EXTREMAL NORMS FOR POSITIVE LINEAR INCLUSIONS.

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Abstract. We consider the joint spectral radius of sets of matrices for discrete or continuous positive linear inclusions and study associated extremal norms. We show that under a matrix-theoretic notion of irreducibility there exist absolute extremal norms. This property is used to extend regularity results for the joint spectral radius. In particular, we see that in the case of positive systems irreducibility in the sense of nonnegative matrices, which is weaker than the usual representation theoretic concept, is sufficient for local Lipschitz properties of the joint spectral radius.

1. Introduction. The joint spectral radius of sets of matrices for discrete or continuous positive linear inclusions and associated extremal norms have been studied in [7, 13, 19, 20, 6, 14, 15, 4], see also the survey [17]. The joint spectral radius plays a key role in characterising growth rates of solutions of inclusions and hence in their stability analysis [10]. Motivated by the practical importance of systems whose state variables are constrained to remain non-negative (given non-negative initial conditions), there has been significant interest recently in studying positive inclusions and positive switched systems [5, 8].

It was shown in [8] that the stability of a 2-dimensional positive inclusion is equivalent to the stability of the convex hull of its associated matrices. Unfortunately this result fails to be true in general and a specific counterexample for 3-dimensional systems has been described in [5]. In studying general linear inclusions, the concepts of extremal and Barabanov norms play an important role as such norms can be used to explicitly characterise the growth rate of an inclusion. Barabanov norms for positive linear inclusions have been recently considered in [18]; an explicit closed-form expression for a Barabanov norm for 2-dimensional discrete positive inclusions is derived.

In this paper, we consider extremal norms for positive inclusions in both the discrete and continuous case. We show that for positive inclusions, a matrix-theoretic notion of irreducibility is sufficient for the existence of an extremal norm. This is novel as the concept of irreducibility we consider is weaker than that used to establish the existence of extremal norms in the work of Barabanov, [1] and others. Our result also relates the existence of an extremal norm to a property of the convex hull of the matrices associated with the inclusion.

We use the above results to extend regularity results for the joint spectral radius [12, 19]. In the case of positive systems irreducibility in the sense of nonnegative matrices is sufficient for local Lipschitz properties of the joint spectral radius. We emphasise that this notion of irreducibility is distinct from that utilised for general inclusions. Our results show that an interesting observation of [11] for the case of nonnegative 2×2 matrices is a consequence of a general property of sets of nonnegative matrices and monotone norms.

In [16] positivity with respect to arbitrary positive cones has been considered. We expect that it will pose no major obstacle to transfer the results of the present paper to general proper cones.

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1.1. Preliminaries. Throughout the paper, $\mathbb{R}$ and $\mathbb{R}^n$ denote the field of real numbers and the vector space of all n-tuples of real numbers, respectively. For $x \in \mathbb{R}^n$ and $i = 1, \dots, n$, x_i denotes the ith coordinate of x. Similarly, $\mathbb{R}^{n\times n}$ denotes the space of $n \times n$ matrices with real entries and for $A \in \mathbb{R}^{n\times n}$, A_{ij} denotes the (i, j)th entry of A. The positive orthant in $\mathbb{R}^n$ is $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, 1 \leq i \leq n\}$; and $\mathbb{R}^{n\times n}_+ := \{A \in \mathbb{R}^{n\times n} : A_{ij} \geq 0, 1 \leq i, j \leq n\}$. The interior of $\mathbb{R}^n_+$ is denoted by $\mathrm{int}\,(\mathbb{R}^n_+)$ and its boundary by $\mathrm{bd}\,(\mathbb{R}^n_+) := \mathbb{R}^n_+ \setminus \mathrm{int}\,(\mathbb{R}^n_+)$. On $\mathbb{R}^n$ we consider the partial order induced by $\mathbb{R}^n_+$. For vectors $x, y \in \mathbb{R}^n$, we write: $x \geq y$ if $x_i \geq y_i$ for $1 \leq i \leq n$; $x > y$ if $x \geq y$ and $x \neq y$; $x \gg y$ if $x_i > y_i, 1 \leq i \leq n$. The absolute value $|x|$ of a vector $x \in \mathbb{R}^n$ is defined by $|x|_i := |x_i|, i = 1, \dots, n$. It will be useful to study norms, which are adapted to the nonnegative setting. A norm on $\mathbb{R}^n$ is called *monotone* if $x \geq y$ implies $\|x\| \geq \|y\|$. This is equivalent to the requirement that $\|x\| = \|\,|x|\,\|$ for all $x \in \mathbb{R}^n$, see [2, Theorem 2], [9, Theorem 5.5.10]. Norms with the latter property are called *absolute* and we will use this name throughout the remainder of the paper.

We say that a matrix A is nonnegative if $A \in \mathbb{R}^{n\times n}_+$. If the off-diagonal entries of A are nonnegative, i.e. if for a suitable $\lambda > 0$ we have $\lambda I + A \in \mathbb{R}^{n\times n}_+$, then A is called Metzler. A matrix $A \in \mathbb{R}^{n\times n}_+$ or a Metzler matrix A is called *reducible* if there exists some permutation matrix P such that $P^T AP$ is of the form

$$\begin{pmatrix} B & C \\ 0 & D \end{pmatrix} \tag{1.1}$$

where $B \in \mathbb{R}^{k\times k}$, $D \in \mathbb{R}^{n-k\times n-k}$ for $1 \leq k \leq n - 1$. If A is not reducible, then it is *irreducible*. An equivalent way of formulating irreducibility is the requirement that for every nonempty proper subset K of $N := \{1, \cdots, n\}$, there exists an $i \in K$, $j \in N \setminus K$ such that $A_{ij} \neq 0$.

2. Linear Inclusions and the Joint Spectral Radius. For a compact set of nonnegative matrices $\mathcal{M}$ we consider the positive linear inclusion

$$x(t + 1) \in \{Mx(t) \mid M \in \mathcal{M}\} . \tag{2.1}$$

In the continuous time setting we define positive linear inclusions as follows. Given a compact set of Metzler matrices $\mathcal{M}$ we consider a linear inclusion of the form

$$\dot{x} \in \{Mx \mid M \in \mathcal{M}\} . \tag{2.2}$$

Solutions of (2.1) corresponding to the initial value x_0 are given by sequences $\{x(t)\}_{t\in\mathbb{N}}$ where for each $t \in \mathbb{N}$ there exists an $A(t) \in \mathcal{M}$ such that $x(t+1) = A(t)x(t)$. The evolution operators generated by $\mathcal{M}$ are therefore the sets

$$\mathcal{S}_t := \{A(t - 1) \dots A(0) \mid A(s) \in \mathcal{M}, s = 0, \dots, t - 1\} ,$$

and the associated matrix semigroup is $\mathcal{S} := \bigcup_{t\in\mathbb{N}} \mathcal{S}_t$, where we set $\mathcal{S}_0 := \{I\}$.

In the continuous case, the evolution operators are given by solutions to the differential equation

$$\dot{\Phi}_\sigma(t) = A(t)\Phi_\sigma(t) , \quad \Phi_\sigma(0) = I ,$$

where $\sigma := A : \mathbb{R}_+ \to \mathcal{M}$ is measurable. (It is possible to consider only piecewise continuous functions A with locally finitely many discontinuities; this neither changes the notions of positivity nor of stability discussed below.) The map σ is called the switching signal defining the differential equation and the notation Φ_σ is a reminder that it is defined via a particular switching signal. In this case the set of time t evolution operators is given by

$$\mathcal{S}_t := \{\Phi_\sigma(t) \mid \sigma : [0,t] \to \mathcal{M} \text{ measurable}\}$$

and again $\mathcal{S} := \bigcup_{t\in\mathbb{R}_+} \mathcal{S}_t$, where we set $\mathcal{S}_0 := \{I\}$.

The well known fact is now that in the discrete time case all matrices $S \in \mathcal{S}$ are nonnegative, if and only if $\mathcal{M} \subset \mathbb{R}_+^{n\times n}$, whereas in the continuous time case all matrices $S \in \mathcal{S}$ are nonnegative if and only if $\mathcal{M}$ contains only Metzler matrices. This is the case we study in the following.

Given a semigroup defined in one of the two ways we have discussed so far, we define the joint spectral radius of $\mathcal{S}$ by setting

$$\rho(\mathcal{S}) := \lim_{t\to\infty} \sup\{\|S\| \mid S \in \mathcal{S}_t\}^{1/t}. \tag{2.3}$$

It is well known that the limit exists, is independent of the norm considered, and characterizes the maximal and uniform exponential growth of solutions to (2.1), resp. (2.2).

For positive systems the interconnection structure of matrices plays a decisive role. Similar to the analysis of nonnegative matrices an associated graph is needed.

2.1. Graphs and Positive Inclusions. Associated to a set of nonnegative or Metzler matrices $\mathcal{M}$ of dimension n we consider the directed graph $\mathcal{G}(\mathcal{M}) = (V,E)$ with vertex set $V = \{1,\dots,n\}$ and edges defined for $i \neq j, 1 \le i,j \le n$ by

$$(i,j) \in E \quad :\Leftrightarrow \quad \exists A \in \mathcal{M} \text{ with } A_{ij} > 0. \tag{2.4}$$

Note that we explicitly do not define edges from a vertex i to itself. This allows to handle continuous and discrete time using the same graph. The graph $\mathcal{G}(\mathcal{M})$ is called strongly connected if for all i,j there is a path form i to j using edges in E.

LEMMA 2.1. *Let $\mathcal{M}$ be a set of nonnegative or Metzler matrices. Then $\mathcal{G}(\mathcal{M})$ is strongly connected if and only if*

(i) discrete time case: for all $i,j \in V$ there exist $A_1,\dots,A_k \in \mathcal{M}$ such that

$$(A_1 \dots A_k)_{ij} > 0.$$

In particular, $\rho(\mathcal{M}) > 0$.

(ii) continuous time case: for all $i,j \in V$ there exist $A_1,\dots,A_k \in \mathcal{M}$ and $t_1,\dots,t_k \ge 0$ such that

$$\left(e^{A_1 t_1} \dots e^{A_k t_k}\right)_{ij} > 0.$$

Proof. Follows by direct calculation. □

We also find it useful to point out the following fact.

LEMMA 2.2. *Let $\mathcal{M}$ be a set of Metzler or nonnegative matrices. Then $\mathcal{G}(\mathcal{M})$ is strongly connected if and only if the convex hull* $\operatorname{conv}\mathcal{M}$ *contains an irreducible matrix.*

3. Extremal Norms. In the analysis of linear inclusions extremal and Barabanov norms play an interesting role. In this paper we restrict our attention to extremal norms. Indeed, in the situations we consider we cannot guarantee that a Barabanov norm exists.

DEFINITION 3.1. *Let $\mathcal{M}$ be a compact set of Metzler or nonnegative matrices. A norm v on $\mathbb{R}^n$ is called extremal for the associated semigroup $\mathcal{S}$, if for all $x \in \mathbb{R}^n$ and all $t \geq 0$ we have*

$$v(Sx) \leq \rho(\mathcal{M})^t v(x)\,, \quad \forall S \in \mathcal{S}_t\,. \tag{3.1}$$

In particular, this means that if the semigroup is exponentially stable, which is equivalent to $\rho(\mathcal{M}) < 1$, then an extremal norm is a Lyapunov function that not only characterizes exponential stability but also the precise growth rate of the system.

The interesting fact is that strong connectedness of $\mathcal{G}(\mathcal{M})$, resp. existence of an irreducible element in the convex hull of $\mathcal{M}$ guarantees the existence of an extremal norm. Note that in general extremal norms need not exist. It is known that existence is equivalent to the boundedness of the semigroup $\{\rho^{-1}(\mathcal{M})S \mid t \geq 0, S \in \mathcal{S}_t\}$, [13], but this is a criterion that is hard to check in general. An additional benefit of the norm constructed here is that it can be chosen to be absolute.

THEOREM 3.2.

(i) Let $\mathcal{M}$ be a compact set of nonnegative matrices. If the graph $\mathcal{G}(\mathcal{M})$ is strongly connected then there exists an absolute extremal norm v for (2.1).

(ii) Let $\mathcal{M}$ be a compact set of Metzler matrices. If the graph $\mathcal{G}(\mathcal{M})$ is strongly connected then there exists an absolute extremal norm v for (2.2).

Proof. The proof is formulated for the semigroup $\mathcal{S}$ and is therefore for both discrete and continuous time. In both cases we know, as a general property of continuous time systems, resp. by Lemma 2.1(i) for the discrete time case, that $\rho(\mathcal{M}) > 0$.

It is a straightforward calculation to see that by considering $\mathcal{M} - \log \rho(\mathcal{M}) I$ in the continuous time case, or $\rho(\mathcal{M})^{-1}\mathcal{M}$ in the discrete time case, we may assume that $\rho(\mathcal{M}) = 1$.

We first show that under our assumptions $\mathcal{S}$ is bounded if $\rho(\mathcal{M}) = 1$. Assume this is not the case. As $\mathcal{G}(\mathcal{M})$ is connected we may by Lemma 2.1 find a constant $c > 0$ such that for all $i, j \in \{1, \ldots, n\}$ there is a matrix $S(i,j) \in \mathcal{S}$ with

$$S(i,j)_{ij} > c\,.$$

Assuming $\mathcal{S}$ is unbounded, we may choose $\bar{S} = \Phi_\sigma(t,0) \in \mathcal{S}$ such that $\bar{S}_{\nu\mu} > 1/c$ for some indices ν, μ. It follows by direct calculation that $\big(S(\mu,\nu)\bar{S}\big)_{\mu\mu} =: a > 1$. As we have for some $s > 0$ that $S(\mu,\nu) \in \mathcal{S}_s$ it follows that $\big(S(\mu,\nu)\bar{S}\big)^k \in \mathcal{S}_{k(t+s)}$ we obtain $\rho(\mathcal{M}) \geq a^{1/(t+s)} > 1$.

So we may assume the generated semigroup $\mathcal{S}$ is bounded. An extremal norm may then be defined in the following way, see also [13]. Let $\|\cdot\|$ be an absolute norm on $\mathbb{R}^n$. Then define for $x \geq 0$

$$v(x) := \sup\{\|Sx\| \mid S \in \mathcal{S}\}\,, \tag{3.2}$$

and extend this definition to $\mathbb{R}^n$ by setting $v(x) := v(|x|), x \in \mathbb{R}^n$. It is clear that v is positively homogeneous and positive definite (as $I \in \mathcal{S}$). As the matrices $S \in \mathcal{S}$ are all nonnegative it follows by absoluteness of $\|\cdot\|$ that for $0 \leq x \leq y$ we have

$0 \le Sx \le Sy$ for $S \in \mathcal{S}$ and so $v(x) \le v(y)$. The triangle inequality for v then follows from

$$v(x+y) = v(|x+y|) \le v(|x|+|y|) = \sup\{\|S(|x|+|y|)\| \mid S \in \mathcal{S}\}$$
$$\le \sup\{\|S|x|\| + \|S|y|)\| \mid S \in \mathcal{S}\} \le v(|x|) + v(|y|) = v(x) + v(y)\,.$$

Thus v is a norm; it is obvious from the definition that v is absolute. Extremality of v then follows by noting that from $0 \le |Sx| \le S|x|$, using the monotonicity of v (and the assumption $\rho(\mathcal{M}) = 1$) we have

$$v(Sx) = v(|Sx|) \le v(S|x|) = \sup\{\|TS|x|\| \mid T \in \mathcal{S}\} \le v(|x|) = v(x)\,. \tag{3.3}$$

□

We note the following immediate corollary.

COROLLARY 3.3.

(i) Let $\mathcal{M}$ be a compact set of nonnegative matrices. If $\operatorname{conv}\mathcal{M}$ *contains an irreducible element $\bar{M}$ then there exists an absolute extremal norm v for* (2.1).

(ii) Let $\mathcal{M}$ be a compact set of Metzler matrices. If $\operatorname{conv}\mathcal{M}$ *contains an irreducible element $\bar{M}$ then there exists an absolute extremal norm v for* (2.2).

Proof. It is clear that $\operatorname{conv}\mathcal{M}$ contains an irreducible matrix $\bar{M}$ if and only if $\mathcal{G}(\mathcal{M})$ is strongly connected. □

4. Regularity of the Joint Spectral Radius. For irreducible matrices $A \in \mathbb{R}_+^{n\times n}$ it is well known that the spectral radius $\rho(A)$ is a simple eigenvalue of A and that all eigenvalues λ of A of modulus equal to the spectral radius are simple. It is a consequence of standard perturbation theory, that under these conditions the spectral radius as a function of the entries of the matrices is Lipschitz continuous on a neighborhood of A. In this section we show that by the previous results the same is true for positive linear inclusions that are irreducible in terms of the graph $\mathcal{G}(\mathcal{M})$. This result complements the result of [19], where it was shown that the joint spectral radius is Lipschitz continuous on the set of compact matrix sets that are *irreducible*. In this context this name is a bit misleading, because a set of matrices is irreducible (a term coming from representation theory this time), if no subspace other than the trivial ones, $\{0\}$ and $\mathbb{R}^n$, is invariant under all matrices in $\mathcal{M}$. This property is not implied by the assumptions in Theorem 3.2. To see this, consider a pair of strictly positive matrices with a common eigenvector (take a set of row stochastic matrices for example). Such a set will automatically satisfy our assumptions but will clearly have a common invariant subspace spanned by the common eigenvector. Hence the set will not be irreducible in the sense used in the work of Barabanov and others. On the other hand, if a set of nonnegative or Metzler matrices has no common invariant subspace, it will be irreducible in our sense. Hence our assumption is strictly weaker than the usual one.

In what follows the Hausdorff distance between compact sets of matrices $\mathcal{M}, \mathcal{N}$ is defined by

$$H(\mathcal{M},\mathcal{N}) := \max\{\max_{A\in\mathcal{M}}\{\operatorname{dist}(A,\mathcal{N})\}, \max_{B\in\mathcal{N}}\{\operatorname{dist}(B,\mathcal{M})\}\}\,.$$

THEOREM 4.1. *(i) In the discrete time case the joint spectral radius is locally Lipschitz continuous on the set*

$$\mathcal{P} := \{\mathcal{M} \mid \mathcal{M} \subset \mathbb{R}_+^{n\times n} \textit{ compact }, \mathcal{G}(\mathcal{M}) \textit{ strongly connected }\}$$

endowed with the Hausdorff metric.
(ii) In the continuous time case the joint spectral radius is locally Lipschitz continuous on the set

$$\mathcal{P} := \{\mathcal{M} \mid \mathcal{M} \subset \left(\mathbb{R}_+^{n\times n} + \mathbb{R}I\right) \ \textit{compact}\ , \mathcal{G}(\mathcal{M}) \ \textit{strongly connected}\ \}$$

endowed with the Hausdorff metric.

In the proof we follow the idea of [19]. There the proof is based on the consideration of the eccentricity of extremal norms corresponding to different sets. In general, the eccentricity of a norm v with respect to a norm $\|\cdot\|$ is defined by

$$\mathrm{ecc}_{\|\cdot\|}(v) := \frac{\max\{v(x) \mid \|x\| = 1\}}{\min\{v(x) \mid \|x\| = 1\}} \,. \tag{4.1}$$

Note that for any $A \in \mathbb{R}^{n\times n}$ we have for the induced operator norm that

$$\frac{1}{\mathrm{ecc}_{\|\cdot\|}(v)} \|A\| \leq v(A) \leq \mathrm{ecc}_{\|\cdot\|}(v)\|A\| \,. \tag{4.2}$$

If both $\|\cdot\|$ and v are monotone the definition in (4.1) reduces to

$$\mathrm{ecc}_{\|\cdot\|}(v) := \frac{\max\{v(x) \mid \|x\| = 1\,, x \geq 0\}}{\min\{v(x) \mid \|x\| = 1\,, x \geq 0\}} \,, \tag{4.3}$$

because both norms are completely determined by their behaviour on the nonnegative orthant $\mathbb{R}^n_+$.

The decisive property is now that the eccentricity of absolute extremal norms is bounded on compact subsets of $\mathcal{P}$. We note that an analogous statement to [20, Lemma 4.1] is false here, because we cannot exclude the possibility of positively homogeneous functions that have an extremality property and vanish on a subspace.

Proposition 4.2. *Let $\mathcal{X} \subset \mathcal{P}$ be compact (as a subset of the metric space $(\mathcal{P}, H)$) and let $\|\cdot\|$ be absolute, then there exists a bound $0 < C < \infty$ such that for all sets $\mathcal{M} \in \mathcal{X}$ there exists an absolute extremal norm v with*

$$\mathrm{ecc}_{\|\cdot\|}(v) < C \,. \tag{4.4}$$

Proof. We show the property locally in a neighborhood of $\mathcal{M} \in \mathcal{P}$, then the assertion follows by a standard compactness argument.

So let $\mathcal{M} \in \mathcal{P}$ and apply Theorem 3.2 to choose an extremal norm for $\mathcal{M}$. We claim that there is a neighborhood of $\mathcal{M}$ in $\mathcal{P}$ for which (4.4) holds. If this is false then we may pick sequences $\mathcal{M}_k \to \mathcal{M}$ and $C_k \to \infty$ such that every absolute extremal norm of $\mathcal{M}_k$ has eccentricity exceeding C_k.

As norms are convex functions, a norm is extremal for $\mathcal{M}$ if and only if it is extremal for $\operatorname{conv} \mathcal{M}$, so that we may assume that all $\mathcal{M}_k$ and $\mathcal{M}$ are convex. In particular, there is by assumption an irreducible matrix $\bar{M} \in \mathcal{M}$ and for k large enough there are irreducible matrices $\bar{M}_k \in \mathcal{M}_k$ with $\bar{M}_k \to \bar{M}$.

As the joint spectral radius is continuous, [19], we know that $\rho(\mathcal{M}_k) \to \rho(\mathcal{M}) > 0$ and so $\rho^{-1}(\mathcal{M}_k)\mathcal{M}_k \to \rho(\mathcal{M})^{-1}\mathcal{M}$, resp. $\mathcal{M} - \log\rho(\mathcal{M})I$ in the continuous time case. As this rescaling does not change extremal norms, we may assume that all joint spectral radii involved are equal to 1.

Let $\|\cdot\|$ be an absolute extremal norm for $\mathcal{M}$. We construct absolute extremal norms v_k for $\mathcal{M}_k$ using (3.2). This is possible by the construction in Theorem 3.2. Note in particular, that this implies $v_k(x) \geq \|x\|$ for all $x \in \mathbb{R}^n$.

Now $\bar{M}$ is irreducible if and only if $K := \bar{M} + \bar{M}^2 + \cdots + \bar{M}^n$ is a positive matrix, (resp. $e^{\bar{M}t}$ is positive), see [3]. As $\bar{M}_k \to \bar{M}$ we may choose a constant $c > 0$ and an index k_0 sufficiently large, such that for all $k \geq k_0$ and all $i, j \in \{1, \dots, n\}$ there exists $1 \leq \ell \leq n$ such that

$$\left(\bar{M}_k\right)_{ij}^{\ell} > c\,, \quad (\text{resp.} \exp\left(\bar{M}_k\right)_{ij} > c)\,. \tag{4.5}$$

If $\mathrm{ecc}_{\|\cdot\|}(v_k) > C_k \to \infty$, then as $v_k(\cdot) \geq \|\cdot\|$ it follows from the definition of v_k that for some k sufficiently large there is an $S \in \mathcal{S}(\mathcal{M}_k)$ such that for suitable $\mu, \nu \in \{1, \dots, n\}$

$$S_{\mu\nu} > 1/c\,. \tag{4.6}$$

As in the final step of the proof of Theorem 3.2 the combination of (4.5) and (4.6) leads to a contradiction to the assumption that $\rho(\mathcal{M}_k) = 1$. This concludes the proof. ∎

Now the proof of Theorem 4.1 can be completed following the steps outlined in [19, 20].

Proof. (of Theorem 4.1) (i) In the discrete-time case, let $\mathcal{X} \subset \mathcal{P}$ be compact, let C be as in Proposition 4.2. Pick $\mathcal{M}, \mathcal{N} \in \mathcal{X}$ and an absolute extremal norm v for $\mathcal{M}$. Recall that by definition this implies for the induced matrix norm, also denoted by v, that $v(A) \leq \rho(\mathcal{M})$ for all $A \in \mathcal{M}$. Then for any $B \in \mathcal{N}$, we may choose $A \in \mathcal{M}$ such that $v(A - B) \leq \mathrm{dist}_v(B, \mathcal{M})$ and we obtain

$$v(B) \leq v(A) + v(B - A) \leq \rho(\mathcal{M}) + H_v(\mathcal{M}, \mathcal{N})\,,$$

where H_v is the Hausdorff distance defined using v. This yields $\rho(\mathcal{N}) \leq \rho(\mathcal{M}) + H_v(\mathcal{M}, \mathcal{N})$. Using (4.2), we see that $H_v(\mathcal{M}, \mathcal{N}) \leq CH(\mathcal{M}, \mathcal{N})$ and by symmetry the assertion follows.

(ii) The continuous time follows as in [19] by noting that the map $\mathcal{M} \to \mathcal{S}_1(\mathcal{M})$ defines a Lipschitz continuous set-valued map. If $\mathcal{M}$ consists of Metzler matrices, then $\mathcal{S}_1(\mathcal{M}) \subset \mathbb{R}_+^{n\times n}$ and the irreducibility property is preserved. In this way the continuous-time case is a direct consequence of the discrete-time case. ∎

Note that the result of Theorem 4.1 does not yield the full force of the statement for single irreducible matrices. There we may obtain Lipschitz continuity of the spectral radius on a neighborhood which may also include matrices not in $\mathbb{R}_+^{n\times n}$. So far our result is restricted to neighborhoods of nonnegative matrix sets, but we expect it can be extended to larger neighborhoods of irreducible sets of nonnegative matrices.

5. Conclusions. In this paper we have considered linear inclusions defining positive systems. We show that under a generalized irreducibility assumption absolute extremal norms exist. As an application local Lipschitz continuity of the joint spectral radius on certain positive linear inclusions is proved.

REFERENCES

[1] N. E. Barabanov, *Lyapunov indicator of discrete inclusions. I–III*, Autom. Remote Control, 49 (1988), pp. 152–157, 283–287, 558–565.

[2] F. Bauer, J. Stoer, and C. Witzgall, *Absolute and monotonic norms*, Numerische Mathematik, 3 (1961), pp. 257–264.
[3] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Classics in Applied Mathematics, SIAM, Philadelphia, PA, USA, 1987.
[4] X. Dai, *Extremal and Barabanov semi-norms of a semigroup generated by a bounded family of matrices*, Journal of Mathematical Analysis and Applications, (2011).
[5] L. Fainshil, M. Margaliot, and P. Chigansky, *On the stability of positive linear switched systems under arbitrary switching laws*, IEEE Transactions on Automatic Control, 54 (2009), pp. 897–899.
[6] N. Guglielmi and M. Zennaro, *An algorithm for finding extremal polytope norms of matrix families*, Lin. Alg. Appl., 428 (2008), pp. 2265–2282.
[7] L. Gurvits, *Stability of discrete linear inclusions*, Lin. Alg. Appl., 231 (1995), pp. 47–85.
[8] L. Gurvits, R. Shorten, and O. Mason, *On the stability of switched positive linear systems*, IEEE Transactions on Automatic Control, 52 (2007), pp. 1099–1103.
[9] R. A. Horn and C. R. Johnson, *Matrix Analysis*, Cambridge University Press, New York, NY, USA, 1985.
[10] R. Jungers, *The Joint Spectral Radius: Theory and Applications*, Springer-Verlag, 2009.
[11] V. Kozyakin, *Structure of extremal trajectories of discrete linear systems and the finiteness conjecture*, Autom. Rem. Control, 68 (2007), pp. 174–209.
[12] V. Kozyakin, *An explicit lipschitz constant for the joint spectral radius*, Linear Algebra and its Applications, 433 (2010), pp. 12–18.
[13] V. S. Kozyakin, *Algebraic unsolvability of problem of absolute stability of desynchronized systems*, Autom. Rem. Control, 51 (1990), pp. 754–759.
[14] M. Maesumi, *Optimal norms and the computation of joint spectral radius of matrices*, Linear Algebra and its Applications, 428 (2008), pp. 2324–2338.
[15] I. Morris, *Criteria for the stability of the finiteness property and for the uniqueness of Barabanov norms*, Lin. Alg. Appl., 433 (2010), pp. 1301–1311.
[16] V. Protasov, R. Jungers, and V. Blondel, *Joint spectral characteristics of matrices: A conic programming approach*, SIAM Journal on Matrix Analysis and Applications, 31 (2010), pp. 2146–2162.
[17] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King, *Stability theory for switched and hybrid systems*, SIAM Review, 49 (2007), pp. 545–592.
[18] R. Teichner and M. Margaliot, *Explicit construction of a Barabanov norm for a class of positive planar discrete-time linear switched systems*, Automatica, 48 (2012), pp. 95–101.
[19] F. Wirth, *The generalized spectral radius and extremal norms*, Lin. Alg. Appl., 342 (2002), pp. 17–40.
[20] F. Wirth, *On the structure of the set of extremal norms of a linear inclusion*, in CDC-ECC'05. 44th IEEE Conference on Decision and Control, and 2005 European Control Conference, 2005, pp. 3019–3024.