

Hardy-Schatten Norms of Systems, Output Energy Cumulants and Linear Quadro-Quartic Gaussian Control

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Abstract—This paper is concerned with linear stochastic control systems in state space. The integral of the squared norm of the system output over a bounded time interval is interpreted as energy. The cumulants of the output energy in the infinite-horizon limit are related to Schatten norms of the system in the Hardy space of transfer functions and the risk-sensitive performance index. We employ a novel performance criterion which seeks to minimize a combination of the average value and the variance of the output energy of the system per unit time. The resulting linear quadro-quartic Gaussian control problem involves the \mathcal{H}_2 and \mathcal{H}_4 -norms of the closed-loop system. We obtain equations for the optimal controller and outline a homotopy method which reduces the solution of the problem to the numerical integration of a differential equation initialized by the standard linear quadratic Gaussian controller.

I. INTRODUCTION

This paper is concerned with linear multi-input multi-output control systems, governed in state space by Ito stochastic differential equations, driven by a standard Wiener process which is regarded as a random disturbance. The integral of the squared Euclidean norm of the system output over a bounded time interval is interpreted as *energy*. In the disturbance attenuation paradigm, the output energy is to be minimized in some sense.

Linear Quadratic Gaussian (LQG) control [1], for example, seeks to minimize the expectation of the output energy which, in the infinite-horizon limit, reduces to the squared \mathcal{H}_2 -norm of the closed-loop system in an appropriate Hardy space of transfer functions. An alternative performance index is employed in the Risk-Sensitive and Minimum Entropy control theories [11]. They utilise the expected value of the exponential of the output energy multiplied by a scaling parameter to adjust the risk sensitivity. Risk-sensitive control extends the LQG approach and is robust with respect to Kullback-Leibler relative entropy bounded uncertainties in the random noise [2].

The risk-sensitive performance index can be represented as a series expansion with respect to the energy scaling parameter. The coefficients of this series are the rates of the asymptotically linear growth of the cumulants of the output energy in the infinite-horizon limit. The cumulant growth rates are directly related to higher-order Schatten norms [12] of the transfer function of the system in an appropriate Hardy space. This allows the risk-sensitive criterion to be viewed as

a linear combination of powers of Hardy-Schatten norms of the system whose weights are governed by the risk-sensitivity parameter in a very specific way. The “reverse engineering” of the risk-sensitive index suggests a wide family of performance criteria in the form of linear combinations of powers of the Hardy-Schatten norms. This gives rise to a class of *output energy cumulant* (OEC) control problems which extend the risk-sensitive paradigm. In fact, the LQG approach can be considered to explore this freedom to a certain degree by retaining the first term (the squared \mathcal{H}_2 -norm of the system) of the risk-sensitive index expansion.

The present paper develops the OEC control idea, outlined above, by employing a performance criterion which seeks to minimize a combination of the average value and the variance of the output energy of the system per unit time. The resulting linear quadro-quartic Gaussian (LQQG) control problem utilizes a *quadro-quartic functional* as a finer truncation of the risk-sensitive performance index which retains the \mathcal{H}_2 and \mathcal{H}_4 -norms of the closed-loop system and the risk-sensitive parameter.

The \mathcal{H}_4 -norm, which involves the Schatten 4-norm of matrices [6] and is referred to as the *quartic norm*, was introduced in [13] as a subsidiary construct in the anisotropy-based robust control theory for discrete-time stochastic systems. In the present study, the quartic norm plays a central role and, in addition to providing the next term in the risk-sensitive index expansion, quantifies (via the \mathcal{H}_4 to \mathcal{H}_2 -norms ratio) the time scale beyond which the infinite-horizon LQG cost starts manifesting itself in sample paths of the output energy of the system.

We consider the LQQG problem in the class of linear stabilizing controllers with the same state dimension as the underlying plant. This allows equations for an optimal controller to be obtained by using Frechet derivatives of the quadro-quartic performance index of the closed-loop system with respect to the state-space realization matrices of the controller. The resulting set of equations depends on the risk sensitivity parameter and yields the standard LQG controller for a zero value of the parameter. We outline a homotopy method which regards the parameter as a fictitious time variable and reduces the solution of the set of equations to a problem involving the numerical integration of an ordinary differential equation (ODE) initialized by the standard LQG controller.

In addition to its extension to the discrete-time case, the LQQG approach may find application in the control of quantum stochastic systems as an alternative to the risk-sensitive control paradigm.

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II. VARIANCE OF OUTPUT ENERGY AND QUARTIC NORM

Suppose $W := (w_t)_{t \in \mathbb{R}}$ is a m -dimensional standard Wiener process (initialised in the infinitely distant past) at the input of a linear time invariant (LTI) system F with a square integrable $\mathbb{R}^{p \times m}$ -valued impulse response function $f := (f_t)_{t \geq 0}$; see Fig. 1. The output $Z := (z_t)_{t \in \mathbb{R}}$ of the



Fig. 1. A LTI system F with input W and output Z .

system is a \mathbb{R}^p -valued Gaussian random process defined by the Ito stochastic integral $z_t := \int_{-\infty}^t f_{t-s} dw_s$. The mean value of Z is zero and the covariance function is

$$\begin{aligned} c_t &:= \mathbf{E}(z_t z_0^T) = \int_0^{+\infty} f_{s+t} f_s^T ds \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega) e^{i\omega t} d\omega = c_{-t}^T, \quad t \geq 0, \end{aligned} \quad (1)$$

where

$$S(\omega) := \widehat{F}(\omega) \widehat{F}(\omega)^* = \int_{-\infty}^{+\infty} c_t e^{-i\omega t} dt \quad (2)$$

is the spectral density of Z . Here, $(\cdot)^* := \overline{(\cdot)}^T$ denotes the complex conjugate transpose of a matrix, and $\widehat{F}(\omega) := F(i\omega) = \int_0^{+\infty} f_t e^{-i\omega t} dt$ is the Fourier transform of the impulse response, that is, the boundary value of the transfer function of the system $F(v) := \int_0^{+\infty} f_t e^{-vt} dt$, with $\text{Re } v > 0$. With f assumed to be square integrable, F belongs to the Hardy space $\mathcal{H}_2^{p \times m}$ of $\mathbb{C}^{p \times m}$ -valued functions of a complex variable, analytic in the right half-plane and endowed with the \mathcal{H}_2 -norm

$$\|F\|_2 := \sqrt{\int_0^{+\infty} \|f_t\|^2 dt} = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \|\widehat{F}(\omega)\|^2 d\omega}. \quad (3)$$

Here, the Plancherel theorem is used, and $\|M\| := \sqrt{\langle M, M \rangle}$ is the Frobenius norm of a matrix M generated by the inner product $\langle M, N \rangle := \text{Tr}(M^* N)$, so that $\|\widehat{F}(\omega)\|^2 = \text{Tr} S(\omega)$ is the trace of the spectral density from (2). In view of (1), $\|F\|_2^2 = \text{Tr} c_0 = \mathbf{E}(|z_t|^2)$ is the variance of the output signal for any t . For a finite time horizon $T > 0$, the random variable

$$\mathcal{E}_T := \int_0^T |z_t|^2 dt \quad (4)$$

is interpreted as the *output energy* of the system F over the time interval $[0, T]$, and

$$\epsilon_T := \mathcal{E}_T / T \quad (5)$$

is the corresponding output energy rate. The mean value of ϵ_T coincides with the squared \mathcal{H}_2 -norm of the system (3): $\mathbf{E} \epsilon_T = \|F\|_2^2$. This ensemble average can manifest itself in sample paths of ϵ_T only by virtue of the law of large numbers, provided that T is large enough. Under additional assumptions on the system F , the rate of the mean square convergence $\text{l.i.m.}_{T \rightarrow +\infty} \epsilon_T = \|F\|_2^2$ is quantified by the

asymptotic behaviour of the variance of ϵ_T . The convergence rate is described by the lemma below in terms of the quantity

$$\|F\|_4 := \sqrt[4]{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|S(\omega)\|^2 d\omega} = \sqrt[4]{2 \int_0^{+\infty} \|c_t\|^2 dt}. \quad (6)$$

This is a continuous-time counterpart of the \mathcal{H}_4 -norm introduced as a subsidiary construct in the anisotropy-based robust control of discrete-time systems [13]. The second equality in (6) follows from the Plancherel theorem applied to the spectral density (2). The systems F with $\|F\|_4 < +\infty$ form a normed space $\mathcal{H}_4^{p \times m}$. The integrand $\|S(\omega)\|^2 = \text{Tr}((\widehat{F}(\omega) \widehat{F}(\omega)^*)^2)$ in (6) is the fourth power of the Schatten 4-norm [6, p. 441] of the matrix $\widehat{F}(\omega)$; see also [12]. The \mathcal{H}_4 -norm $\|F\|_4$ will be referred to as the *quartic norm* of the system F .

Lemma 1: Let $F \in \mathcal{H}_2^{p \times m} \cap \mathcal{H}_4^{p \times m}$. Then the variance of the output energy rate (5) of the system behaves asymptotically as

$$\text{var}(\epsilon_T) \sim 2\|F\|_4^4 / T, \quad T \rightarrow +\infty. \quad (7)$$

Proof: By applying Lemma 6 of Appendix A to the Gaussian random vectors z_s and z_t and using (1), it follows that $\text{cov}(|z_s|^2, |z_t|^2) = 2\|c_{s-t}\|^2$. Hence, the variance of the output energy (4) can be computed as

$$\begin{aligned} \text{var}(\mathcal{E}_T) &= \int_{[0, T]^2} \text{cov}(|z_s|^2, |z_t|^2) ds dt \\ &= 2 \int_{[0, T]^2} \|c_{s-t}\|^2 ds dt = 4T \int_0^T (1 - u/T) \|c_u\|^2 du, \end{aligned} \quad (8)$$

where use is made of the property $c_t = c_{-t}^T$ and the invariance of the Frobenius norm of a matrix under the transpose. Since the assumption $F \in \mathcal{H}_4^{p \times m}$ ensures the square integrability of the covariance function (1), then

$$\lim_{T \rightarrow +\infty} \int_0^T (1 - u/T) \|c_u\|^2 du = \int_0^{+\infty} \|c_u\|^2 du \quad (9)$$

holds by Lebesgue's dominated convergence theorem. Since $\partial_T \int_0^T (1 - u/T) \|c_u\|^2 du = T^{-2} \int_0^T u \|c_u\|^2 du \geq 0$, the convergence is monotonic. Now, (7) is obtained by using (5) and combining (8) and (9) with (6): $\text{var}(\epsilon_T) = \text{var}(\mathcal{E}_T) / T^2 \sim 4 \int_0^{+\infty} \|c_u\|^2 du / T = 2\|F\|_4^4 / T$ as $T \rightarrow +\infty$. \square

In view of a central limit theorem for quadratic functionals of Gaussian processes [4, Theorem 2], the relation (7) provides the scaling factor for the asymptotic standard normality of the random variable $\sqrt{T/2}(\epsilon_T - \|F\|_2^2) / \|F\|_4^2$ as $T \rightarrow +\infty$. Heuristically, the root mean square deviation of ϵ_T from its mean value $\|F\|_2^2$ is relatively small if

$$T \gg T_* := 2(\|F\|_4 / \|F\|_2)^4. \quad (10)$$

The right-hand side of (10) quantifies the time horizon beyond which the \mathcal{H}_2 -norm $\|F\|_2$ manifests itself in the sample paths of the output energy of the system. On the other hand, for $T \ll T_*$, the ergodic properties of the system output Z do not expose themselves since the expected value $\mathbf{E} \epsilon_T = \|F\|_2^2$ of the output energy rate is “indistinguishable” in the background of random fluctuations whose standard

deviation can be estimated by using (7) as $\sqrt{\text{var}(\epsilon_T)} \sim \|F\|_4^2 \sqrt{2/T} \gg \mathbf{E}\epsilon_T$. Thus, the squared \mathcal{H}_2 -norm as the average output energy loses its significance for quantifying the disturbance attenuation capabilities of the system on short time scales $T \ll T_*$. The critical time horizon T_* defined by (10) is similar to the *integral time scale* of measurements in turbulent flows [3, pp. 50–51]. As an example, let Z be an Ornstein-Uhlenbeck process generated from a standard Wiener process W by a single-input single-output system F according to the SDE

$$dz_t = az_t dt + \sqrt{2|a|} dw_t, \quad (11)$$

parameterized by $a < 0$. The covariance function (1) of Z is $c_t = e^{a|t|}$, and the \mathcal{H}_2 and \mathcal{H}_4 -norms of the system F , defined by (3) and (6), are $\|F\|_2 = 1$ and $\|F\|_4 = |a|^{-1/4}$. Therefore, the critical time horizon (10) takes the form $T_* = 2/|a|$ and coincides with the typical transient time; see Fig. 2.

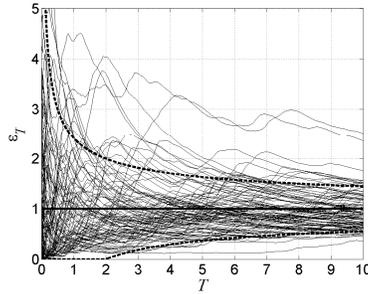


Fig. 2. 100 sample paths of ϵ_T versus $T \leq 10$ for the Ornstein-Uhlenbeck process generated by (11) with $a = -1$, so that the critical time horizon beyond which ϵ_T exposes relative proximity to the limit value $\|F\|_2^2 = 1$ (horizontal bold line) is $T_* = 2$. The dashed bold lines localize the typical values of ϵ_T which form a “tube” of half-width $\sqrt{T_*/T}$ about the limit.

III. CUMULANTS OF OUTPUT ENERGY AND HARDY-SCHATTEN NORMS

For a finite time horizon $T > 0$, let C_T denote a Toeplitz integral operator whose kernel is specified by the covariance function (1). A \mathbb{R}^p -valued integrable function $\psi := (\psi_t)_{0 \leq t \leq T}$ is mapped by C_T to $\varphi := (\varphi_s)_{0 \leq s \leq T}$ as $\varphi_s := \int_0^T c_{s-t} \psi_t dt$. Suppose θ is a real parameter satisfying $0 < \theta < 1/\rho(C_T)$, where $\rho(\cdot)$ is the spectral radius. In view of the Fredholm formula [12, Theorem 3.10 on p. 36] (see also [4] and references therein),

$$\ln \mathbf{E} e^{\theta \mathcal{E}_T / 2} = -\frac{1}{2} \text{Tr} \ln(I - \theta C_T) = \frac{1}{2} \sum_{k \geq 1} \theta^k \text{Tr}(C_T^k) / k, \quad (12)$$

where I is the identity operator. The trace of the k -fold iterate of C_T is computed as

$$\begin{aligned} \text{Tr}(C_T^k) &= \int_{[0, T]^k} \text{Tr}(c_{t_0-t_1} c_{t_1-t_2} \times \dots \\ &\quad \times c_{t_{k-2}-t_{k-1}} c_{t_{k-1}-t_0}) dt_0 \times \dots \times dt_{k-1}. \end{aligned} \quad (13)$$

The expectation in (12) is the moment-generating function of \mathcal{E}_T , and hence,

$$\begin{aligned} \ln \mathbf{E} e^{\theta \mathcal{E}_T / 2} &= \sum_{k \geq 1} (\theta/2)^k \mathbf{K}_k(\mathcal{E}_T) / k! \\ &= \theta (\mathbf{E} \mathcal{E}_T + \theta \text{var}(\mathcal{E}_T) / 4) / 2 + O(\theta^3), \quad \theta \rightarrow 0. \end{aligned} \quad (14)$$

Here, $\mathbf{K}_k(\xi) := \partial_v^k \ln \mathbf{E} e^{v\xi} \big|_{v=0} = P_k(\mathbf{E}\xi, \dots, \mathbf{E}(\xi^k))$ denotes the k th cumulant of a random variable ξ , which is related with the first k moments of ξ via a universal polynomial P_k . The first three of these polynomials are $P_1(\mu_1) = \mu_1$, $P_2(\mu_1, \mu_2) = \mu_2 - \mu_1^2$ and $P_3(\mu_1, \mu_2, \mu_3) = \mu_3 - 3\mu_1\mu_2 + 2\mu_1^3$. By comparing the power series in (12) and (14) and using the identity $(2r)!! = r!2^r$, it follows that the k th cumulant of the output energy (4) of the system is related to the trace (13) as

$$\mathbf{K}_k(\mathcal{E}_T) = (2k - 2)!! \text{Tr}(C_T^k). \quad (15)$$

Using (2) and extending (3) and (6), we define, for a positive integer k , a higher order Hardy norm of the system F by

$$\|F\|_{2k} := \sqrt[2k]{\frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr}(S(\omega)^k) d\omega}, \quad (16)$$

which reproduces the \mathcal{H}_2 and \mathcal{H}_4 -norms for $k = 1, 2$. Here, $\sqrt[2k]{\text{Tr}(S(\omega)^k)}$ is the Schatten $2k$ -norm [6, p. 441] of the matrix $\widehat{F}(\omega)$. The resulting Hardy-Schatten space $\mathcal{H}_{2k}^{p \times m}$ is equipped with the norm $\|\cdot\|_{2k}$. Similarly to the \mathcal{H}_2 -norm, the \mathcal{H}_{2k} -norms (16) are all invariant under replacing the system F with its dual F^\dagger ,

$$\|F^\dagger\|_{2k} = \|F\|_{2k}, \quad k \geq 1, \quad (17)$$

where F^\dagger has the transposed impulse response $(f_t^\dagger)_{t \geq 0}$. Indeed, the transpose of a square matrix does not modify its spectrum, and for conformable complex matrices X and Y , the matrices XY and YX share nonzero eigenvalues. Therefore, with the dependence on the frequency ω omitted for brevity, $\text{Tr}((\widehat{F}^\dagger(\widehat{F}^\dagger)^*)^k) = \text{Tr}(((\widehat{F}^* \widehat{F})^k)^\dagger) = \text{Tr}((\widehat{F} \widehat{F}^*)^k)$, and hence (17) follows. By the Szegő limit theorem for Toeplitz operators [5], under additional integrability conditions,

$$\lim_{T \rightarrow +\infty} (\text{Tr} \chi(C_T) / T) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \text{Tr} \chi(S(\omega)) d\omega. \quad (18)$$

Here, χ is a function of a complex variable, satisfying $\chi(0) = 0$ and analytic in a neighbourhood of the interval $[0, \|F\|_\infty^2]$, with $\|F\|_\infty$ the \mathcal{H}_∞ -norm of F . In view of (15), the application of (18) to elementary polynomials $\chi(v) := v^k$ yields the asymptotically linear growth of the output energy cumulants with respect to time: $\lim_{T \rightarrow +\infty} (\mathbf{K}_k(\mathcal{E}_T) / T) = (2k - 2)!! \|F\|_{2k}^{2k}$, provided that $F \in \bigcap_{j=1}^k \mathcal{H}_{2j}^{p \times m}$, with Lemma 1 being a particular case for $k = 2$. The application of (18) to $\chi(v) := (2/\theta) \ln(1 - \theta v)$, with $0 < \theta < \|F\|_\infty^{-2}$, gives

$$\begin{aligned} \frac{2}{\theta} \lim_{T \rightarrow +\infty} \frac{\ln \mathbf{E} e^{\theta \mathcal{E}_T / 2}}{T} &= -\frac{1}{2\pi\theta} \int_{-\infty}^{\infty} \ln \det(I_p - \theta S(\omega)) d\omega \\ &= \sum_{k \geq 1} \theta^{k-1} \|F\|_{2k}^{2k} / k = \mathbf{Q}_\theta(F) + O(\theta^2), \quad \theta \rightarrow 0, \end{aligned} \quad (19)$$

where I_p denotes the identity matrix of order p , and

$$\mathbf{Q}_\theta(F) := \|F\|_2^2 + \theta \|F\|_4^4 / 2. \quad (20)$$

The expected exponential-of-quadratic functional $\mathbf{E} e^{\theta \mathcal{E}_T / 2}$ in (19) is used as a performance criterion in the risk-sensitive

and minimum entropy control theories [11]. The *quartic norm* $\|F\|_4$ provides the next correction to the squared \mathcal{H}_2 -norm $\|F\|_2^2$ in the series expansion (19) for small θ . Therefore, the *quadro-quartic* functional \mathbf{Q}_θ , defined by (20), can be regarded as a finer truncation of the risk-sensitive performance index.

IV. QUADRO-QUARTIC FUNCTIONAL IN STATE SPACE

Let F be a strictly proper LTI system with an m -dimensional standard Wiener process W at the input, p -dimensional output Z and n -dimensional state X governed by an Ito SDE:

$$dx_t = Ax_t dt + Bdw_t, \quad z_t = Cx_t, \quad (21)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$ are constant matrices. The state-space representation will be written as

$$F = (A, B, C) = \begin{matrix} & \begin{matrix} \leftarrow n & \rightarrow \leftarrow m \end{matrix} \\ \begin{matrix} n \\ p \end{matrix} & \begin{matrix} \downarrow & \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \\ \downarrow & \end{matrix} \end{matrix}, \quad (22)$$

where we have also shown the dimensions, and the horizontal and vertical separators serve to avoid confusion with an ordinary block matrix. The dual system is $F^\dagger = (A^T, C^T, B^T)$. If the matrix A is Hurwitz, then the mutually dual controllability and observability Gramians P and Q of (22) are unique solutions of the algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \quad (23)$$

In what follows, an important role is played by the matrix

$$H := QP, \quad (24)$$

whose spectrum is formed by the squared Hankel singular values of the system (22). We will write $\|X\|_M := \sqrt{\text{Tr}(X^T M X)}$ for the weighted Frobenius (semi-) norm of a real matrix X generated by a positive (semi-) definite matrix M .

Lemma 2: Let F be an asymptotically stable system with the state-space realization (22). Then the quartic norm (6) is expressed in terms of the Gramians P , Q from (23) and the matrix H from (24) as

$$\begin{aligned} \|F\|_4^4 &= 2\|(A, PC^T, C)\|_2^2 = 2\|PC^T\|_Q^2 \\ &= 2\|(A, B, B^T Q)\|_2^2 = 2\|QB\|_P^2 = -4\text{Tr}(A^T H^2). \end{aligned} \quad (25)$$

Proof: Let Z be a stationary Gaussian random process generated by (21), with W a standard Wiener process. Then the steady-state covariance function (1) is

$$c_t = Ce^{At} PC^T, \quad t \geq 0. \quad (26)$$

Here, we use the fact that the controllability Gramian is the steady-state covariance matrix of the state of the system: $P = \text{cov}(x_t)$. Since the function c_t in (26) coincides with the impulse response of the system (A, PC^T, C) , then (6) yields $\|F\|_4^4 = 2\|(A, PC^T, C)\|_2^2 = 2\text{Tr}(CPQP C^T) = 2\|PC^T\|_Q^2$, which proves the first two equalities in (25). Here, we have also used the property that the system (A, PC^T, C) shares the matrices A , C with the underlying system (22) and hence, inherits from F the observability

Gramian Q . The remaining three equalities in (25) follow from the first two by the invariance of the \mathcal{H}_2 and \mathcal{H}_4 -norms under taking the dual of a system, and by the duality of the controllability and observability Gramians. \square

The controllability and observability Gramians Φ , Ψ of a subsidiary system $(A, PC^T, B^T Q)$, which satisfy the algebraic Lyapunov equations

$$A\Phi + \Phi A^T + PC^T C P = 0, \quad A^T \Psi + \Psi A + QB B^T Q = 0, \quad (27)$$

will be referred to as the controllability and observability *Schattenians* of the system (22). The representations (25) imply that

$$\|F\|_4^4 = 2\text{Tr}(C\Phi C^T) = 2\text{Tr}(B^T \Psi B),$$

and hence, the significance of the Schattenians Φ , Ψ for the quartic norm is analogous to the role which the Gramians P , Q play for the \mathcal{H}_2 -norm.

Theorem 1: Let F be an asymptotically stable system with the state-space realization (22). Then the quadro-quartic functional (20) is expressed in terms of the Gramians P , Q from (23) and the matrix H from (24) as

$$\begin{aligned} \mathbf{Q}_\theta(F) &= \|(A, [B \quad \sqrt{\theta} PC^T], C)\|_2^2 \\ &= \text{Tr}((BB^T + \theta PC^T C P)Q) \\ &= \left\| \left(A, B, \begin{bmatrix} C \\ \sqrt{\theta} B^T Q \end{bmatrix} \right) \right\|_2^2 = \text{Tr}((C^T C + \theta QB B^T Q)P) \\ &= -2\text{Tr}(A^T H(I_n + \theta H)). \end{aligned} \quad (28)$$

Proof: Substitution of $\|F\|_2 = \sqrt{\text{Tr}(B^T Q B)}$ and the first two equalities from (25) into (20) yields

$$\begin{aligned} \mathbf{Q}_\theta(F) &= \|(A, B, C)\|_2^2 + \theta \|(A, PC^T, C)\|_2^2 \\ &= \|(A, [B \quad \sqrt{\theta} PC^T], C)\|_2^2 = \text{Tr}((BB^T + \theta PC^T C P)Q), \end{aligned}$$

which establishes the first two equalities in (28). The third and fourth representations of the quadro-quartic functional are obtained from the first two by the duality argument or directly from the third and fourth equalities in (25). The last representation of $\mathbf{Q}_\theta(F)$ in (28) follows from the previous ones by using the Lyapunov equations (23):

$$\begin{aligned} \mathbf{Q}_\theta(F) &= \text{Tr}((BB^T + \theta PC^T C P)Q) \\ &= -\text{Tr}((AP + PA^T + \theta P(A^T Q + QA)P)Q) \\ &= -2\text{Tr}(A^T QP + \theta A^T (QP)^2) = -2\text{Tr}(A^T H(I_n + \theta H)). \end{aligned}$$

\square

V. LINEAR QUADRO-QUARTIC GAUSSIAN CONTROL PROBLEM

Consider a plant with a m_1 -dimensional standard Wiener process W as the input disturbance and a m_2 -dimensional input control signal U . The outputs of the system are a p_1 -dimensional to-be-controlled signal Z and a p_2 -dimensional observation signal Y . Also, the system has a n -dimensional state X . These processes are governed by

$$dx_t = Ax_t dt + B_1 dw_t + B_2 u_t dt, \quad (29)$$

$$z_t = C_1 x_t + D_{12} u_t, \quad (30)$$

$$dy_t = C_2 x_t dt + D_{21} dw_t. \quad (31)$$

Here, $A \in \mathbb{R}^{n \times n}$, $B_k \in \mathbb{R}^{n \times m_k}$, $C_j \in \mathbb{R}^{p_j \times n}$, $D_{jk} \in \mathbb{R}^{p_j \times m_k}$, with $D_{11} = 0$ and $D_{22} = 0$. The control signal U is generated at the output of a controller K with input Y . We consider a strictly proper LTI controller

$$K = \begin{array}{c} n \uparrow \\ \downarrow m_2 \\ \left[\begin{array}{c|c} a & b \\ \hline c & 0 \end{array} \right], \end{array} \quad (32)$$

with a n -dimensional state Ξ . It is driven by the observation Y and produces the output U as

$$d\xi_t = a\xi_t dt + bdy_t, \quad u_t = c\xi_t, \quad (33)$$

where $a \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^{n \times p_2}$, $c \in \mathbb{R}^{m_2 \times n}$. The closed-loop system

$$F := \begin{array}{c} 2n \uparrow \\ \downarrow p_1 \\ \left[\begin{array}{c|c} \mathcal{A} & \mathcal{B} \\ \hline \mathcal{C} & 0 \end{array} \right] = \left[\begin{array}{cc|c} a & bC_2 & bD_{21} \\ B_2c & A & B_1 \\ \hline D_{12}c & C_1 & 0 \end{array} \right], \end{array} \quad (34)$$

governed by (29)–(33) and depicted in Fig. 3, has the $2n$ -

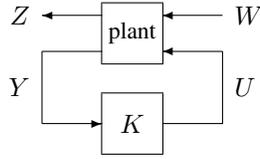


Fig. 3. The closed-loop system F with input W and output Z .

dimensional combined state (Ξ, X) . We formulate a *linear quadro-quartic Gaussian* (LQQG) control problem as the minimization of the functional (20) over n -dimensional controllers (32) such that the matrix \mathcal{A} of the closed-loop system in (34) is Hurwitz:

$$\mathbf{Q} := \mathbf{Q}_\theta(F) = -2\text{Tr}(\mathcal{A}^T H (I_{2n} + \theta H)) \rightarrow \min, \quad K \text{ stabilizes } F. \quad (35)$$

Here, $\theta \geq 0$ is a given parameter as before, and use is made of Theorem 1, so that the matrix H is associated by (24) with the Gramians P, Q of the closed-loop system satisfying the algebraic Lyapunov equations

$$AP + PA^T + BB^T = 0, \quad A^T Q + QA + C^T C = 0. \quad (36)$$

In the case $\theta = 0$, the LQQG problem (35) reduces to the standard linear quadratic Gaussian (LQG) control problem. For $\theta > 0$, the LQQG problem is a compromise between minimizing the mean value and the variance of the output energy per unit time, with θ becoming the relative weight of the quartic norm.

VI. MATRICES WITH Γ -SHAPED SPARSITY

Since it is convenient to assemble the state-space realization matrices into a matrix with “ Γ -shaped” sparsity, we denote the set of real $(r+p) \times (r+m)$ -matrices with zero bottom-right block of size $(p \times m)$ by

$$\mathbf{\Gamma}_{r,m,p} := \left\{ \begin{bmatrix} \rho & \sigma \\ \tau & 0 \end{bmatrix} : \rho \in \mathbb{R}^{r \times r}, \sigma \in \mathbb{R}^{r \times m}, \tau \in \mathbb{R}^{p \times r} \right\}. \quad (37)$$

This is a linear subspace of $\mathbb{R}^{(r+p) \times (r+m)}$ which inherits the Frobenius inner product of matrices. Let $\mathbf{\Pi}_{r,m,p}$ denote the

orthogonal projection onto $\mathbf{\Gamma}_{r,m,p}$ which pads the bottom-right $(p \times m)$ -block of a $(r+p) \times (r+m)$ -matrix with zeros:

$$\mathbf{\Pi}_{r,m,p} \left(\begin{bmatrix} \rho & \sigma \\ \tau & \varpi \end{bmatrix} \right) = \begin{bmatrix} \rho & \sigma \\ \tau & 0 \end{bmatrix}. \quad (38)$$

The dependence of the closed-loop system matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ on the controller matrices a, b, c in (34) can be written as

$$\mathbf{\Gamma} := \begin{bmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & 0 \end{bmatrix} = \mathbf{\Gamma}_0 + \mathbf{\Gamma}_1 \gamma \mathbf{\Gamma}_2, \quad \gamma := \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}. \quad (39)$$

The affine map $\mathbf{\Gamma}_{n,p_2,m_2} \ni \gamma \mapsto \mathbf{\Gamma} \in \mathbf{\Gamma}_{2n,m_1,p_1}$ is specified completely by three matrices

$$\mathbf{\Gamma}_0 := \begin{bmatrix} 0_n & 0 & 0 \\ 0 & A & B_1 \\ 0 & C_1 & 0 \end{bmatrix}, \quad \mathbf{\Gamma}_1 := \begin{bmatrix} I_n & 0 \\ 0 & B_2 \\ 0 & D_{12} \end{bmatrix}, \quad \mathbf{\Gamma}_2 := \begin{bmatrix} I_n & 0 & 0 \\ 0 & C_2 & D_{21} \end{bmatrix}, \quad (40)$$

where 0_n denotes the $(n \times n)$ -matrix of zeros.

VII. EQUATIONS FOR OPTIMAL CONTROLLER

We now obtain necessary conditions of optimality in the class (32) of n -dimensional stabilizing controllers K for the LQQG problem (35). To this end, we compute the Frechet derivatives of the quadro-quartic functional of the closed-loop system F as a composite function $\gamma \mapsto \mathbf{\Gamma} \mapsto \mathbf{Q}$ of the controller matrices a, b, c and equate the derivatives to zero. The differentiation is carried out in two steps: we first consider $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to be independent variables, and then take into account their dependence on a, b, c .

Lemma 3: The Frechet derivatives of the quadro-quartic functional \mathbf{Q} with respect to the closed-loop system matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$, assembled into the matrix $\mathbf{\Gamma}$ in (39), are computed as

$$\partial_{\mathbf{\Gamma}} \mathbf{Q} := \begin{bmatrix} \partial_{\mathcal{A}} \mathbf{Q} & \partial_{\mathcal{B}} \mathbf{Q} \\ \partial_{\mathcal{C}} \mathbf{Q} & 0 \end{bmatrix} = 2 \begin{bmatrix} R & \Omega \mathbf{B} \\ \mathcal{C} \Upsilon & 0 \end{bmatrix}. \quad (41)$$

Here,

$$\Upsilon := P + \theta(PH + \Phi), \quad (42)$$

$$\Omega := Q + \theta(HQ + \Psi), \quad (43)$$

$$R := H + \theta(H^2 + Q\Phi + \Psi P), \quad (44)$$

with P, Q the Gramians from (36); the matrix H is given by (24), and Φ, Ψ are the controllability and observability Schattenians of F satisfying the algebraic Lyapunov equations

$$\mathcal{A}\Phi + \Phi\mathcal{A}^T + PC^T CP = 0, \quad \mathcal{A}^T \Psi + \Psi\mathcal{A} + QB B^T Q = 0. \quad (45)$$

Proof: By recalling (20) and applying Lemmas 7, 8 of Appendices B, C to the closed-loop system F , it follows that

$$\begin{aligned} \partial_{\mathbf{\Gamma}} \mathbf{Q} &= \partial_{\mathbf{\Gamma}} (\|F\|_2^2) + \theta \partial_{\mathbf{\Gamma}} (\|F\|_4^4) / 2 \\ &= 2 \begin{bmatrix} H & Q\mathbf{B} \\ \mathcal{C}P & 0 \end{bmatrix} + 2\theta \begin{bmatrix} H^2 + Q\Phi + \Psi P & (HQ + \Psi)\mathbf{B} \\ \mathcal{C}(PH + \Phi) & 0 \end{bmatrix} \end{aligned}$$

which, in view of the notations (42)–(44), implies (41). \square

The Gramians P, Q of the closed-loop system and related matrices (that is, $H, \Phi, \Psi, \Upsilon, \Omega, R$) inherit the four $(n \times n)$ -block structure of the matrix \mathcal{A} in (34). The blocks are numbered as follows:

$$\mathcal{A} := \begin{bmatrix} \overleftarrow{\mathcal{A}}_{11} & \overleftarrow{\mathcal{A}}_{12} \\ \overleftarrow{\mathcal{A}}_{21} & \overleftarrow{\mathcal{A}}_{22} \end{bmatrix} \begin{matrix} \uparrow_n \\ \downarrow_n \end{matrix} = \begin{bmatrix} \overleftarrow{\mathcal{A}}_{\bullet 1} & \overleftarrow{\mathcal{A}}_{\bullet 2} \end{bmatrix} \begin{matrix} \uparrow_{2n} \\ \downarrow_{2n} \end{matrix} = \begin{bmatrix} \overleftarrow{\mathcal{A}}_{1\bullet} \\ \overleftarrow{\mathcal{A}}_{2\bullet} \end{bmatrix} \begin{matrix} \uparrow_n \\ \downarrow_n \end{matrix}. \quad (46)$$

In this notation, the $(\cdot)_{11}$ blocks are associated with the controller state, and the $(\cdot)_{22}$ blocks pertain to the plant state.

Lemma 4: The Frechet derivatives of the quadro-quartic functional \mathbf{Q} of the closed-loop system (34) with respect to the controller matrices a, b, c , assembled into the matrix γ in (39), are computed as

$$\begin{aligned} \partial_\gamma \mathbf{Q} &= \begin{bmatrix} \partial_a \mathbf{Q} & \partial_b \mathbf{Q} \\ \partial_c \mathbf{Q} & 0 \end{bmatrix} \\ &= 2 \begin{bmatrix} R_{11} & R_{12} C_2^T + \Omega_{1\bullet} \mathcal{B} D_{21}^T \\ B_2^T R_{21} + D_{12}^T \mathcal{C} \Upsilon_{\bullet 1} & 0 \end{bmatrix}, \quad (47) \end{aligned}$$

where the matrices Υ, Ω, R are defined by (42)–(44).

Proof: Since \mathbf{Q} is a composite function of a, b, c which enter this functional through the matrices $\mathcal{A}, \mathcal{B}, \mathcal{C}$ of the closed-loop system F , the chain rule yields

$$\partial_\gamma \mathbf{Q} = (\partial_\gamma \Gamma)^\dagger (\partial_\Gamma \mathbf{Q}) = \Pi_{n,p_2,m_2} (\Gamma_1^T \partial_\Gamma \mathbf{Q} \Gamma_2^T). \quad (48)$$

Here, $(\cdot)^\dagger$ denotes the adjoint of a linear operator in the sense of the Frobenius inner product of matrices, and Π_{n,p_2,m_2} is the orthogonal projection onto the subspace Γ_{n,p_2,m_2} defined by (37)–(38). Indeed, the first variation of the affine map Γ , defined by (39), is $\delta\Gamma = \Gamma_1(\delta\gamma)\Gamma_2$. Hence, $\delta\mathbf{Q} = \text{Tr}(\partial_\Gamma \mathbf{Q} \delta\Gamma^T) = \text{Tr}(\partial_\Gamma \mathbf{Q} (\Gamma_1(\delta\gamma)\Gamma_2)^T) = \text{Tr}(\Gamma_1^T \partial_\Gamma \mathbf{Q} \Gamma_2^T \delta\gamma^T) = \text{Tr}(\Pi_{n,p_2,m_2} (\Gamma_1^T \partial_\Gamma \mathbf{Q} \Gamma_2^T) \delta\gamma^T)$, which establishes (48). Substitution of the matrices Γ_1 and Γ_2 from (40) into the right-hand side of (48) yields

$$\begin{aligned} \partial_\gamma \mathbf{Q} &= \Pi_{n,p_2,m_2} \left(\begin{bmatrix} I_n & 0 & 0 \\ 0 & B_2^T & D_{12}^T \end{bmatrix} \begin{bmatrix} \partial_a \mathbf{Q} & \partial_b \mathbf{Q} \\ \partial_c \mathbf{Q} & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & C_2^T \\ 0 & D_{21}^T \end{bmatrix} \right) \\ &= \begin{bmatrix} (\partial_a \mathbf{Q})_{11} & (\partial_a \mathbf{Q})_{12} C_2^T + (\partial_b \mathbf{Q})_{11} D_{21}^T \\ B_2^T (\partial_a \mathbf{Q})_{21} + D_{12}^T (\partial_c \mathbf{Q})_{11} & 0 \end{bmatrix}. \quad (49) \end{aligned}$$

Here, in view of (41),

$$\partial_a \mathbf{Q} = 2R, \quad (\partial_b \mathbf{Q})_{11} = 2\Omega_{1\bullet} \mathcal{B}, \quad (\partial_c \mathbf{Q})_{11} = 2\mathcal{C} \Upsilon_{\bullet 1}, \quad (50)$$

and the block numbering (46) is used. The assertion (47) of the lemma now follows from (49) and (50). \square

Necessary conditions for optimality in the class of controllers (32) for the LQQG problem (35) are now obtained by equating the blocks of the matrix $\partial_\gamma \mathbf{Q}$ in (47) to zero:

$$R_{11} = 0, \quad (51)$$

$$R_{12} C_2^T + \Omega_{1\bullet} \mathcal{B} D_{21}^T = 0, \quad (52)$$

$$B_2^T R_{21} + D_{12}^T \mathcal{C} \Upsilon_{\bullet 1} = 0. \quad (53)$$

VIII. OBSERVATION-STATE AND STATE-FEEDBACK MATRICES

Lemma 5: Suppose the matrix D_{21} is of full row rank, and D_{12} is of full column rank. Also, let (32) be a stabilizing controller with a minimal state-space realization. Then the

top-left blocks of the matrices P, Q from (36) and Υ, Ω from (42), (43) are all positive definite:

$$P_{11} \succ 0, \quad Q_{11} \succ 0, \quad \Upsilon_{11} \succ 0, \quad \Omega_{11} \succ 0. \quad (54)$$

Proof: Since $\theta \geq 0$, and the matrices $PH = PQQP, HQ = QPQ$, associated with the Gramians P, Q , and the Schattenians Φ, Ψ from (45) are all positive semi-definite, then (42) and (43) imply that $\Upsilon \succcurlyeq P$ and $\Omega \succcurlyeq Q$. Hence, the same ordering holds for the top-left blocks of these matrices: $\Upsilon_{11} \succcurlyeq P_{11}$ and $\Omega_{11} \succcurlyeq Q_{11}$. Therefore, the last two relations in (54) will follow from the first two. We will now prove that $P_{11} \succ 0$ under the assumptions that D_{21} is of full row rank and (a, b) is controllable. Indeed, P_{11} is the covariance matrix of the controller state:

$$\begin{aligned} P_{11} &= \text{cov}(\xi_t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} g(\omega) \Lambda(\omega) g(\omega)^* d\omega, \\ g(\omega) &:= (i\omega I_n - a)^{-1} b, \quad (55) \end{aligned}$$

where $\Lambda(\omega) := h(\omega)h(\omega)^*$ is the spectral density associated with the observation signal Y from (31), with $h(\omega) := D_{21} + [0 \ C_2] (i\omega I_{2n} - \mathcal{A})^{-1} \mathcal{B}$. From $\lim_{\omega \rightarrow \infty} \Lambda(\omega) = D_{21} D_{21}^T$, it follows that if D_{21} is of full row rank, then $\Lambda(\omega) \succ 0$ for all sufficiently large ω , say $|\omega| > \omega_0$. Now, if P_{11} is singular, then $v^T P_{11} v = 0$ for some nonzero $v \in \mathbb{R}^n$. In this case, (55) yields $0 = v^T P_{11} v \geq (2\pi)^{-1} \int_{|\omega| > \omega_0} \|g(\omega)^* v\|_{\Lambda(\omega)}^2 d\omega$, which, in view of $\Lambda(\omega) \succ 0$ over the high frequency range, implies that $v^T g(\omega) = 0$ for all $|\omega| > \omega_0$. Hence, by considering the first n terms of the Laurent series $v^T g(\omega) = \sum_{k=1}^{+\infty} v^T a^{k-1} b / (i\omega)^k$ at infinity [8, Lemma 2.3 on pp. 16–17], it follows that the rank of the matrix $[b \mid \dots \mid a^{n-1} b]$ is less than n , and the pair (a, b) is not controllable. Thus, the full row rank of D_{21} and the controllability of (a, b) indeed ensure $P_{11} \succ 0$. By duality, a similar reasoning shows that the observability of (a, c) and the full column rank condition on D_{12} imply $Q_{11} \succ 0$. \square

Theorem 2: Suppose the matrix D_{21} is of full row rank, and D_{12} is of full column rank. Then the matrices b and c of an optimal controller (32) in the LQQG problem (35) with a minimal state-space realization satisfy

$$b = -\Omega_{11}^{-1} (R_{12} C_2^T + \Omega_{12} B_1 D_{21}^T) (D_{21} D_{21}^T)^{-1}, \quad (56)$$

$$c = -(D_{12}^T D_{12})^{-1} (B_2^T R_{21} + D_{12}^T C_1 \Upsilon_{21}) \Upsilon_{11}^{-1}, \quad (57)$$

where the matrices Υ, Ω, R are defined by (42)–(44).

Proof: Substitution of the matrices \mathcal{B} and \mathcal{C} from (34) into (52) and (53) brings these equations to the form

$$R_{12} C_2^T + (\Omega_{11} b D_{21} + \Omega_{12} B_1) D_{21}^T = 0, \quad (58)$$

$$B_2^T R_{21} + D_{12}^T (D_{12} c \Upsilon_{11} + C_1 \Upsilon_{21}) = 0. \quad (59)$$

By Lemma 5, the matrices Υ_{11} and Ω_{11} are nonsingular. Therefore, left multiplication of both sides of (58) by Ω_{11}^{-1} and right multiplication by $(D_{21} D_{21}^T)^{-1}$ yields (56). Similarly, right multiplication of both sides of (59) by Υ_{11}^{-1} and left multiplication by $(D_{12}^T D_{12})^{-1}$ yields (57). \square

Under the assumptions of Theorem 2, the modified set of equations for the state-space realization matrices of an optimal controller in the LQQG problem (35) is formed

by the algebraic Lyapunov equations (36), (45) and by the algebraic equations (51), (56), (57). In the case $\theta = 0$, these equations can be shown to yield the two independent Riccati equations for the standard LQG controller.

IX. HOMOTOPY METHOD

With the matrix γ from (39), we associate a linear subspace of Γ_{n,p_2,m_2} by

$$\mathbb{T}(\gamma) = \left\{ \begin{bmatrix} \tau a - a\tau & \tau b \\ -c\tau & 0 \end{bmatrix} : \tau \in \mathbb{R}^{n \times n} \right\}. \quad (60)$$

This is the tangent space generated by the group of transformations $(a, b, c) \mapsto (\sigma a \sigma^{-1}, \sigma b, c \sigma^{-1})$ (where $\sigma \in \mathbb{R}^{n \times n}$ are arbitrary nonsingular matrices), which leave the transfer function of the controller (32), and hence, the input-output operator of the closed-loop system (39), unchanged. The matrix $\partial_\gamma \mathbf{Q}$, associated with the controller K , belongs to the orthogonal complement $\mathbb{T}(\gamma)^\perp$ of $\mathbb{T}(\gamma)$ to Γ_{n,p_2,m_2} in the sense of the Frobenius inner product. We say that the controller delivers a strong local minimum to the quadro-quartic functional \mathbf{Q} in (35) if, in addition to the equality $\partial_\gamma \mathbf{Q} = 0$, it also makes the second order Frechet derivative $\partial_\gamma^2 \mathbf{Q} = \partial_\gamma^2(\|F\|_2^2) + \theta \partial_\gamma^2(\|F\|_4^4)/2$ positive definite on the subspace $\mathbb{T}(\gamma)^\perp$. Now, suppose there exists a smooth map $0 \leq \theta \mapsto \gamma_*(\theta) \in \Gamma_{n,p_2,m_2}$ such that $\gamma_*(\theta)$ is a strong local minimum of the quadro-quartic functional \mathbf{Q}_θ of the closed-loop system F in the sense above, so that $\partial_\gamma \mathbf{Q}_\theta|_{\gamma=\gamma_*(\theta)} = 0$. By differentiating the last equality with respect to θ , it follows that

$$\partial_\gamma^2 \mathbf{Q}_\theta(\gamma'_*) + \partial_\gamma(\|F\|_4^4)/2 = 0. \quad (61)$$

Here, $\gamma'_*(\theta) := \partial_\theta \gamma_*(\theta)$ and use is made of the identity $\partial_\theta \mathbf{Q}_\theta = \|F\|_4^4/2$ which follows from (20) and, in view of the interchangeability of the derivatives in θ and γ , implies that $\partial_\theta \partial_\gamma \mathbf{Q}_\theta = \partial_\gamma(\|F\|_4^4)/2 \in \mathbb{T}(\gamma)^\perp$. Since the matrix $\gamma_*(\theta)$ is defined up to the orbit of the transformation group, then $\gamma'_*(\theta) := \partial_\theta \gamma_*(\theta)$ is defined modulo the subspace $\mathbb{T}(\gamma_*(\theta))$ from (60). Therefore, (61), which is a linear equation with respect to $\gamma'_*(\theta)$, can be restricted to the subspace $\mathbb{T}(\gamma_*(\theta))^\perp$. As long as $\gamma_*(\theta)$ is a strong local minimum of \mathbf{Q}_θ , so that the self-adjoint operator $\partial_\gamma^2 \mathbf{Q}$ is positive definite (and hence, invertible) on $\mathbb{T}(\gamma_*(\theta))^\perp$, the equation (61) is equivalent to

$$\gamma'_*(\theta) = -\mathbf{L}^{-1}(\partial_\gamma(\|F\|_4^4))/2, \quad (62)$$

where \mathbf{L} is the restriction of $\partial_\gamma^2 \mathbf{Q}$ to the subspace $\mathbb{T}(\gamma)^\perp$. The equation (62) is an ODE, with $\theta \geq 0$ playing the role of fictitious time. The initial value $\gamma_*(0)$ is provided by the state-space realization triple of the standard LQG controller. The computation of a LQQG controller for $\theta > 0$ can be carried out by numerically integrating the homotopy ODE (62) initialized at $\gamma_*(0)$. The operator \mathbf{L} involves Frechet differentiation of solutions of algebraic Lyapunov equations with respect to their coefficients, and the inverse \mathbf{L}^{-1} can be computed by using the vectorization of matrices [10]. The state-space formulae of the homotopy algorithm and other details of its implementation will be reported in a subsequent publication.

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APPENDIX

A. Covariance of squared norms of Gaussian random vectors

Lemma 6: Let ξ and η be jointly Gaussian random vectors with zero mean. Then the covariance of their squared Euclidean norms is expressed in terms of the Frobenius norm of their cross-covariance matrix by

$$\mathbf{cov}(|\xi|^2, |\eta|^2) = 2\|\mathbf{cov}(\xi, \eta)\|^2. \quad (\text{A.1})$$

Proof: By applying the representation [7] for the mixed moments of Gaussian random variables in terms of their covariances to the entries of the vectors ξ and η , it follows that $\mathbf{E}(\xi_i^2 \eta_j^2) = \mathbf{E}(\xi_i \xi_i) \mathbf{E}(\eta_j \eta_j) + \mathbf{E}(\xi_i \eta_j) \mathbf{E}(\xi_i \eta_j) + \mathbf{E}(\xi_i \eta_j) \mathbf{E}(\xi_i \eta_j) = \mathbf{E}(\xi_i^2) \mathbf{E}(\eta_j^2) + 2(\mathbf{cov}(\xi_i, \eta_j))^2$. Therefore,

$$\begin{aligned} \mathbf{E}(|\xi|^2 |\eta|^2) &= \sum_{i,j} \mathbf{E}(\xi_i^2 \eta_j^2) \\ &= \mathbf{E}(|\xi|^2) \mathbf{E}(|\eta|^2) + 2 \sum_{i,j} (\mathbf{cov}(\xi_i, \eta_j))^2, \end{aligned} \quad (\text{A.2})$$

where the rightmost sum is $\|\mathbf{cov}(\xi, \eta)\|^2$. The relation (A.1) is now obtained by substituting (A.2) into $\mathbf{cov}(|\xi|^2, |\eta|^2) := \mathbf{E}(|\xi|^2 |\eta|^2) - \mathbf{E}(|\xi|^2) \mathbf{E}(|\eta|^2)$. Note that (A.1) can also be established by using [9, Lemma 6.2]. \square

B. State space formula for Frechet derivative of \mathcal{H}_2 -norm

Lemma 7: The Frechet derivative of the squared \mathcal{H}_2 -norm $E := \|F\|_2^2$ of the system (22), with A Hurwitz, is computed as

$$\partial_\Gamma E = 2 \begin{bmatrix} H & QB \\ CP & 0 \end{bmatrix}, \quad \Gamma := \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}. \quad (\text{B.1})$$

Here, the matrix H is associated by (24) with the Gramians P, Q from (23).

Proof: The Frechet derivative $\partial_{\Gamma}E$ inherits the block structure of the matrix Γ :

$$\partial_{\Gamma}E = \begin{bmatrix} \partial_A E & \partial_B E \\ \partial_C E & 0 \end{bmatrix}. \quad (\text{B.2})$$

We will now compute the blocks of this matrix. To calculate $\partial_A E$, let B and C be fixed. Then the first variation of E with respect to A is $\delta E = \text{Tr}(C^T C \delta P) = -\text{Tr}((A^T Q + Q A) \delta P) = -\text{Tr}(Q(A \delta P + (\delta P) A^T)) = \text{Tr}(Q((\delta A) P + P \delta A^T)) = 2\text{Tr}(H \delta A^T)$, which implies that

$$\partial_A E = 2H. \quad (\text{B.3})$$

Here, use has also been made of the first variation of the Lyapunov equation for P with constant B which yields $A \delta P + (\delta P) A^T + (\delta A) P + P \delta A^T = 0$. To compute $\partial_B E$, we fix A and C . Then the observability Gramian Q , which is a function of A and C , is also constant, and the first variation of E with respect to B is $\delta E = \text{Tr}(Q \delta(B B^T)) = \text{Tr}(Q((\delta B) B^T + B \delta B^T)) = 2\text{Tr}(Q B \delta B^T)$, and hence,

$$\partial_B E = 2Q B. \quad (\text{B.4})$$

The derivative $\partial_C E$ is calculated by a similar reasoning. Assuming A and B (and so also the controllability Gramian P) to be fixed, the first variation of E with respect to C is $\delta E = \text{Tr}(P \delta(C^T C)) = \text{Tr}(P((\delta C^T) C + C^T \delta C)) = 2\text{Tr}(C P \delta C^T)$, which implies that

$$\partial_C E = 2C P. \quad (\text{B.5})$$

Substitution of (B.3)–(B.5) into (B.2) yields (B.1). \square

C. Frechet differentiation of quartic norm in state space

Lemma 8: The Frechet derivative of the fourth power $N := \|F\|_4^4$ of the quartic norm of the system (22), with A Hurwitz, is computed as

$$\partial_{\Gamma} N = 4 \begin{bmatrix} H^2 + Q\Phi + \Psi P & (HQ + \Psi)B \\ C(PH + \Phi) & 0 \end{bmatrix}. \quad (\text{C.1})$$

Here, the matrix H is associated by (24) with the Gramians P, Q from (23), and Φ, Ψ are the Schattenians from (27).

Proof: We will compute the Frechet derivative of N by using the representation

$$N = 2E_1 = 2E_2, \quad E_1 := \|F_1\|_2^2, \quad E_2 := \|F_2\|_2^2, \quad (\text{C.2})$$

of the \mathcal{H}_4 -norm from Lemma 2 in terms of the squared \mathcal{H}_2 -norms of the subsidiary systems $F_1 := (A, B, B^T Q)$ and $F_2 := (A, P C^T, C)$ as composite functions of the matrices A, B, C . Since the controllability and observability Gramians of F_1 are P and Ψ , and the controllability and observability Gramians of F_2 are Φ and Q , then application of Lemma 7 from Appendix B to the systems F_1 and F_2 yields

$$\partial_{\Gamma_1} E_1 = 2 \begin{bmatrix} \Psi P & \Psi B \\ B^T H & 0 \end{bmatrix}, \quad \Gamma_1 := \begin{bmatrix} A & B \\ B^T Q & 0 \end{bmatrix}, \quad (\text{C.3})$$

$$\partial_{\Gamma_2} E_2 = 2 \begin{bmatrix} Q\Phi & H C^T \\ C\Phi & 0 \end{bmatrix}, \quad \Gamma_2 := \begin{bmatrix} A & P C^T \\ C & 0 \end{bmatrix}. \quad (\text{C.4})$$

Suppose the matrices A and C are fixed and hence, so also is Q . Then (C.3) implies that the first variation of E_1 with respect to B is

$$\begin{aligned} \delta E_1 &= 2\text{Tr}(\Psi B \delta B^T) + 2\text{Tr}(B^T H \delta(B^T Q)^T) \\ &= 2\text{Tr}((\Psi B + Q H^T B) \delta B^T) \\ &= 2\text{Tr}((\Psi + H Q) B \delta B^T), \end{aligned} \quad (\text{C.5})$$

where the identity $Q H^T = Q P Q = H Q$ has also been used. From (C.2) and (C.5), it follows that

$$\partial_B N = 4(H Q + \Psi) B. \quad (\text{C.6})$$

Suppose the matrices A and B are fixed and hence, so also is P . Then (C.4) implies that the first variation of E_2 with respect to C is

$$\begin{aligned} \delta E_2 &= 2\text{Tr}(H C^T \delta(P C^T)^T) + 2\text{Tr}(C \Phi \delta C^T) \\ &= 2\text{Tr}((C \Phi + C H^T P) \delta C^T) \\ &= 2\text{Tr}(C(\Phi + P H) \delta C^T), \end{aligned} \quad (\text{C.7})$$

where the identity $H^T P = P Q P = P H$ has also been used. From (C.2) and (C.7), it follows that

$$\partial_C N = 4C(P H + \Phi). \quad (\text{C.8})$$

Now, let B and C be constant. Then, in view of (C.3), the variation of E_1 with respect to A is

$$\begin{aligned} \delta E_1 &= 2\text{Tr}(\Psi P \delta A^T) + 2\text{Tr}(B^T H \delta(B^T Q)^T) \\ &= 2\text{Tr}(\Psi P \delta A^T) + 2\text{Tr}(H^T B B^T \delta Q). \end{aligned} \quad (\text{C.9})$$

The first variation of the Lyapunov equation for Q in (23) with C constant yields $A^T \delta Q + (\delta Q) A + (\delta A)^T Q + Q \delta A = 0$. Therefore,

$$\begin{aligned} \text{Tr}((\delta Q) B B^T H) &= -\text{Tr}((\delta Q)(A P + P A^T) H) \\ &= -\text{Tr}((\delta Q) A P H) - \text{Tr}((\delta Q) P A^T H) \\ &= \text{Tr}((A^T \delta Q + (\delta A)^T Q + Q \delta A) P H) - \text{Tr}((\delta Q) P A^T H) \\ &= 2\text{Tr}(H^2 \delta A^T) + \text{Tr}(P(H A^T - A^T H) \delta Q) \\ &= 2\text{Tr}(H^2 \delta A^T) + \text{Tr}(P(C^T C P - Q B B^T) \delta Q) \\ &= 2\text{Tr}(H^2 \delta A^T) - \text{Tr}((A \Phi + \Phi A^T) \delta Q) - \text{Tr}(H^T B B^T \delta Q) \\ &= 2\text{Tr}((H^2 + Q \Phi) \delta A^T) - \text{Tr}(H^T B B^T \delta Q) \\ &= \text{Tr}((H^2 + Q \Phi) \delta A^T). \end{aligned} \quad (\text{C.10})$$

Here, we have also used the definition of the controllability Schattenian Φ in (27), and the identity $H A^T - A^T H = C^T C P - Q B B^T$ which is obtained from (23) and (24) as $0 = Q(A P + P A^T + B B^T) - (A^T Q + Q A + C^T C) P = H A^T - A^T H + Q B B^T - C^T C P$. Substitution of (C.10) into (C.9) yields $\delta E_1 = 2\text{Tr}((H^2 + Q \Phi + \Psi P) \delta A^T)$, which, in view of (C.2), implies that

$$\partial_A N = 4(H^2 + Q \Phi + \Psi P). \quad (\text{C.11})$$

The representation (C.1) now follows from (C.6), (C.8) and (C.11). \square