Risk-Sensitive Dissipativity and Relevant Control Problems

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Abstract— **The paper is focused on control-affine stochastic Itô systems with control-quadratic storage functions. The concept of dissipativity with risk-sensitive storage function (RSSF) is proposed, with dissipativity criterion derived involving generalized Hamilton-Jacobi-Bellman inequalities. The proof utilizes a certain version of stochastic Artstein's inequality. Connections to risk-sensitive suboptimal control, the theory of games, invariant probabilistic measure and deterministic $\mathcal{H}_\infty$-control are established. In linear-quadratic case the results are expressed via linear matrix inequalities (LMI). An example is provided.**

I. INTRODUCTION

A. Dissipativity

The concept of dissipativity, imported to control theory from physics by J.C. Willems [1], provides intuitively clear framework to analyze and synthesize systems using the notions of generalized energy functions, *videlicet*, *storage function* and *supply rate*. A certain system is dissipative if storage function and supply rate satisfy dissipation inequality describing generalized energy losses. Deterministic version of the theory was demonstrated efficient in nonlinear stabilization problems and numerous applications; among them are control problems for mechanical and electrical systems (robots, electric motors, power converters, ships, diesel engines), chemical processes, power systems and others. The reader is offered getting acquainted with the monograph [2] and references therein for further details on the subject.

In last years many investigators developed alternative approaches to dissipation in stochastic systems. With the purpose of being brief, let us summarize the most relevant control problems solved on the basis of stochastic dissipativity. They are: stochastic stabilization [3], stochastic $\mathcal{H}_2$-control and robust control [4], stochastic ergodic control [5], stochastic $\mathcal{H}_\infty$-control [6]-[7], robust simultaneous stabilization of deterministic systems [8], stochastic stabilization and $\mathcal{L}_2$-control of delayed systems [9].

B. Risk-Sensitivity

On the other hand, *risk-sensitive optimal control theory* should be identified as one of the most studied fields. D.H. Jacobson [10] was the first to embed parameterized exponential function into cost criterion associated with stochastic optimal control problem in linear-quadratic case. In his pioneering paper exponential performance indices were shown to result in optimal controllers that were dependent on a rate of exogenous disturbances. Further research let employ these specific features of the models with exponential cost criteria in applications, *prima facie*, economics [11][1] and missile guidance [12].

In last decades risk-sensitivity was discovered to build a certain "bridge" between stochastic and deterministic approaches to treating exogenous disturbances in control theory. In particular, it was proven in [13] that deterministic $\mathcal{H}_\infty$-control problem (disturbance attenuation problem) could be reduced to a soft-constrained differential game. Meanwhile, optimal control problem studied in [10] was equivalent to the same game. Similar connections for disturbance attenuation problem were derived later in general cases (nonlinear systems or nonquadratic cost criteria), see [14]-[15]. To finish brief overview of the matter, let us cite the recent publication [16] (see sources therein, either).

C. Scope of the Paper

Unfortunately, known versions of stochastic dissipativity do not directly address the issue of relation between storage function and risks (often modeled by Wiener processes). The aforesaid makes the authors motivated to suggest an alternative approach of stochastic dissipativity with risk-sensitive storage function (RSSF). Interesting observations are expected to arise from such kind of dissipativity. We focus on control-affine Itô diffusions with control-quadratic storage functions. Criterion of dissipativity with RSSF is derived, provided state-feedback control ensuring dissipativity is selected according to certain version of stochastic Artstein's inequality. Storage function and the above-mentioned state-feedback control are shown dependent on intensity of risks. This enables to relate our version of dissipativity to risk-sensitive suboptimal control [17], deterministic games and $\mathcal{H}_\infty$-control theory [13], invariant probabilistic measure [18], as well as to stochastic games [17]. A numerical example is provided.

II. FORMULATION OF THE PROBLEM

A. System Description

We have in mind the following control-affine system governed by stochastic differential Itô equation:

$$\begin{cases} dx_t = [f_1(x_t) + f_2(x_t)u_t]dt + \varepsilon D(x_t)d\mathcal{W}_t, \\ x_0 = x, \ t \in [0, \infty). \end{cases} \tag{1}$$

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[1]Connections to economic problems predetermined the name of risk-sensitivity for this field in optimal control theory

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Here $x_t \in \mathbb{R}^n$ is the state vector, $u_t \in \mathbb{R}^p$ is the control vector, $\mathcal{W}_t \in \mathbb{R}^m$ stands for a standard Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural filtration $\mathcal{F}_t (t \geq 0)$. The symbol $\varepsilon > 0$ denotes a certain parameter representing *intensity of risks* described by $\mathcal{W}_t$. The initial state x is deterministic, the functions $f(x,u) = f_1(x) + f_2(x)u$ and $D(x)$ are continuous, with $D(x)D^T(x) > 0 \; \forall x \in \mathbb{R}^n$.

Consider control-quadratic function

$$L(x_t, u_t) = L_0(x_t) + |u_t|^2, \tag{2}$$

associated with system (1). We will call it by *supply rate*, if for any admissible control $u_t : \mathbb{E}^x \int_0^t |L(x_s,u_s)| ds < \infty$. Here $\mathbb{E}^x$ indicates the operator of conditional expectation under the initial condition x.

B. The Set of Admissible Controls

As inputs we use Markov controls $u_t = \varphi(x_t)$ with continuous function φ. Given $u_t = \varphi(x_t)$, assume there exists a unique solution to (1) representing strong Markov process with respect to $\mathcal{F}_t$. Standard Lipschitz condition and condition of linear growth, imposed on $f(x,u)_{u=\varphi(x)}$ and $D(x)$ [19], ensure such class of controls being nonempty. The set of admissible controls $\Phi^2_{\mathcal{F}}(\mathbb{R}^p)$ includes $u_t = \varphi(x_t)$ such that

$$\|u\|^2_{\Phi^2_{\mathcal{F}}} = \mathbb{E}^x \int_0^t |u_s|^2 ds < \infty \; \forall t \in [0, \infty). \tag{3}$$

C. The Subject of Investigation

Our aim is to suggest a new version of dissipativity for system (1), (2) which would lead to risk-sensitive nature of corresponding storage functions.

We present the main result in Section III (i.e., dissipation inequality and criterion). Some interesting connections to other control theories are discussed in Section IV, while a numerical example is provided in Section V. Conclusions and possible future works are outlined in Section VI.

III. DISSIPATIVITY WITH RISK-SENSITIVE STORAGE FUNCTION

We begin with some technical aspects. Let $\mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^+)$ be the set of twice continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}^+$. Given $V(x) \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^+)$, denote $V_x(x) = [\partial V(x)/\partial x_1, \ldots, \partial V(x)/\partial x_n]$, $V_{xx}(x) = \left[\partial^2 V(x)/\partial x_i \partial x_j\right]_{n \times n}$. For $V(x) \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^+)$ and $\mathcal{F}_t$-measurable function u introduce the differential operator

$$\mathcal{A}_u V(x) = V_x(x) f(x,u) + \varepsilon^2 tr[V_{xx}(x) D(x) D^T(x)]/2. \tag{4}$$

For any predetermined u (4) represents the generator of random process (1) in $\mathbb{R}^n$. Since $D(x)D^T(x) > 0$, the operator $\mathcal{A}_u$ is strictly elliptical so as (1) is locally controlled by $\mathcal{W}_t$. For further discussion of random processes we refer the reader to [19] and the bibliography therein.

Consider the following dissipation inequality.

$$\begin{aligned}\mathbb{E}^x \exp V^{(\varepsilon)}(x_t) \leq \exp V^{(\varepsilon)}(x) + \\ + \mathbb{E}^x \int_0^t [-L(x_s,u_s) + \lambda^{(\varepsilon)}(x_s)] \exp V^{(\varepsilon)}(x_s) ds.\end{aligned} \tag{5}$$

Definition 1: System (1), (2) is called dissipative with risk-sensitive storage function (RSSF) on time interval $[0, \infty)$ if there exist a nonnegative continuous storage function $V^{(\varepsilon)} : \mathbb{R}^n \to \mathbb{R}^+$ and a nonnegative continuous supply increase function $\lambda^{(\varepsilon)} : \mathbb{R}^n \to \mathbb{R}^+$ such that (5) holds for any finite time t and any solutions $x_t, u_t \in \Phi^2_{\mathcal{F}}(\mathbb{R}^p)$ to (1) with the initial condition $x_0 = x \in \mathbb{R}^n$.

This notion of stochastic dissipativity involves exponential of storage function and supply increase function yet not considered by other researchers. Let us prove necessary and sufficient conditions for stochastic dissipativity with RSSF (dependence of corresponding functions on x is omitted in most cumbersome formulas to save space).

Theorem 1: System (1), (2) is dissipative with RSSF $V^{(\varepsilon)}$ on time interval $[0, \infty)$ under intensity ε of risks, if there exists a nonnegative continuous function $V^{(\varepsilon)} \in \mathcal{C}^2(\mathbb{R}^n;\mathbb{R}^+)$ solving the generalized Hamilton-Jacobi-Bellman (GHJB) inequality

$$V_x^{(\varepsilon)} f_1 + L_0 - \left[|f_2^T V_x^{(\varepsilon)T}|^2 - 2\varepsilon^2 |D^T V_x^{(\varepsilon)T}|^2\right]/4 \leq 0. \tag{6}$$

In this case the supply increase function and Markov control ensuring dissipativity with RSSF $V^{(\varepsilon)}$ are defined by

$$\lambda^{(\varepsilon)}(x) = \varepsilon^2 tr[V_{xx}^{(\varepsilon)}(x) D(x) D^T(x)]/2, \tag{7}$$

$$\varphi_1^{(\varepsilon)}(x) = -f_2^T(x) V_x^{(\varepsilon)T}(x)/2. \tag{8}$$

Inversely, suppose system (1), (2) is dissipative with RSSF $V^{(\varepsilon)}$ on time interval $[0, \infty)$ under intensity ε of risks. If Markov control ensuring dissipativity with RSSF $V^{(\varepsilon)}$ is selected according to the condition

$$\inf_u \{\mathcal{A}_u V^{(\varepsilon)} + L(x,u)\} \leq 0, \tag{9}$$

then RSSF $V^{(\varepsilon)}$ satisfies GHJB inequality (6), while the supply increase function and Markov control ensuring dissipativity with RSSF $V^{(\varepsilon)}$ are defined by (7) and (8), respectively.

Proof. We outline key steps only. Starting from dissipation inequality (5) and applying the generalized Itô formula [19]

$$\mathbb{E}^x V^{(\varepsilon)}(x_t) - V^{(\varepsilon)}(x) = \mathbb{E}^x \int_0^t [\mathcal{A}_u V^{(\varepsilon)}(x_s)] ds, \tag{10}$$

we obtain

$$\mathcal{A}_u \exp V^{(\varepsilon)} \leq [-L(x,u) + \lambda^{(\varepsilon)}] \exp V^{(\varepsilon)}. \tag{11}$$

Next, note that

$$\mathcal{A}_u \exp V^{(\varepsilon)} = \exp V^{(\varepsilon)} [\mathcal{A}_u V^{(\varepsilon)} + \varepsilon^2 |D^T V_x^{(\varepsilon)T}|^2/2]. \tag{12}$$

Thus, one arrives at the following relation:

$$\mathcal{A}_u V^{(\varepsilon)} + L(x,u) + \varepsilon^2 |D^T V_x^{(\varepsilon)T}|^2/2 \leq \lambda^{(\varepsilon)}. \tag{13}$$

Selecting the control in (13) based on condition (10), the reader derives formulas (6)-(8) and verifies the second part of theorem. We emphasize that the described steps may be performed in the inverse sequence; therefore, the first part is proven in much the same way.

Remark 1: GHJB inequality (6) has the extra term $\varepsilon^2|D^T(x)V_x^{(\varepsilon)^T}(x)|^2/2$ explicitly depending on intensity ε of risks; this states risk-sensitive nature of the storage function $V^{(\varepsilon)}(x)$ and Markov control ensuring dissipativity with RSSF $V^{(\varepsilon)}(x)$.

Remark 2: Formula (13) may be interpreted as an extension of stochastic Artstein's theorem [20]; it leads to undeniable relation between control Lyapunov functions and storage functions.

To conclude Section III, we adopt the formulas to special case of linear systems with quadratic supply rates, i.e.,

$$f_1(x_t) = Ax_t, f_2(x_t) = B, D(x_t) = D, L_0(x_t) = x_t^T Q x_t. \quad (14)$$

Obviously, the storage function becomes quadratic:

$$V^{(\varepsilon)}(x_t) = x_t^T Z^{(\varepsilon)} x_t,\ Z^{(\varepsilon)} = Z^{(\varepsilon)^T} \geq 0. \quad (15)$$

Expression (6) is reduced to generalized quadratic Riccati inequality (subject to the matrix $Z^{(\varepsilon)}$)

$$A^T Z^{(\varepsilon)} + Z^{(\varepsilon)} A + Q - Z^{(\varepsilon)}[BB^T - 2\varepsilon^2 DD^T]Z^{(\varepsilon)} \leq 0. \quad (16)$$

Supply increase function (7) and Markov control (8) are then calculated by

$$\lambda^{(\varepsilon)}(x) = \varepsilon^2 tr[Z^{(\varepsilon)} DD^T], \quad (17)$$

$$u = -B^T Z^{(\varepsilon)} x = \varphi_2^{(\varepsilon)}(x). \quad (18)$$

Standard Schur complement [21] makes it possible to transduce (16) to the linear form

$$\begin{bmatrix} -H^{(\varepsilon)}A^T - AH^{(\varepsilon)} + BB^T - 2\varepsilon^2 DD^T & H^{(\varepsilon)} \\ H^{(\varepsilon)} & Q^{-1} \end{bmatrix} \geq 0, \quad (19)$$

subject to symmetric matrix $H^{(\varepsilon)} = Z^{(\varepsilon)^{-1}} \geq 0$. This enables utilization of LMI solvers for obtaining numerical solution to the problem, see Section V.

IV. CONNECTIONS TO ERGODIC CONTROL, RISK-SENSITIVE SUBOPTIMAL CONTROL, GAMES, AND $\mathcal{H}_\infty$-CONTROL

In this section we demonstrate that the framework of dissipativity with RSSF could be efficiently involved to solve a number of control problems.

A. Ergodic Control Problem

We have to find Markov control $u_t = \varphi(x)$ stabilizing system (1), (2) in the sense of *unique invariant measure*

$$\mu_\varphi(C) = \int_{\mathbb{R}^n} \pi(t, x, C)\mu_\varphi(dx), \quad (20)$$

defined on σ-algebra $\mathcal{Y}$ of Borel sets in $\mathbb{R}^n$. Here π denotes the transition function of (1), $C \in \mathcal{Y}$. The measure is assumed to satisfy $\mathbb{E}_\varphi\{|x|^2\} = \int_{\mathbb{R}^n} |x|^2 \mu_\varphi(dx) < \infty$.

The following result takes place.

Lemma 1: Within the conditions given in the first part of dissipativity criterion in Section III, let the function L_0 be such that for sufficiently large x

$$\varepsilon^2|V_x^{(\varepsilon)}D|^2/2 \geq 1 + \varepsilon^2 tr[V_{xx}^{(\varepsilon)} DD^T]/2 - L_0 - |V_x^{(\varepsilon)} f_2|^2/4. \quad (21)$$

Then the control (8) solves the stated ergodic control problem for system (1), (2).

Proof. It could be easily verified that the condition (21), accompanied by nonrestrictive assumptions imposed on storage functions, imply positive recurrence of the process (1) under Markov control (8), see positivity criterion in [18]. This, in turn, ensures stabilization in the required sense.

Remark 3: Obviously, (21) is met by $L_0(x) \geq 0$.

Remark 4: In linear-quadratic case (1), (2), (14) formula (21) takes the form

$$Q \leq -Z^{(\varepsilon)}[BB^T + 2\varepsilon^2 DD^T]Z^{(\varepsilon)}. \quad (22)$$

B. Risk-Sensitive Suboptimal Control Problem and Stochastic Game

Consider stochastic control-affine system

$$\begin{cases} dx_t = [f_1(x_t) + f_2(x_t)u_t]dt + D(x_t)d\mathcal{W}_t, \\ x_0 = x,\ t \in [0, \infty), \end{cases} \quad (23)$$

defined similar to (1). Introduce parameterized control-quadratic cost function

$$\mathcal{J}(u) = \lim_{T\to\infty} T^{-1} \ln \mathbb{E}_x \exp\{\gamma^{-2}\times \times \int_0^T (L_0(x_s) + |u_s|^2)ds\} \to \min, \quad (24)$$

for Markov control $u = u(x) \in \mho$ and given scalar $\gamma > 0$; $\mho$ stands for the set of admissible controls.

Optimal control problem (23), (24) was considered by T. Runolfsson [17] in a general setting, with nonaffine system and nonquadratic cost function in control (to be within the limits of the manuscript, we do not focus on assumptions for problem (23)-(24), nor on description of $\mho$ referring the reader to original paper [17]). Following his technique, define the auxiliary stochastic system

$$dx_t^v = [f(x_t^v, u(x_t^v)) + D(x_t^v)v(x_t^v)]dt + D(x_t^v)d\mathcal{W}_t, \quad (25)$$

where $v : \mathbb{R}^n \to \mathbb{R}^k$. Denote by $\mathcal{A}_{u,v}$ the generator of (25) and let Σ_u be the set of all v's such that for some probability measure $\mu^{u,v} \in \mathcal{M}_\mu$:

$$\int_{\mathbb{R}^n} (\mathcal{A}_{u,v}h)(x)\mu^{u,v}(dx) = 0,\ h \in G_u \subset \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}). \quad (26)$$

Here $\mathcal{M}_\mu$ stands for the set of all probability measures, while G_u is a certain countable dense set. It is shown in [17] that

$$\mathcal{J}^* = \inf_{u\in\mho} \mathcal{J}(u) = \inf_{u\in\mho} \sup_{v\in\Sigma_u} \int_{\mathbb{R}^n} \left(\gamma^{-2}L - |v|^2/2\right) \mu^{u,v}(dx). \quad (27)$$

Thus, risk-sensitive optimal control problem (23)-(24) is equivalent to stochastic differential game (25), (27). Rewrite optimality conditions in the form

$$\mathcal{J}^* = \inf_{u\in\mho} \sup_{v\in\Sigma_u} \left[\mathcal{A}_{u,v}\phi + \gamma^{-2}L - |v|^2/2\right], \quad (28)$$

with function $\phi(x)$ being in the domain of $\mathcal{A}_{u,v}$. Next, evaluating *supremum* over v's and *infimum* over u's, one evidently has

$$v^* = D^T(x)\phi_x^T(x), \quad (29)$$

$$u^* = -\gamma^2 f_2^T(x)\phi_x^T(x)/2. \quad (30)$$

Substitute (29), (30) in (28) to obtain

$$\mathcal{J}^* = f_1\phi_x + \gamma^{-2}L_0 - \left[\gamma^2|f_2^T\phi_x^T|^2 - 2|D^T\phi_x^T|^2\right]/4+ \\ + tr[\phi_{xx}DD^T]. \quad (31)$$

We introduce the function

$$\phi(x) = \gamma^{-2}\widetilde{V}(x), \quad (32)$$

and arrive to the following formulas:

$$f_1\widetilde{V}_x + L_0 - \left[|f_2^T\widetilde{V}_x^T|^2 - 2\gamma^{-2}|D^T\widetilde{V}_x^T|^2\right]/4 = 0, \quad (33)$$

$$\mathcal{J}^* = \gamma^{-2}tr[\widetilde{V}_{xx}DD^T], \quad (34)$$

$$u^* = -f_2^T(x)\widetilde{V}_x^T(x)/2. \quad (35)$$

Now, instead of optimal control problem (23), (24) let us study the suboptimal one; i.e., assume it is required to find suboptimal pair $(u^\star, \mathcal{J}^\star)$ such that $\mathcal{J}(u^\star) \leq \mathcal{J}^\star$. To solve it, consider the inequality

$$f_1\widetilde{V}_x + L_0 - \left[|f_2^T\widetilde{V}_x^T|^2 - 2\gamma^{-2}|D^T\widetilde{V}_x^T|^2\right]/4 \leq 0. \quad (36)$$

Evidently, if we utilize the solution to (36) and substitute it into the formulas (34), (35) to calculate $\mathcal{J}^\star$ and $u^\star$, these will be the solution to suboptimal problem in question.

Recall system (1) and require supply rate (2) to satisfy

$$L_0(x) \geq 0. \quad (37)$$

This guarantees that the function $L(x, u)$ is compact and bounded below. Next, notice that (6) coincides with (36) if

$$\gamma = \varepsilon^{-1}. \quad (38)$$

Thus, the suboptimal control $u^\star$ and suboptimal value $\mathcal{J}^\star$ can be constructed based on Markov control ensuring dissipativity with RSSF (8) and supply increase function (7):

$$u^\star = \varphi_1^{(\varepsilon)}(x), \quad (39)$$

$$\mathcal{J}^\star = 2\lambda^{(\varepsilon)}(x). \quad (40)$$

Derived formulas demonstrate that our theory has relationship to the theories of risk-sensitivity and stochastic differential games and may be involved to solve risk-sensitive suboptimal control problems.

C. $\mathcal{H}_\infty$-Control Problem and Deterministic Game

Risk-sensitive storage functions may be used in robust control problems. Study deterministic control-affine system

$$\begin{cases} \dot{x}(t) = f_1(x(t)) + f_2(x(t))u(t) + D(x(t))w(t), \\ x(0) = x_0 \neq 0, \ t \in [0, \infty), \end{cases} \quad (41)$$

where x, u, w are the state, control and disturbance vectors defined on Hilbert spaces $\Gamma_x, \Gamma_u, \Gamma_w$, respectively [13]; x_0 plays the role of additional disturbance. Let system (41) be associated with the performance criterion

$$\mathcal{I}(u, w) = \int_0^\infty (L_0(x(s)) + |u(s)|^2)ds, \ L_0 \geq 0. \quad (42)$$

Robust $\mathcal{H}_\infty$-control problem (notably, *disturbance attenuation problem*) consists in the following [13]. For a given scalar $\gamma > 0$, find state-feedback control

$$u = \eta^{(\gamma)}(x) \quad (43)$$

such that for any $w \in \Gamma_w$ and $x_0 \in \mathbb{R}^n$ the following inequality is satisfied:

$$\mathcal{I}(\eta^{(\gamma)}, w) \leq \gamma^2[||w||^2 + q_0(x_0)]. \quad (44)$$

Here q_0 stands for a positive function, $||w|| = \sqrt{\int_0^\infty |w(s)|^2 ds}$ is the norm on Γ_w. It is shown in [13] (technicalities omitted here) that the solution to (41)-(44) can be constructed as the solution to *soft-constrained differential game* with payoff function

$$\mathcal{I}^{(\gamma)}(u, w) = \mathcal{I}(u, w) - \gamma^2||w||^2 - \gamma^2 q_0(x_0). \quad (45)$$

The first player strives to minimize (45) via the strategy $u = \eta^{(\gamma)}(x)$, while the second one seeks to maximize it using the policy $w = \nu^{(\gamma)}(x)$ (*worst-case design*). Instead of the Isaacs equality for the upper value $\widehat{V}^{(\gamma)}(x)$ of the game let us concentrate attention on the Isaacs inequality

$$\inf_{u\in\Gamma_u} \sup_{w\in\Gamma_w} \Big\{\partial\widehat{V}^{(\gamma)}/\partial x\,[f_1(x) + f_2(x)u + D(x)w] + \\ + L(x, u) - \gamma^2|w|^2\Big\} \leq 0. \quad (46)$$

It is clear that the maximizing disturbance and minimizing control are defined by

$$w = D^T(x)/(2\gamma^2)\partial\widehat{V}^{(\gamma)^T}/\partial x = \nu^{(\gamma)}(x), \quad (47)$$

$$u = -f_2^T(x)\partial\bar{V}^{(\gamma)^T}/(2\partial x) = \eta^{(\gamma)}(x). \quad (48)$$

Therefore, Isaacs inequality (46) takes the form

$$\partial\widehat{V}^{(\gamma)}/\partial x f_1 + L_0 - \left[\left|f_2^T\partial\widehat{V}^{(\gamma)^T}/\partial x\right|^2 - \\ -\gamma^{-2}\left|D^T\partial\widehat{V}^{(\gamma)^T}/\partial x\right|^2\right]/4 \leq 0. \quad (49)$$

Again, we emphasize that, provided supply rate (2) meets (37), then (6) implies (49) if

$$\varepsilon = (\gamma\sqrt{2})^{-1}. \quad (50)$$

Disturbance-attenuating state-feedback control is, in fact,

$$\eta^{(\gamma)}(x) = \varphi_1^{(\varepsilon)}(x). \quad (51)$$

As it is proven in [13], if the solution to (49) satisfies $\sup_x\{\widehat{V}^{(\gamma)} - \gamma^2 q_0\} = 0$ for the function $q_0(x)$, then $\eta^{(\gamma)}(x)$ solves $\mathcal{H}_\infty$-control problem (41)-(44). This underlines robust properties of Markov control (8) ensuring dissipativity with RSSF and relates our version of dissipativity to deterministic theories of differential games and $\mathcal{H}_\infty$-control.

Remark 5: In linear-quadratic case the function $q_0(x)$ turns out to be quadratic form $q_0(x_t) = x_t^T Q_0 x_t, Q_0 > 0$; the corresponding constraint on it could be simplified to

$$\gamma^2 Q_0 - Z^{(\gamma)} \geq 0. \quad (52)$$

V. A NUMERICAL EXAMPLE

To illustrate the obtained results in linear-quadratic case, we have selected MATLAB software. Interface package YALMIP [22] and solver SeDuMi [23] have been involved to process linear matrix inequalities. Consider the problem to ensure dissipativity for system (1), (2), (14); set the matrices and intensity of risks as follows.

$$A = \begin{bmatrix} -7 & 0 \\ 0 & -7 \end{bmatrix},\ B = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},\ Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$
$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},\ \varepsilon = 2.5. \quad (53)$$

Let us find the minimal solution to (19), i.e., $tr[H^{(\varepsilon)}] \to \min$ (in fact, we are interested in maximum-valued matrix $Z^{(\varepsilon)}$ being inverse to $H^{(\varepsilon)}$). Under calculation accuracy $\beta = 10^{-9}$ we have obtained

$$Z^{(2.5)} = \begin{bmatrix} 0.1580 & 0 \\ 0 & 0.1580 \end{bmatrix}. \quad (54)$$

Markov control (18), that corresponds to $Z^{(2.5)}$, is therefore specified by

$$u = \varphi_2^{(2.5)}(x) = -K^{(2.5)}x,\ K^{(2.5)} = \begin{bmatrix} 0.3160 & 0 \\ 0 & 0.3160 \end{bmatrix}. \quad (55)$$

Supply increase function (17) is then equal to

$$\lambda^{(2.5)}(x) = 1.975. \quad (56)$$

Since $Q \geq 0$, the condition (9) is met. The dynamics of system (1), (2), (14) with parameters (53) under control (55) has been modeled in MATLAB Simulink Toolbox; the initial condition and simulation time have been defined by $x_0 = [20\ 20]^T$ and $t_{\max} = 15$ s, respectively. The results are presented by Fig. 1. Obviously, the corresponding random process is stable in the sense of unique invariant measure.

Next, system (23), (24), (14) with parameters (53), having the values of ε and γ related by (38), has been simulated under control (55). To analyze cost function (24) on time horizon $[0, t_{max}]$, the following normalized rate has been utilized:

$$\Delta(t_{\max}) = \frac{\frac{\varepsilon^2}{t_{\max}} \int_0^{t_{\max}} \left(x_\tau^T Q x_\tau + |u_\tau^\star|^2 \right) d\tau}{2\varepsilon^2 \mathrm{tr}\left[Z^{(\varepsilon)} D D^T \right]} = $$
$$= \frac{\int_0^{t_{\max}} x_\tau^T \left[Q + K^{(2.5)} K^{(2.5)^T} \right] x_\tau d\tau}{2 t_{\max} \mathrm{tr}\left[Z^{(2.5)} D D^T \right]}, \quad (57)$$

where x is the realization of x_t. As far as $Q \geq 0$, (55) and (56) form the suboptimal pair $(-K^{(2.5)}x, 2\lambda^{(2.5)}(x))$. For $t_{max} = [10, 25, 50, 100, 200, 300, 500]$ s the curve (57) has been plotted, see Fig. 2.

Finally, linear version (14) of system (41), (42) with parameters (53), where ε and γ are correlated via (50), has been simulated to study robust properties of the control (55). The disturbance $\xi(t)$ has been specified by

$$\xi(t) = [\xi_i(t)],\ \xi_i(t) = \alpha_i e^{-\beta_i t},\ \beta_i > 0,\ i = 1, ..., 3. \quad (58)$$

The matrix Q_0 (see (52)) has been selected according to condition

$$\frac{1}{12.5} Q_0 - Z^{(2.5)} \geq 0, \quad (59)$$

and has been given by

$$Q_0 = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}. \quad (60)$$

We have used the initial condition $x_0 = [15\ 15]^T$, disturbance coefficients $\alpha = (-75, 200, 100)$ and $\beta = (-3, -0.5, -1)$, as well as weighting matrix (60) and simulation time $t_{\max} = 10$ s. The reader may find the curve of the partial sums

$$\mathcal{S}(t) = \frac{2\varepsilon^2 \int_0^t \left(x(s)^T Q x(s) + |u^{\circledast}(s)|^2 \right) ds}{\int_0^t |\xi(s)|^2 ds + x_0^T Q_0 x_0} = $$
$$= \frac{12.5 \int_0^t x(s)^T \left[Q + K^{(2.5)} K^{(2.5)^T} \right] x(s) ds}{\int_0^t |\xi(s)|^2 ds + x_0^T Q_0 x_0} \quad (61)$$

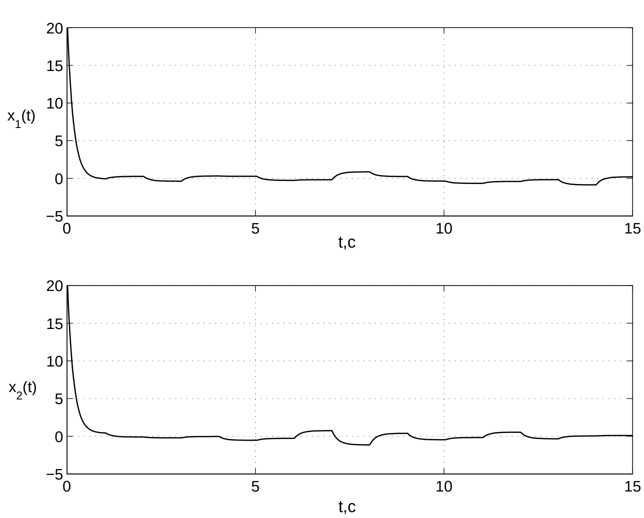

Fig. 1. The state vector x_t under the control $u = -K^{(2.5)}x$

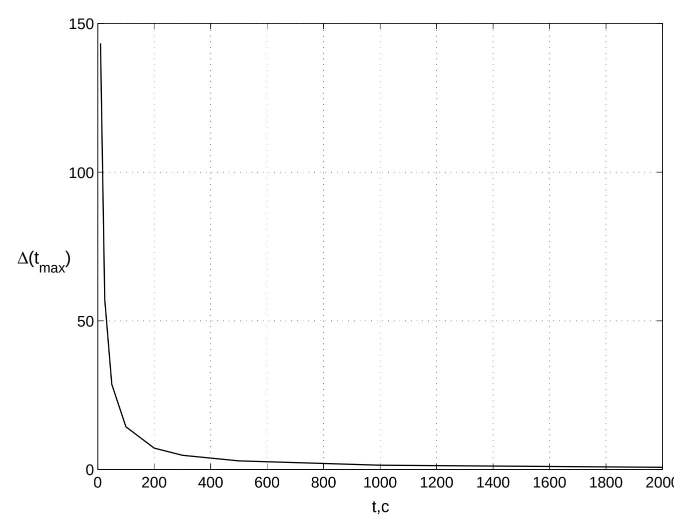

Fig. 2. The rate (55) under the control $u = -K^{(2.5)}x$ evaluated for various simulation time

of $\mathcal{H}_\infty$-norm under the control $u = -K^{(2.5)}x$ (Fig. 3) For convenience the rate (61) is normalized, see (44). All the figures agree with theoretical expectations.

VI. CONCLUSIONS AND FUTURE WORKS

This paper has discussed stochastic control-affine systems defined by Itô differential equations. The authors have suggested an alternative approach to dissipation and proposed new notion of dissipativity with RSSF. Criterion for a system in question to be dissipative in the suggested sense has been formulated and proven, provided state-feedback control ensuring dissipativity is selected according to certain version of stochastic Artstein's inequality. Of particular interest is the fact that the proof involves a certain generalization of stochastic Artstein's formula. In linear-quadratic case the final results have been expressed via linear matrix inequalities.

The development of dissipativity with RSSF has made it possible to relate dissipativity to differential (deterministic and stochastic) games. It has been shown that Markov control ensuring dissipativity with RSSF stabilizes the system in the sense of unique invariant probabilistic measure. In addition, it has been demonstrated that this control serves the solution to risk-sensitive suboptimal control problem, as well as to robust $\mathcal{H}_\infty$-control problem for deterministic systems with exogenous disturbances. Numerical example has been given.

Future works could be directed towards extension of the obtained results to general cases (e.g., nonaffine controlled Itô diffusions or nonquadratic supply rates).

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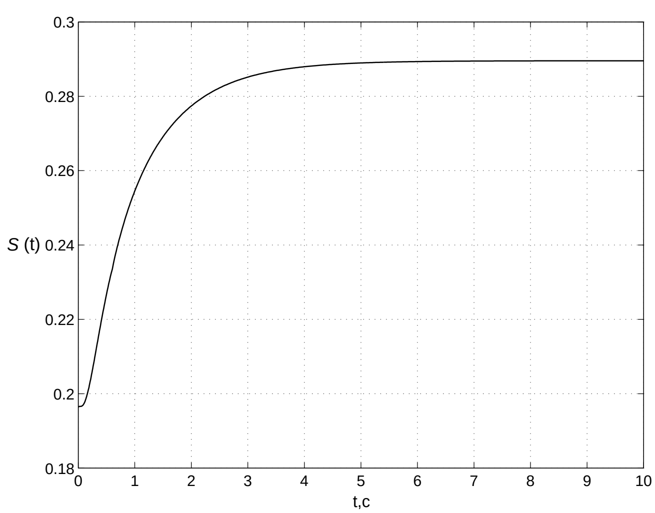

Fig. 3. The rate (61) under the control $u = -K^{(2.5)}x$

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