On the controllability of diffusively coupled multi-agent networks with switching topologies

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1 Introduction

The past few years has witnessed much research effort in distributed and cooperative control of multi-agent systems with linear diffusive couplings [1]. Among all relevant problems, it is very interesting to know whether the overall system behavior can be affected by only a small fraction of the agents. This question can be answered by addressing the controllability of the system after this fraction of agents are assigned as leaders, which are under the forcing of the external control inputs. The controllability problem has been formulated to, for instance, control a formation of mobile robots with the aim that, by manipulating the trajectories of the leaders, all the robots can move from any initial positions to any desired final positions within finite time [1].

The controllability of diffusively coupled multi-agent systems has been studied in [2]. There we assume that the underlying graph topology of the couplings in a system is time-invariant and reveal the role that the graph topology plays in the controllability of the given multi-agent system by using two classes of graph partitions.

In this paper, we consider the same problem as in [2], but we drop the time-invariant topology assumption. Thus, we are considering a more realistic scenario when the topology of the multi-agent system switches with time. Beyond being more realistic, switching topologies have an advantage in rendering a controllable system, i.e. a multi-agent system with switching topologies can become controllable even if each of these topologies leads to an uncontrollable system.

Here, graph partitions are still employed to study the controllability of a system with switching topologies. By referring to the controllability results of switched linear systems, we provide the lower and the upper bounds of the controllable subspace of the system in terms of graph partitions. Moreover, we show that the two bounds are tight by finding examples where the bounds can be achieved.

The rest of this abstract is organized as follows. Section 2 quickly reviews the controllability/reachability results in switched linear systems. Then we introduce the system of our interest in Section 3. As the main tools in controllability problem, graph partitions are studied in Section 4. And the main result of our paper will be mentioned in Section 5.

2 Controllability of switched linear systems

Let \((A_i, B_i) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{m \times n}\) with \(i = \{1, 2, \ldots, p\}\) be given matrices. Let \(S\) be the set of switching signals defined as the set of right-continuous, piecewise constant functions \(\sigma : \mathbb{R}_+ \rightarrow \{1, 2, \ldots, p\}\) which have finitely many discontinuities over every finite interval.

Consider the switched linear system

\[
\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t)
\]
where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, and $\sigma \in S$ is a switching signal. Let $x^{\xi,\sigma,u}$ denote the unique state trajectory of the system (1) for the initial state $\xi$ (i.e. $x^{\xi,\sigma,u}(0) = \xi$), the switching signal $\sigma$, and the input $u$.

We say that a state $\eta \in \mathbb{R}^n$ is

- reachable if there exist $\sigma \in S$, locally-integrable $u$, and a positive number $T$ such that $x^{0,\sigma,u}(T) = \eta$,

- controllable if there exist $\sigma \in S$, locally-integrable $u$, and a positive number $T$ such that $x^{\eta,\sigma,u}(T) = 0$.

Let $R$ and $C$ denote the set of all reachable and controllable states, respectively. We say that the system (1) is

- reachable if $R = \mathbb{R}^n$,

- controllable if $C = \mathbb{R}^n$,

- completely controllable if $R = C = \mathbb{R}^n$.

As it is shown in [3], the sets of reachable and controllable states coincide and are subspaces. To elaborate more, let $\langle A \mid B \rangle$ denote the smallest $A$-invariant subspace that contains $\text{im} B$. In a similar fashion, let $\langle \{A_1,A_2,\ldots,A_p\} \mid \{B_1,B_2,\ldots,B_p\} \rangle$ denote the smallest subspace that is invariant under $A_i$ and contains $\text{im} B_i$ for all $i = \{1,2,\ldots,p\}$. Then, it can be shown that

$$R = C = \langle \{A_1,A_2,\ldots,A_p\} \mid \{B_1,B_2,\ldots,B_p\} \rangle.\]$$

In the sequel, we are interested in particular kind of switched linear systems for which each subsystem constitutes a diffusively coupled multi-agent networks.

### 3 Diffusively coupled multi-agent systems with switching topologies

Let $G_i = (V,E_i)$ with $i \in \{1,2,\ldots,p\}$ be simple undirected graphs where $V = \{1,2,\ldots,n\}$ is the vertex set and $E_i \subseteq V \times V$ is the edge set. For each $G_i$, we group vertices into two categories: $V^i_1 = \{\ell^i_1,\ldots,\ell^i_m\} \subseteq V$ and $V^i_2 = V \setminus V^i_1$. Then the diffusively coupled leader-follower multi-agent system associated with $G_i$ is given by

$$\dot{x}_j = - \sum_{(j,k) \in E_i} (x_j - x_k) \text{ if } j \in V^i_1,$$

$$\dot{x}_j = - \sum_{(j,k) \in E_i} (x_j - x_k) + u_j \text{ if } j \in V^i_2,$$

where $x_j \in \mathbb{R}$ is the state of agent $j$, $u_j \in \mathbb{R}$ is the external input to agent $j$, $V^i_1$ is the leader set, and $V^i_2$ is the follower set. Here, a leader refers to an agent for which one can apply an external input and a follower refers to an agent which is not a leader.

By defining $x = \text{col}(x_1,\ldots,x_n)$ and $u = \text{col}(u_1,\ldots,u_n)$, this system can be written into the compact form of

$$\dot{x} = -L_i x + M_i u \quad (2)$$

where $L_i$ is the Laplacian matrix of $G_i$ and $M_i \in \mathbb{R}^{n \times n}$ is defined by

$$[M_i]_{jk} = \begin{cases} 1 & \text{if } j = \ell^i_k \\ 0 & \text{otherwise}. \end{cases}$$

In this paper, we consider the switched multi-agent system given by

$$\dot{x}(t) = -L_{\sigma(t)} x(t) + M_{\sigma(t)} u(t) \quad (3)$$

and investigate its controllability properties in terms of the topologies of underlying graphs. As such, graph partitions will play a central role in the sequel.
4 Graph partitions

We quickly review certain graph partitions that will be used later. More details can be found in [2]. Any nonempty subset of the vertex set $V$ is called a cell. A collection of mutually disjoint cells $\pi = \{C_1, \ldots, C_r\}$ is said to be a partition of $V$ if $\bigcup_{j=1}^{r} C_j = V$. Let $\Pi$ denote the set of all the partitions of $V$. A partition $\pi_1$ is said to be finer than a partition $\pi_2$, denoted by $\pi_1 \leq \pi_2$, if each cell of $\pi_1$ is a subset of some cell of $\pi_2$.

Let $\Pi_{\text{AEP}}(G)$ denote the set of all almost equitable partitions of a given graph $G = (V, E)$ [2]. For the collection of graphs $\{G_1, \ldots, G_p\}$, define

$$
\Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p) = \{\pi \mid \pi \in \Pi_{\text{AEP}}(G_i) \text{ and } \pi \leq \pi_i \text{ for all } i\}
$$

where $\pi_i$ is a given partition of $G_i$. It can be shown that there exists a unique partition, say $\pi^* = \pi^*(\pi_1, \ldots, \pi_p)$, such that

$$
\begin{aligned}
\pi^* &\in \Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p) \\
\pi &\leq \pi^* \text{ for all } \pi \in \Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p) \\
\pi &\leq \pi' \text{ for all } \pi \in \Pi_{\text{AEP}}(\pi_1, \ldots, \pi_p) \implies \pi^* \leq \pi'.
\end{aligned}
$$

We denote the distance partition relative to a vertex $v$ of a graph $G$ by $\pi_D(v; G)$ (see [2] for more details).

5 Main results

Let $\mathcal{K}$ denote the controllable subspace of the system (3), that is

$$
\mathcal{K} = \langle \{L_1, L_2, \ldots, L_p\} \mid \{M_1, M_2, \ldots, M_p\} \rangle.
$$

Then, we prove that

$$
\mathcal{K} \subseteq \text{im } P(\pi^*(\pi_1^L, \ldots, \pi_p^L))
$$

where $\pi_i^L$ is the partition $\{\{l_i^1\}, \ldots, \{l_i^{m_i}\}, V_i^p\}$ and $P(\pi)$ denotes the characteristic matrix of the partition $\pi$. After this upper bound in terms of the almost equitable partitions, we prove the following lower bound on the dimension of the controllable subspace $\mathcal{K}$:

$$
\dim(\mathcal{K}) \geq \max_{i \in \{1, 2, \ldots, p\}} \max_{k \in \{1, 2, \ldots, m_i\}} \text{card}(\pi_D(l_i^k; G_i)).
$$

Moreover, these two bounds are tight in the sense that there are examples for which one of them is attained whereas the other holds strictly. Such examples will also be provided in the final submission.

References

