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Quantum Stochastic Stability and Weak-$*$ Convergence of System Observables

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***Abstract*— The evolution of open quantum systems can be described using quantum stochastic differential equations (QSDEs). The solution of QSDEs leads to a one parameter semigroup of completely positive operators with which one can associate a minimal quantum Markov dilation. In this paper, we use a Lyapunov type theorem to prove asymptotic stability in the weak-$*$ operator norm for such minimal Markov dilations provided some assumptions are satisfied. This theorem uses the fact that the unit ball in the space of bounded operators on a Banach space is weak-$*$ compact.**

I. INTRODUCTION

Recent advances in quantum and nano technology are driving theoretical and experimental research towards quantum feedback control (see e.g. [1], [2], [3], [4], [5], [6], [7], [8]). There are several examples of quantum control systems that include feedforward, feedback and parallel interconnections of quantum and classical components [9]. Since quantum feedback systems may include components, such as optical amplifiers that are active, questions of network stability are of considerable importance.

Recently, Somaraju and Petersen [10] examined some Lyapunov type stability results for open quantum systems (also see [11] for similar results). We extend the results in [10] to prove asymptotic weak convergence of the system states to a set determined by the Lyapunov function. The results in this paper are largely based on the work of Kushner [12] on the stability of classical Markov processes.

The remainder of this paper is organised as follows: in the following section, we briefly recall quantum stochastic processes and associated Markov dilations that may be used to describe the evolution of open quantum systems. The interested reader is referred to excellent monographs by Parthasarathy [13], [14] for further details on this topic (also see [9]). In Section III we recall, without proof, previous results on Lyapunov stability that appeared in [10] and then prove our main results. Conclusions are given in the final Section.

II. OPEN QUANTUM SYSTEMS

We consider an open quantum system $\mathscr{S}$ with physical variable space $\mathcal{A}_{\mathscr{S}}$, which is a von Neumann sub-algebra of the set of bounded operators, $\mathcal{B}(\mathcal{H})$ on an underlying Hilbert space $\mathcal{H}$. The self-energy of the system is described by a Hamiltonian H which is a self-adjoint operator in $\mathcal{A}_{\mathscr{S}}$. This system interacts with a heat bath through n field channels which are given by the quantum stochastic processes

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_n \end{pmatrix}, \quad \Lambda = \begin{pmatrix} A_{11} & \dots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \dots & A_{nn} \end{pmatrix}.$$

Here A_i, $i = 1, 2, \dots, n$ and A_{ij}, $i, j = 1, 2, \dots, n$ are elements of $\mathcal{F}$, a space of operators on a specific Hilbert space Γ, called the Fock space. Physically, the Fock space is a Hilbert space that describes an indefinite number of bath quanta (photons) and the A_i, A_{ij} are operators that describe the annihilation of photons and scattering of photons between different field channels. We assume that these processes are canonical, and satisfy the quantum Ito rule:

$$dA_j(t)dA_k(t)^* = \delta_{jk}dt, \; dA_{jk}(t)dA_l(t)^* = \delta_{kl}dA_j(t)^*,$$
$$dA_j(t)dA_{kl}(t) = \delta_{jk}A_l(t), \; dA_{jk}(t)dA_{lm}(t) = \delta_{kl}dA_{jm}(t).$$

Here, $(\cdot)^*$ denotes the Hilbert-adjoint of an operator.

The systems coupling with the bath is determined by the scattering matrix S and the coupling operators L, where

$$L = \begin{pmatrix} L_1 \\ \vdots \\ L_n \end{pmatrix}, \quad S = \begin{pmatrix} S_{11} & \dots & S_{1n} \\ \vdots & \ddots & \vdots \\ S_{n1} & \dots & S_{nn} \end{pmatrix}$$

and S_{ij} and L_i are operators in $\mathcal{A}_{\mathscr{S}}$ and the scattering matrix satisfies

$$\sum_i S_{ki} S_{ji}^* = \sum_i S_{ik}^* S_{ij} = \delta_{kj}$$

Given a system operator $X \in \mathcal{A}_S$, its evolution in the Heisenberg picture is defined as $X(t) = j_t(X)$ and satisfies

$$\begin{aligned} dX(t) &= \mathcal{L}_{L(t),H(t)}(X(t))dt \\ &+ dA^\dagger(t)S^\dagger(t)[X(t), L(t)] + [L^\dagger(t), X(t)]S(t)dA(t) \\ &+ \text{trace}\{(S^\dagger(t)X(t)S(t) - X(t))d\Lambda(t)\} \end{aligned} \quad (1)$$

Here, $[\cdot, \cdot]$ denotes the commutator of two operators and for a matrix $S = (S_{ij})$ of operators $S^\dagger = (S_{ji}^*)$ and

$$\mathcal{L}_{L(t),H(t)}(X(t)) = \frac{1}{2}L^\dagger[X(t), L(t)] + \frac{1}{2}[L^\dagger(t), X(t)]L(t) - i[X(t), H(t)]$$

is the *Lindblad* generator of the system. In the following, for ease of notation we drop the subscript and write $\mathcal{L}$ for the system's generator.

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A. Markovian evolution

If the system's Lindblad generator $\mathcal{L}$ is bounded then we can set $T_t = e^{t\mathcal{L}}$. T_t is a one parameter semigroup of bounded operators on $\mathcal{H}$ generated by $\mathcal{L}$[1].

In order to describe the Markov nature of the evolution, we use the specific structure of the Fock space Γ. Specifically, there exists a collection of operators $\{\mathcal{F}_t, t \geq 0\}$ such that $\mathcal{F}_t \subset \mathcal{A}_{\mathscr{S}} \otimes \mathcal{F}$ is generated by the noises $A_i(s)$ and $A_{i,j}(s), i,j = 1,2,\ldots,n, s \leq t$. Roughly, the space $\mathcal{F}_t$ corresponds to the set of all events up to time t. There is an associated *vacuum conditional expectation* $\mathbb{E}_t : \mathcal{A}_{\mathscr{S}} \otimes \mathcal{F} \to \mathcal{F}_t$. The Markovian property is encapsulated in the relation

$$\mathbb{E}_s j_t(X) = j_s(T_{s-t}X)$$

for all $X \in \mathcal{A}_{\mathscr{S}}$ and $0 \leq s \leq t$.

Moreover, the conditional expectation can be used to state a quantum version of Dynkin's formula

$$\mathbb{E}_s[j_t(X)] = X(s) + \int_s^t \mathbb{E}_s[j_{t'}(\mathcal{L}(X))]dt'. \qquad (2)$$

A precise definition of the conditional expectation and an explanation of the above formula is beyond the scope is this article. Interested readers are referred to [13, Ch. 26] and [14].

B. Markov Stop times

Dynkin's formula (2) plays a central role in Lyapunov stability theory. In formula (2), the integration limits are deterministic quantities s and t. However, it is possible to generalise this formula to situations wherein the integration limit t is a random time. In this subsection, we discuss the strong Markov process for which the integration limit may be a random time. We begin by introducing the notion of a stop (Markov) time in Definition 2.1.

Definition 2.1: [14, p. 111] A *stoptime (or Markov time)* τ for the flow j_t is a spectral measure on $[0,\infty]$ with values in orthogonal projections in $\mathcal{A}_{\mathscr{S}} \otimes \mathcal{F}$ satisfying the condition

$$[\tau([0,s]), j_t(X)] = 0, \ \forall \ s \leq t \text{ and } X \in \mathscr{A}.$$

The projection $\tau([0,t])$ is to be interpreted as the event of stopping the Markov process has occurred at or before time t. We denote by $\mathbf{1}_E$ the event $\tau(E)$ for all Borel subsets E of $[0,\infty]$. For any two stoptimes τ_1, τ_2 that commute (i.e. $[\tau_1([0,a]), \tau_2([0,b])] = 0$) we can define the minimum $\tau_1 \wedge \tau_2$ and maximum $\tau_1 \vee \tau_1$ stoptimes, of τ_1 and τ_2 as

$$\begin{aligned} \mathbf{1}_{\tau_1 \wedge \tau_2 \leq t} &= \mathbf{1}_{\tau_1 \leq t} + \mathbf{1}_{\tau_2 \leq t} - \mathbf{1}_{\tau_1 \leq t}\mathbf{1}_{\tau_2 \leq t}, \\ \mathbf{1}_{\tau_1 \vee \tau_2 \leq t} &= \mathbf{1}_{\tau_1 \vee \tau_2 \leq t}\mathbf{1}_{\tau_1 \vee \tau_2 \leq t} \ \forall t. \end{aligned}$$

Also, if $t \geq 0$, then denote the 'deterministic' stoptime τ such that $\tau(\{t\}) = \mathbf{1}$ by t. Typical examples of stop times include first exit (and entry) times of the adapted process $X(t)$ from a set.

Now, if the semigroup T_t generated by T_t is strongly continuous (i.e. continuous in the strong operator topology) then it can be shown [14, Theorem 16.5] that Dynkin's formula applies with a random time τ as an integration limit:

$$\mathbb{E}_s[j_\tau(X)] = X(s) + \int_s^\tau \mathbb{E}_s[j_\tau(\mathcal{L}(X))]dt'. \qquad (3)$$

C. Quantum State

We assume that the system is initially decoupled from the bath. We further assume that the system is initially in a pure state $\phi \in \mathcal{H}$, even though the results in this paper can easily be generalised to mixed states determined by a trace-class operator ρ of unit trace. The bath is assumed to be in vacuum state ψ and the complete state of the system and bath, is given by $\phi \otimes \psi$

The state $\phi \otimes \psi$ can be used to define a continuous linear functional $\mathbb{P} : \mathscr{A}_{\mathscr{S}} \otimes \mathcal{F} \to \mathbb{C}$ as

$$\mathbb{P}(X) = \langle \phi \otimes \psi, X\phi \otimes \psi \rangle, \forall X \in \mathscr{A}_{\mathscr{S}} \otimes \mathcal{F}. \qquad (4)$$

Here, $\langle \cdot, \cdot \rangle$ denotes the inner product in the tensor product space $\mathcal{H} \otimes \Gamma$.

Physically, for self-adjoint X, $\mathbb{P}(X)$ is the quantum expectation of a measurable physical quantity which is parameterized by the operator X. We can think of projections in $\mathscr{A}_{\mathscr{S}} \otimes \mathcal{F}$ as events and for a projection $P \in \mathscr{A}_{\mathscr{S}} \otimes \mathcal{F}$, $\mathbb{P}(P) \in [0,1]$ is the probability of event P. The pair $(\mathscr{A}_{\mathscr{S}} \otimes \mathcal{F}, \mathbb{P})$ is called a quantum probability space (see e.g. [16]).

III. Main Results

We collect the following assumptions together for future reference.

- A1 Let $\mathcal{H}, \Gamma, \mathcal{F}$ and $\mathcal{F}_t$ be as described in Section II and let $\mathcal{A}_{\mathscr{S}} = \mathcal{B}(\mathcal{H})$. For $X \in \mathcal{A}_{\mathscr{S}}$, the Heisenberg evolution of X is given by (1) and $X(0) = X$. Assume that the system is initially in state $\phi \otimes \psi$, where ψ is quantum vacuum noise and let $\mathbb{P}$ be as defined in (4).
- A2 Let $V : \mathcal{B}(\mathcal{H})^n \to \mathcal{B}(\mathcal{H})$ be of the form

$$\mathbf{X} \mapsto \sum_{i=1}^{p} Y_1 Y_2 \ldots Y_{q_i}.$$

  Here p and q_i, $i = 1,\ldots,p$ are finite integers and $Y_1,\ldots,Y_{q_i} \in \{X_1,\ldots,X_n,X_1^*,\ldots X_n^*\}$, $i = 1,\ldots,p$. Suppose $V(\mathbf{X})$ is non-negative (i.e. for all $\psi \in \mathcal{H}, \mathbf{X} \in \mathcal{B}(\mathcal{H})^n$, $\langle \psi, V(\mathbf{X})\psi \rangle \geq 0$) and continuous in the set $Q_m = \{\mathbf{X} \in \mathcal{B}(\mathcal{H})^n : V(\mathbf{X}) \leq m\mathbf{1}\}$. Let $t_0 = \inf\{t : j_t(\mathbf{X}) \notin Q_m\}$ and suppose τ_m determined by $\tau_m(\{t_0\}) = \mathbf{1}$ is a stoptime on $(\mathcal{H}, F_t, j_t)$.
- A3 $V(\mathbf{X})$ is in the domain of $\mathcal{L} = \mathcal{L}_{L(t),H(t)}$, the Lindblad generator of the system for all $\mathbf{X} \in Q_m$.
- A4 The semigroup T_t, generated by $\mathcal{L}$ is strongly continuous. Therefore, the map $t \mapsto j_t(X)\psi$ is continuous for all $X \in \mathcal{K}$ and $\psi \in \mathcal{H}$ [17, Proposition 5.3].

The following Lemma appeared in [10].

Lemma 3.1: Suppose (A1)-(A3) are satisfied and let $j_{\tau\wedge t}(\mathcal{L}(V(X)) \leq 0$. Then, $V(j_{\tau\wedge t}(\mathbf{X}))$ is a nonnegative supermartingale in the sense that $\mathbb{E}_0[V(j_{\tau\wedge t}(\mathbf{X}))] \leq$

[1] This concept may be extended to the case of unbounded $\mathcal{L}$ as well (See e.g. [15]).

$V(j_{\tau\wedge 0}(\mathbf{X}))$ and for $\lambda \leq m$, and if $j_0(\mathbf{X}) \in Q_m$, then

$$\mathbb{P}\{\sup_{\infty > t \geq 0} V(j_{\tau\wedge 0}(\mathbf{X})) \geq \lambda\} \leq \frac{\|V(j_{\tau\wedge 0}(\mathbf{X}))\|}{\lambda} \quad (5)$$

We now prove an elementary lemma that will be used to prove our main result on asymptotic stability.

Lemma 3.2: Let $V(X) \geq 0$ be bounded in an open region $Q \subset \mathcal{B}(\mathcal{H})$ assume (A4). Suppose τ is the first exit time from the open set $P \subset Q$ and let $V(X)$ be in the domain of $\mathcal{L}$. If $\mathcal{L}(V(X)) \leq -b < 0$, then $\mathbb{E}_0(\tau) \leq \|V(X)\|/b$.

Proof: Using Dynkin's formula (3) we get

$$\begin{aligned} V(X) - \mathbb{E}_0[j_\tau(V(X))] &= -\int_0^\tau \mathbb{E}_0[j_\tau(\mathcal{L}(X))]dt' \\ &\geq b\mathbb{E}_0[\tau]. \end{aligned}$$

The result now follows from the non-negativity of $V(X)$ and the definition of $\|\cdot\|$ in $\mathcal{B}(\mathcal{H})$. ■

Before we state the main results of our paper, we briefly recall the definition of the weak-$*$ operator topology. Given a Hilbert space $\mathcal{H}$ with an orthonormal basis $\{e_i\}$ and $\rho : \mathcal{H} \to \mathcal{H}$ is a bounded linear operator then the trace of ρ is defined to be

$$\mathrm{trace}\{\rho\} = \sum_i \langle e_i, \rho e_i \rangle.$$

The set of all operators with finite trace is denoted by $\mathcal{T}_1(\mathcal{H})$ and this space is in fact a Banach space with norm

$$\|\cdot\|_1 = \mathrm{trace}\{\cdot\}.$$

Any T in $\mathcal{B}(\mathcal{H})$, the space of bounded linear operators on $\mathcal{H}$, defines a linear functional f_T on $\mathcal{T}_1(\mathcal{H})$ through the relation

$$f_T(\rho) = \mathrm{trace}\{T\rho\}, \forall \rho \in \mathcal{T}_1(\mathcal{H}). \quad (6)$$

It can be shown that the space $\mathcal{B}(\mathcal{H})$ is isometrically isomorphic to the dual space of $\mathcal{T}_1(\mathcal{H})$ (see e.g. [13, ch. 9]).

The topology induced on $\mathcal{B}(\mathcal{H})$ by the predual space $\mathcal{T}_1(\mathcal{H})$ is defined to be the *weak-$*$* topology of $\mathcal{B}(\mathcal{H})$. Specifically, any $\rho \in \mathcal{T}_1(\mathcal{H})$ induces a seminorm $s_\rho(\cdot) = |f_\cdot(\rho)|$ where $f_\cdot$ is as defined in (6). The topology induced by this family of seminorms is in fact the weak-$*$ topology of $\mathcal{B}(\mathcal{H})$.

We can now state the main result of our paper. In the following theorem and proof when we make any statements regarding convergence of an element in $\mathcal{B}(\mathcal{H}\otimes\Gamma)$, we always mean convergence with regards to the weak-$*$ topology.

Theorem 3.3 (Asymptotic Stability): Assume (A1)-(A4) and suppose $\mathcal{L}V(X) = -k(X) \leq 0$. Set $P_m(X) = Q_m \cap \{X : k(X) = 0\}$ and let $N_\epsilon(P_m)$ denote the ϵ-neighborhood of P_m relative to Q_m in the weak-$*$ norm. That is, $X \in N_\epsilon(P_m)$ if $X \in Q_m$ and for all $\rho \in \mathcal{T}_1(\mathcal{H})$, $\|\rho\|_1 \leq 1$, there exists an $X' \in P_m$ such that

$$s_\rho(X - X') < \epsilon.$$

Now suppose there exists a d_0 such that for all $0 \leq d < d_0$, there exists an ϵ_d such that for we have $k(X) \geq d > 0$ for X in $Q_m \setminus N_{\epsilon_d}(P_m)$[2]. If we further assume that Q_m is bounded, then

$$\mathbb{P}\{X_T \to P_m\} \geq 1 - \|V(X)\|/m. \quad (7)$$

The convergence is in the weak-$*$ topology in $\mathcal{B}(\mathcal{H})$.

Proof: Choose d_1 and d_2 such that $d_0 > d_1 > d_2 > 0$ and let ϵ_i correspond to d_i such that $k(X) \geq d_i$ for all $X \in Q_m \setminus N_{\epsilon_i}(P_m)$. Furthermore we choose ϵ_i such that $N_{\epsilon_2}(P_m)$ is a proper subset of $N_{\epsilon_1}(P_m)$.

Now define $T_X(t, \epsilon_i)$ as follows: 1) if $t \leq \tau_m$ then set $T_X(t, \epsilon_i)$ to the total time spent in $Q_m \setminus N_{\epsilon_i}(P_m)$ after time t and before the first exit time τ_m from the set Q_m and 2) if $t > \tau_m$ then set $T_X(t, \epsilon_i) = 0$. From Lemma 3.2 $\mathbb{E}_0(T_X(t, \epsilon_i))$ is bounded and therefore the stop-time $T_X(t, \epsilon_i) < \infty$ with probability 1. Hence, $T_X(t, \epsilon_i) \to 0$ as $t \to \infty$ with probability 1.

By Lemma 3.1, $j_t(X)$ remains in the interior of Q_m for all time with probability not less than $1 - \|V(X)\|/m$. Therefore, we are left with two scenarios. a) As $t \to \infty$, $X(t) \to N_{\epsilon_2}(P_m)$ with probability no less than $1 - \|V(X)\|/m$ or b) $X(t)$ oscillates between the two sets N_{ϵ_2} and $Q_m \setminus N_{\epsilon_1}$ infinitely often with time spent in $Q_m \setminus N_{\epsilon_1}$ becoming arbitrarily small as $t \to \infty$ with probability not less than $1 - \|V(X)\|/m$. We now prove that the second scenario leads to a contradiction.

Fix $\delta_1 > 0$. Then because Q_m is assumed to be bounded, the closure of Q_m is weak-$*$ compact. Because the Markov semigroup T_t is strongly continuous, $j_t(X)$ is a continuous function of time. Therefore, for all ρ in the predual of $\mathcal{B}(\mathcal{H})$, with $\|\rho\| \leq 1$, we have

$$\sup_{x \in Q_m \setminus N_{\epsilon_2}(P_m)} \mathbb{P}_x\{\sup_{h \geq s \geq 0} s_\rho(j_s(X) - X) \geq \epsilon_1 - \epsilon_2\} < \delta_1. \quad (8)$$

Therefore, for all t, if $j_t(X) \notin N_{\epsilon_1}(P_m)$ then for all $s \in [t, t+h]$, $j_s(X) \notin N_{\epsilon_2}(P_m)$ with probability not less than $1 - \delta_1$.

But, because $T_X(t, \epsilon_2) \to 0$ as $t \to \infty$, for all $\delta_2 > 0$, there exists a t_0 such that for all $t > t_0$

$$\mathbb{P}\{T_X(t, \epsilon_2) > h\} < \delta_2. \quad (9)$$

Because δ_1 and δ_2 may be made arbitrarily small, under scenario b), (8) and (9) are contradictory. Therefore,

$$\mathbb{P}\{x_s \to N_{\epsilon_2}\} \geq 1 - \frac{\|V(X)\|}{m}.$$

The theorem now follows from the fact that ϵ_2 can be made arbitrarily small. ■

We have a simple Corollary to the above theorem.

Corollary 3.4: If the conditions of the above theorem hold for all $m \geq 0$, then set $P = \bigcup_{m=1}^\infty P_m$ and let $N_\epsilon(P)$ be the weak-$*$ neighborhood of P relative to $Q = \bigcup_{m=1}^\infty Q_m$. If there exists a d such that for each $d, 0 \leq d \leq d_0$, there is an ϵ_d such that $k(X) \geq d$ if $X \in Q \setminus N_{\epsilon_d}(P)$ then $j_t(X) \to P$ with probability 1.

[2]Note that the statement of this theorem is trivially true as a consequence of Lemma 3.1 if $k(X)$ is identically zero in Q_m

IV. Conclusion

In this paper, we prove an asymptotic convergence result using a Lyapunov function. The theorem uses the fact that the unit ball in the set of bounded operators on a Hilbert space is weak-$*$ compact.

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