Health Department and with the Group NALAG of the Department of Computer Science, KU Leuven, Leuven, Belgium Laurent.Sorber@cs.kuleuven.be

2Mikael Sørensen is with the Group Science, Engineering and Technology of KU Leuven-Kulak, Kortrijk, Belgium, with the Stadius Center for Dynamical Systems, Signal Processing and Data Analytics of the Department of Electrical Engineering (ESAT), KU Leuven, Leuven, Belgium and with iMinds Future Health Department Mikael.Sorensen@kuleuven-kulak.be

3Marc Van Barel is with the Group NALAG of the Department of Computer Science, KU Leuven, Leuven, Belgium Marc.VanBarel@cs.kuleuven.be

4Lieven De Lathauwer is with the Group Science, Engineering and Technology of KU Leuven-Kulak, Kortrijk, Belgium, with the Stadius Center for Dynamical Systems, Signal Processing and Data Analytics of the Department of Electrical Engineering (ESAT), KU Leuven, Leuven, Belgium and with iMinds Future Health Department Lieven.DeLathauwer@kuleuven-kulak.be
We discuss applications in signal processing [23], [24], [25].

REFERENCES


