21st International Symposium on
Mathematical Theory of Networks and Systems
July 7-11, 2014. Groningen, The Netherlands

Extended abstract: Time minimization versus energy minimization in the one-input controlled Kepler problem with weak propulsion

Bernard Bonnard[1], Helen C. Henninger [2] and Jean-Baptiste Pomet[2]
Keywords: Low thrust transfer, minimum-time control. AMS subject classification: 49K30 , 49J15[‡]

Let q be the position of a satellite in an earth-centered reference frame e_i, e_j, e_k. We consider the one-input controlled Kepler equation describing orbital transfers normalized as

$$\ddot{q} = -\frac{q}{\|q\|^3} + u \tag{1}$$

in the case of weak propulsion: that is, the control magnitude is constrained by $|u| \leq \varepsilon$, a small parameter. If K is the mechanical energy of the uncontrolled system, $K = \frac{1}{2}\|\dot{q}\|^2 - 1/\|q\|$ and $C = q \wedge \dot{q}$ is the momentum of the uncontrolled system ($\wedge$ is the standard cross-product of three-vectors), then the elliptic domain in the $(q, \dot{q})$ space is

$$X = \{K < 0,\, C \neq 0\}.$$

This domain is called the elliptic domain because it is foliated by ellipses of the free motion. On it, one can choose coordinates x related to first integrals of the uncontrolled motion that describe the geometry of the foliating ellipses. Rescaling the control with $u = \varepsilon v$ to introduce the small parameter, there are two minimization problems which can be associated to the system (1), both of which are which are equally applicable e.g. for the launch of satellites or space vehicles: energy minimization and time minimization.

The energy minimization problem has already been studied in [1], however, the corresponding time minimization problem has not yet been considered. We provide a qualitiative analysis of the time-minimal case, covering the same topics of geodesic convexity and smoothness of trajectories considered in [1]. To do this, we make use of the Pontryagin principle [2], and use averaging with respect to a fast variable to provide an approximation of the trajectories. This kind of averaging process is covered in ([1], [3]-[8]). It is appropriate in this problem because of the assumption of weak propulsion ([9]-[10]); in general, high propulsion transfer problems require very different optimization techniques and produce different (often simpler) optimal control strategies to low propulsion transfer problems, for example as in [11].

As well as the coordinate x, we use the longitude l defining the position of the spacecraft on its orbits. In these coordinates, the system (1) restricted to the planar case (no motion in the e_k-direction) can be written as

$$\frac{dx}{dt} = u_t F(x,l) \quad \frac{dl}{dt} = \Omega(x,l). \tag{2}$$

This system is represented in the tangential/normal frame and the control u_t is in the tangential direction. The planar system (2) under tangential control is still fully controllable in the elliptic domain as is the planar two-input case in this domain [13].

One way to introduce averaging is to use the so-called ‘mean eccentric anomaly’. The eccentric anomaly E is related to l and the orbital elements e (eccentricity) and ω (argument of the periapsis) by

$$\tan\frac{(l-\omega)}{2} = \sqrt{\frac{1+e}{1-e}}\tan\frac{E}{2} \tag{3}$$

and the mean eccentric anomaly is $E - e\sin E$; the Kepler equation (third Kepler law) implies, when the control is zero,

$$E - e\sin E = nt\,,$$

$t = 0$ being the time at the pericenter, and n the mean motion. Introducing $x_0 = (E - e\sin E)/n$, one has $\dot{x}_0 = 1$ if $u_t = 0$. In the coordinates (x, x_0), the system (2) becomes

$$\frac{dx}{dt} = u_t F'(x,x_0), \quad \frac{dx_0}{dt} = 1 + u_t G(x,x_0).$$

**This work was partially supported by Thales Alenia Space

†1Bernard Bonnard is with Institut de Mathématiques de Bourgogne, 9 avenue Savary, 21078 Dijon, France bernard.bonnard@u-bourgogne.fr

‡2 Helen C. Henninger and Jean-Baptiste Pomet are with INRIA Sophia Antipolis - Mediteranée, 2004, route des Lucioles - BP 93 06902 Sophia Antipolis Cedex, France helen_clare.henninger@inria.fr

The trajectories parameterized by x_0 are solutions of

$$\frac{dx}{dx_0} = \frac{\varepsilon v F'(x,x_0)}{1+\varepsilon v G(x,x_0)},$$

which is approximated for small ε by

$$\frac{dx}{dx_0} = \varepsilon v F'(x,x_0).$$

In this parametrization, the energy and time minimization problems become respectively

- energy : $\mathrm{Min}_v \int_0^{x_0} \varepsilon^2 v^2$
- time : $\mathrm{Min}_v \ x_0, |v| \leq 1$.

Applying the Pontryagin maximum principle leads to the analysis of the Hamiltonians

$$\begin{aligned} H_E(x,p,x_0) &= H'(x,p,x_0)^2 \\ H_T(x,p,x_0) &= \sqrt{H'(x,p,x_0)^2} \end{aligned}$$

in the energy and time cases, respectively, where $H'(x,p,l) = \frac{1}{2}(\langle p, F'(x,l)\rangle)^2$, $\langle p, F'(x,l)\rangle$ is the Hamiltonian lift of the vector field $F'(x,l)$, and H' is periodic with respect to x_0 with period $2\pi/k$. The respective averaged Hamiltonians are

$$\begin{aligned} \bar{H}_E(x,p) &= \frac{k}{2\pi}\int_0^{2\pi/k} H_E(x,p,x_0)dx_0 \\ \bar{H}_T(x,p) &= \frac{k}{2\pi}\int_0^{2\pi/k} H_T(x,p,x_0)dx_0. \end{aligned}$$

In the energy case, the Hamiltonian $\bar{H}_E$ is associated to a Riemannian metric whose coefficients can be computed explicitly. This method was exploited in [1] using a C^0- approximation in a time duration $1/\varepsilon$; the geodesic flow is Liouville integrable and the orbital transfer towards circular orbits is related to a flat metric in suitable explicit coordinates. Hence in this case the minimizing solutions can be easily computed (straight lines in such coordinates).

Despite the formal analogy with the energy case, the Hamiltonian $\bar{H}_T$ is associated to a nonsmooth Finsler metric [12]. Due to technical problems in going from Riemannian to nonsmooth Finsler geometry the computations of time-optimal transfer towards circular orbits is a complicated problem. Particularly, complicated extremals are related to the switching surface Σ : $H_T = 0$. Observe that in the single-input case the control is given by $u_t = \mathrm{sign}\,(H_T(x,p))$ and meeting the surface Σ transversally corresponds to a regular switching [13]. More complicated singularities can occur in the non-transversal case, for instance in relation with singular trajectories of the system (contained by definition in the surface Σ).

We make a qualitative description of the time minimum transfers under single input, and compare this to the single-input energy case as computed in [1]. The variable $x = (n,e)$ (only two variables are needed because the transfer is planar and to circular orbits). These main results were determined:

Theorem. *The Hamiltonian $\bar{H}_T$ is C^1 but not C^2 on $\mathscr{D} = \{(n,e,p_n,p_e) \in (-1,1)\times\mathbb{R}^3 \ : \ \frac{3np_n}{-3nep_n-2(1-e^2)p_e} = 1\}$. At every other point $(n,e,p_n,p_e) \in ((-1,1)\times\mathbb{R}^3)\backslash\mathscr{D}$, it is smooth. There are exactly 2 directions of the covector at which the singularity occurs for any fixed $(n,e) \in \mathbb{R}\times(-1,1)$.*

Theorem. *For any (n^0,e^0) and (n^1,e^1) in the elliptic domain $\mathscr{X} = \{(n,e),\ 0 < n < +\infty,\ -1 < e < 1\}$, there exist a time $T \geq 0$ and a solution $t \mapsto (n(t),e(t),p_n(t),p_e(t))$ of the associated Hamiltonian system*

$$\dot{n} = \frac{\partial \bar{H}_T}{\partial p_n},\ \dot{e} = \frac{\partial \bar{H}_T}{\partial p_e},\ \dot{p}_n = -\frac{\partial \bar{H}_T}{\partial n},\ \dot{p}_e = -\frac{\partial \bar{H}_T}{\partial e}$$

defined from $[0,T]$ to $\mathscr{X}$, such that $(n(0),e(0)) = (n^0,e^0)$ and $(n(T),e(T)) = (n^1,e^1)$.

We have shown that in the time-minimal case, trajectories are nonsmooth in two distinct directions of the adjoint vector. The convexity result demonstrates that, in contrast to the energy-minimization problem under the transfer to circular orbits, any final circular orbit may be reached time-optimally from any initial elliptic orbit under control in only the tangential direction. Thus there are orbital manoeuvres that can be made optimally under time minimization which cannot be made optimally when energy is minimized. Indeed, by [1], only transfers to a fairly restricted subset of circular orbits may be performed in a way that optimizes energy. Thus time-optimization strategies for satellite trajectories appear to be more flexible than energy-optimal strategies in that they allow transfers to a much wider range of final circular orbits. Analysis of trajectories for this single-input case also gives a relevant insight into the 'full' (two-input) planar time-minimal case.

References

[1] B. Bonnard, J.-B. Caillau and R. Dujol. Energy minimization of single input orbit transfer by averaging and continuation. *Bull. Sci. Math.* 130(8) (2006), 707–719.

[2] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko. The mathematical theory of optimal processes. Translated from the Russian by K. N. Trirogoff; edited by L. W. Neustadt. Interscience Publishers John Wiley & Sons, Inc. New York-London, 1962.

[3] A. Bombrun. *Les Transferts Orbitaux à Faible Pousée: Optimalité et Feedback*. Thèse de doctorat, Ecole des Mines de Paris, Mar 2007.

[4] A. Bombrun and J.-B. Pomet. The averaged control system of fast oscillating control systems. *SIAM J. Control Optim.*, 51(3):2280-2350, 2013.

[5] B. Bonnard and J.-B. Caillau. Geodesic flow of the averaged controlled Kepler equation. *Forum Mathematicum*. 21(50) (2009), 797-814, Sept 2009

[6] F. Chaplais. Averaging and deterministic optimal control. *SIAM J. Control Optim*. 25(3): 767-780, 1987.

[7] S. Geffroy. *Généralisation des Techniques de Moyennation en Contrôl Optimale - Application aux Problèmes de Rendez-Vous Orbitaux en Poussée Faible*. Thèse de doctorat, Institute national polytechnique de toulouse, Toulouse, France, Oct 1997.

[8] S. Geffroy and R. Epenoy. Optimal low-thrust transfers with constraints - generalization of averaging techniques. *Acta astronautica*, 41(3):133-149, 1997.

[9] T. N. Edelbaum. Optimum low-thrust rendezvous and station-keeping. *AIAA J.*, 2:1196-1201, 1964.

[10] T. N. Edelbaum. Optimum power-limited orbit transfer in strong gravity fields. *AIAA J.*, 3: 921-925, 1965.

[11] T. N. Edelbaum. Propulsion requirements for controllable satellites. *ARS J.*, 31: 1079-1089, 1961.

[12] D. Bao, S.-S. Chern, and Z. Shen. *An introduction to Riemann-Finsler geometry*, Volume 200 of Graduate Texts in Mathematics. Springer-Verlag, New York, 2000.

[13] B. Bonnard and D. Sugny. *Optimal Control with Applications in Space and Quantum Dynamics* volume 5 of *AIMS Series on Applied Mathematics*. AIMS, 2012.