A stability/instability trichotomy for non-negative Lur’e systems*

Adam Bill 1, Chris Guiver 2, Hartmut Logemann 1 and Stuart Townley 2

Abstract—We identify a stability/instability trichotomy for a class of non-negative continuous-time Lur’e systems. Asymptotic as well as input-to-state stability concepts (ISS) are considered. The presented trichotomy rests on Perron-Frobenius theory, absolute stability theory and recent ISS results for Lur’e systems.

I. INTRODUCTION

Let $A = (a_{ij}) \in \mathbb{R}^{n \times n}$ and $b, c \in \mathbb{R}^n$ and consider the corresponding single-input single-output non-negative linear system

$$\dot{x} = Ax + bu, \quad x(0) = \xi \in \mathbb{R}_+^n; \quad y = c^T x. \tag{1}$$

We assume that

(A1) $A$ is Metzler, $b, c \in \mathbb{R}_+^n$ and $b, c \neq 0$ holds.

We recall that $A = (a_{ij})$ is Metzler if $a_{ij} \geq 0$ for $i \neq j$ (all off-diagonal elements are non-negative).

System (1) is said to be non-negative if (A1) holds and $u \geq 0$. Non-negative systems of the form (1) occur naturally in biological, ecological and economic contexts.

We impose the following assumptions.

(A2) $A$ is Hurwitz.

(A3) There exist non-negative numbers $\alpha$ and $\kappa$ such that $\alpha I + A + \kappa bc^T$ is primitive.

Recall that (A3) means that the matrix $(\alpha I + A + \kappa bc^T)^k$ is a positive matrix for some $k \in \mathbb{N}$.

In the following, let $G$ denote the transfer function of (1), that is, $G(s) := c^T (sI - A)^{-1} b$.

Lemma 1.1: Assume that (A1)-(A3) hold. Then $G(0) > 0$ and $\|G\|_{\infty} = G(0)$.

A proof of Lemma 1.1 can be found in [1].

Applying nonlinear non-negative feedback $u = f(y)$ to (1), where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz, leads to the following non-negative Lur’e system

$$\dot{x} = Ax + bf(c^T x), \quad x(0) = \xi \in \mathbb{R}_+^n. \tag{2}$$

We assume that the following assumption holds.

(A4) $f : \mathbb{R}_+ \to \mathbb{R}_+$ is locally Lipschitz and $f(0) = 0$.

II. LYAPUNOV STABILITY RESULTS

In this section, we present results which describe the stability/instability properties in each of three cases, where “stability” is interpreted in the sense of Lyapunov.

Let $x(\cdot ; \xi)$ denote the unique maximally defined forward solution of (2) with maximal interval of existence $[0, \omega_\xi)$, where $0 < \omega_\xi \leq \infty$.

The proposition below relates to Case 1. It follows from well known results in absolute stability theory, see, for example, [3].

Proposition 2.1: Assume that (A1)-(A4) hold.

(a) If $f(z)/z \leq p$ for all $z > 0$, then the equilibrium 0 is stable in the large in the sense that there exists $\Gamma \geq 1$ such that, for every $\xi \in \mathbb{R}_+^n$, $\omega_\xi = \infty$ and

$$\|x(t; \xi)\| \leq \Gamma \|\xi\| \quad \forall t \geq 0.$$
(b) If \( f(z)/z < p \) for all \( z > 0 \), then the equilibrium 0 is globally asymptotically stable. In particular, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to 0 \) as \( t \to \infty \).

(c) If \( \sup_{z>0} f(z)/z < p \), then the equilibrium 0 is globally exponentially stable, that is, there exist \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and
\[
\|x(t; \xi)\| \leq Ne^{-\nu t}\|\xi\| \quad \forall t \geq 0.
\]

In Case 2, the solutions of (2) diverge to \( \infty \) for every non-zero initial condition. More precisely, we have the following result.

**Theorem 2.2:** Assume that (A1)-(A4) hold and let \( \xi \in \mathbb{R}_+^n \), \( \xi \neq 0 \). Then, there exists \( N \geq 1 \) such that \( x(t; \xi) \to \infty \) as \( t \to \infty \).

We proceed to consider Case 3.

**Theorem 2.3:** Assume that (A1)-(A4) hold.

(a) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and
\[
\left| \frac{f(z) - f(y^*)}{z - y^*} \right| \leq p \quad \forall z \geq 0, \; z \neq y^*
\]
then \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) is an equilibrium of (2) and \( x^* \) is stable in the sense that there exists \( \Gamma \geq 1 \) such that, for every \( \xi \in \mathbb{R}_+^n \), \( \omega_\xi = \infty \) and
\[
\|x(t; \xi) - x^*\| \leq \Gamma \|\xi - x^*\| \quad \forall t \geq 0.
\]

(b) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and
\[
\left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall z \geq 0, \; z \neq y^*
\]
then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is globally asymptotically stable in the sense that \( x^* \) is stable in the large (see statement (a) of this theorem) and, for every \( \xi \in \mathbb{R}_+^n \) such that \( \xi \neq 0 \), \( \omega_\xi = \infty \) and \( x(t; \xi) \to x^* \) as \( t \to \infty \).

(c) If there exists \( y^* > 0 \) such that \( f(y^*) = py^* \),
\[
\left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p \quad \forall z > 0, \; z \neq y^*
\]
and
\[
\limsup_{y \to y^*} \left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p,
\]
and if
\[
\liminf_{z \to 0} \frac{f(z)}{z} > p,
\]
then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is “semi-globally” exponentially stable in the sense that, for every compact set \( K \subset \mathbb{R}_+^n \) with \( 0 \notin K \), there exists \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in K \), \( \omega_\xi = \infty \), and
\[
\|x(t; \xi) - x^*\| \leq Ne^{-\nu t}\|\xi - x^*\| \quad \forall t \geq 0.
\]

(d) If (3) holds and there exists \( y^* > 0 \) such that \( f(y^*) = py^* \) and, for every \( \varepsilon > 0 \),
\[
\sup_{z > \varepsilon, z \neq y^*} \left| \frac{f(z) - f(y^*)}{z - y^*} \right| < p,
\]
then \( 0 \) and \( x^* = -pA^{-1}by^* \in \mathbb{R}_+^n \) are the only equilibria of (2) and \( x^* \) is “quasi-globally” exponentially stable in the sense that, for every \( \delta > 0 \) there exist \( N \geq 1 \) and \( \nu > 0 \) such that, for every \( \xi \in \mathbb{R}_+^n \) with \( \|\xi\| \geq \delta \), \( \omega_\xi = \infty \) and
\[
\|x(t; \xi) - x^*\| \leq Ne^{-\nu t}\|\xi - x^*\| \quad \forall t \geq 0.
\]

We remark that “global” exponential stability of \( x^* \) (in the sense that there exist \( N \geq 1 \) and \( \nu > 0 \) such that (4) is satisfied for all \( \xi \in \mathbb{R}_+^n \) with \( \xi \neq 0 \)) does not hold. This is an immediate consequence of the following result which follows from continuity properties of the flow generated by the Lur’e system (2).

**Proposition 2.4:** Assume that (A1)-(A4) hold and that there exists \( y^* > 0 \) such that \( f(y^*) = py^* \). Then, for every sequence \( t_n \in \mathbb{R}_+ \) such as \( t_n \to \infty \) as \( n \to \infty \), there exists a sequence \( (\xi_n) \) in \( \mathbb{R}_+^n \) with \( \xi_n \neq 0 \) and \( \xi_n \to 0 \) as \( n \to \infty \) and such that
\[
\lim_{n \to \infty} \frac{\|x(t_n; \xi_n) - x^*\|}{\|\xi_n - x^*\|} = 1,
\]
where \( x^* = -pA^{-1}by^* \).

Discrete-time results similar to statement (b) of Proposition 2.1, Theorem 2.2 and statement (b) of Theorem 2.3 can be found in [7].

Proofs of the results in Section II can be found in [1].

**III. INPUT-TO-STATE STABILITY RESULTS**

Finally, we investigate the stability behaviour of (2) subject to non-negative disturbances, that is, we analyze the input-to-state stability (ISS) properties of the forced Lur’e system
\[
\dot{x} = Ax + b(f(x^T x) + d), \quad x(0) = \xi \in \mathbb{R}_+^n, \quad (5)
\]
where \( d : \mathbb{R}_+ \to \mathbb{R}_+ \) is locally essentially bounded. The unique maximally defined forward solution of (5) is denoted by \( x(\cdot; \xi, d) \).

For an overview of ISS theory, the reader is referred to [6]. We recall some terminology and notation relating to comparison functions. Let \( K \) denote the set of all continuous functions \( \varphi : \mathbb{R}_+ \to \mathbb{R}_+ \) such that \( \varphi(0) = 0 \) and \( \varphi \) is strictly increasing. Moreover, define \( \mathcal{K}_{\infty} := \{ \varphi \in K : \lim_{s \to \infty} \varphi(s) = \infty \} \). We denote by \( \mathcal{K} \) the set of functions in two variables \( \psi : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \) with the following properties:
\[
\psi(\cdot, t) \in K \text{ for all } t \geq 0, \text{ and } \psi(s, \cdot) \text{ is nonincreasing with } \lim_{s \to \infty} \psi(s, t) = 0 \text{ for all } s \geq 0.
\]

The following proposition is a consequence of recent ISS results for Lur’e systems, see [4], [5].

**Proposition 3.1:** Assume that (A1)-(A4) hold. If there exists \( \rho \in \mathcal{K}_{\infty} \) such that
\[
f(z) \leq pz - \rho(z) \quad \forall z \geq 0,
\]
then the equilibrium 0 of the unforced Lur’e system (2) is ISS in the sense that there exist ψ ∈ KL and ϕ ∈ K such that for all $\xi \in \mathbb{R}^n_+$ and all non-negative $d \in L^\infty_{loc}(\mathbb{R}^+)$, $x(\cdot; \xi, d)$ is defined on $\mathbb{R}^+$ and

$$\|x(t; \xi, d)\| \leq \psi(||\xi||, t) + \varphi(||d||_{L^\infty(0,1)}) \quad \forall t \geq 0.$$ 

The following lemma shows that, under suitable assumptions, the equilibrium $x^*$ has stability properties which are similar to ISS.

**Theorem 3.2:** Assume that (A1)-(A4) hold and that there exists $y^* > 0$ such that $f(y^*) = py^*$ and

$$\left|\frac{f(z) - f(y^*)}{z - y^*}\right| < p \quad \forall z > 0, \ z \neq y^*, \quad (6)$$

Furthermore, assume that (3) holds and

$$pz - f(z) \to \infty \quad \text{as} \quad z \to \infty. \quad (7)$$

Then $0$ and $x^* = -pA^{-1}by^* \in \mathbb{R}^n_+$ are the only equilibria of the unforced Lur’e system (2) and $x^*$ is “quasi ISS” in the sense that, for every $\delta > 0$, there exist $\psi \in KL$ and $\varphi \in K$ such that for all $\xi \in \mathbb{R}^n_+$ with $||\xi|| \geq \delta$ and all non-negative $d \in L^\infty_{loc}(\mathbb{R}^+)$, $x(\cdot; \xi, d)$ is defined on $\mathbb{R}^+$ and

$$\|x(t; \xi, d) - x^*\| \leq \psi(||\xi - x^*||, t) + \varphi(||d||_{L^\infty(0,1)}) \quad \forall t \geq 0.$$ 

(8)

To relate the conditions (6) and (7) to those in Proposition 3.1, we note that if (6) and (7) hold, then, for every $\varepsilon > 0$, there exists $\rho \in K_\infty$ such that

$$|f(z) - f(y^*)| \leq \rho|z - y^*| - \rho|z - y^*| \quad \forall z \geq \varepsilon, \ z \neq y^*.$$ 

The proof of Theorem 3.2 is based on Proposition 3.1 and the following lemma.

**Lemma 3.3:** Assume that (A1)-(A4) hold. If (3) is satisfied and there exists $y^* > 0$ such that $f(y^*) = py^*$ and (6) holds, then, for every $\delta > 0$, there exist constants $\eta > 0$ and $\tau \geq 0$ such that for all $\xi \in \mathbb{R}^n_+$ with $||\xi|| \geq \delta$ and all non-negative $d \in L^\infty_{loc}(\mathbb{R}^+)$, $x(\cdot; \xi, d)$ is defined on $\mathbb{R}^+$ and

$$c^T x(t; \xi, d) \geq \eta \quad \forall t \geq \tau.$$ 

This lemma also plays a key role in the proof of statements (b)-(d) of Theorem 2.3 (with disturbance $d = 0$). Detailed proofs of Proposition 3.1, Theorem 3.2 and Lemma 3.3 can be found in [1].

Finally, it follows from Proposition 2.4 that “global” ISS of $x^*$ (in the sense that there exist $\psi \in KL$ and $\varphi \in K$ such that (8) is satisfied for all $\xi \in \mathbb{R}^n_+$ with $\xi \neq 0$ and all non-negative $d \in L^\infty_{loc}(\mathbb{R}^+)$) does not hold.

**References**


