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# A Unified Approach to Controllability of Closed and Open Quantum Systems

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***Abstract*— Based on Lie-algebraic methods from nonlinear control theory, we present a unified approach to control problems of finite dimensional closed and open quantum systems. In particular, we provide a simplified treatment of different controllability notions for closed quantum systems as well as new accessibility results for open quantum systems described by the Lindblad-Kossakowski master equation. To derive controllability and accessibility results, we exploit known results on the classification of all Lie groups which act transitively on Grassmann manifolds, and respectively, on $\mathbb{R}^d \setminus \{0\}$. For the special case of open quantum systems of coupled spin-$\frac{1}{2}$ particles, we obtain a remarkably simple characterization of accessibility.**

## I. INTRODUCTION

In this paper we explore the possibilities of steering a quantum system from an initial state to a target state in finite time. The evolution of closed quantum systems is described by the Schrödinger equation which is a linear differential equation on a Hilbert space **H**, or more generally, by the Liouville-von Neumann master equation

$$\dot{\rho} = -\mathrm{i}\big[H,\rho\big] \quad , \quad \rho(0) = \rho_0, \tag{1}$$

where $H \in \mathbb{C}^{N\times N}$ is Hermitian and is called the Hamiltonian. The system (1) then evolves on the *unitary orbit* of the initial density operator $\rho_0$. In contrast, for real-world open quantum systems interacting with the environment, one has to take the dissipation or relaxation effects into account. In this case, the evolution takes place on the convex set of *all density operators* $\mathscr{P}$ and is governed by the so-called Lindblad-Kossakowski master equation [2], [13], [21]

$$\dot{\rho} = -\mathrm{i}\big[H,\rho\big] + \mathscr{L}_D(\rho) \quad , \quad \rho(0) = \rho_0 \in \mathscr{P}, \tag{2}$$

where the linear operator

$$\mathscr{L}_D(\rho) := \sum_{j=1}^{N_c} V_j\rho V_j^\dagger - \frac{1}{2}V_j^\dagger V_j\rho - \frac{1}{2}\rho V_j^\dagger V_j, \tag{3}$$

with $V_j \in \mathbb{C}^{N\times N}$, models the interaction with the environment. Note that $N_c$ corresponds to an arbitrary finite number of *interaction channels*. In (2), the presence of interaction (i.e. $\mathscr{L}_D \neq 0$) destroys the isospectral flow of the Liouville equation (1). Now, consider that control inputs $u_k(t) \in \mathbb{R}$ enter the Hamiltonian $H$ in (2) as

$$H = H_0 + \sum_{k=1}^{m} u_k(t)H_k,$$

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where $H_0$ and $H_k$ are Hermitian matrices representing the internal Hamiltonian of the system and external control Hamiltonian, respectively. Then one obtains a class of bilinear control systems which is of particular interest for many applications in quantum control, e.g. [1], [10], [17].

In our paper, we focus on fundamental control-theoretical issues dealing with controllability and accessibility of bilinear quantum control systems (2). For closed quantum systems (i.e. $\mathscr{L}_D = 0$), the reachable set $\mathscr{R}(\rho_0)$ of (2) is clearly restricted to the unitary orbit of $\rho_0$. Thus, it is natural to ask: Can one reach the entire unitary orbit of $\rho_0$ and what are nesessary and sufficient conditions on $H$ to guarantee this property? For open quantum systems (i.e. $\mathscr{L}_D \neq 0$), the situation is quite different. In this case, the reachable set $\mathscr{R}(\rho_0)$ is indeed no longer confined to the unitary orbit of $\rho_0$. But nevertheless, controllability on $\mathscr{P}$ fails for the above type of controlled Lindblad-Kossakowski master equation, e.g. [4], [10]. Therefore, a meaningful question for open quantum system is the issue of accessibility, i.e. whether the reachable sets $\mathscr{R}(\rho_0)$ have non-empty interior in $\mathscr{P}$. To obtain a unified approach to these problems, we consider a suitable Lie group action such that (2) can be regarded as an induced system coming from a bilinear control system on an associated Lie group. Hence, controllability and accessibility questions for finite dimensional quantum systems can be reduced to characterizing all Lie groups which act transitively on certain homogeneous spaces [10].

Explicitly, our results are as follow. First, we characterize the controllability of closed quantum systems via the classification of all matrix Lie groups acting transitively on Grassmann manifolds [10], [23]. Thus, different notions of controllability such as pure state, projective state, and density operator controllability are treated in a uniform way compared to previous approaches [3], [22]. Second, we derive necessary and sufficient accessibility conditions for *unital* open quantum systems (i.e. $\mathscr{L}_D(I_N) = 0$) using the classification of all matrix Lie groups acting transitively on their underlying real vector space except the origin [7]. Moreover, for the *non-unital* Lindblad-Kossakowski master equation (i.e. $\mathscr{L}_D(I_N) \neq 0$), we obtain a similar, but only sufficient accessibility criterion. Finally, for the special case of coupled spin-$\frac{1}{2}$ systems, a remarkably simple characterization of accessibility arises from these conditions. Our results correct some earlier statements in [4], [5].

The paper is organized as follows. Section II deals with the infinitesimal generator of the Lindblad-Kossakowski master equation. Results for controllability and accessibility are derived in Section III and IV, respectively. Section V concludes.

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## II. THE LINDBLAD-KOSSAKOWSKI MASTER EQUATION

Let $\mathbb{C}^{N\times N}$ denote the set of all complex $N\times N$ matrices and let $\mathbf{H}=\mathbb{C}^N$ denote the Hilbert space of all complex N-tuples $(z_1,\ldots,z_N)^\top$, endowed with the standard Hermitian inner product. Moreover, we use $(\cdot)^\dagger$ to denote the conjugate transpose of a matrix. The set of $N\times N$ skew-Hermitian matrices with trace zero is denoted by $\mathfrak{su}(N)$. The states of finite dimensional $N$-level quantum systems are then completely described by their density operators $\rho$, i.e. the state space is given by the compact convex set

$$\mathscr{P}:=\left\{\rho\in\mathbb{C}^{N\times N}|\ \rho=\rho^\dagger\geq 0\ ,\ \mathrm{Tr}(\rho)=1\right\}\tag{4}$$

of all positive semidefinite self-adjoint operators on $\mathbf{H}$ with trace one. In the sequel, $[\cdot,\cdot]$ denotes the usual matrix commutator, i.e. $[A,B]=AB-BA$.

The evolution of an open quantum system subject to interaction with the environment has to deal with relaxation and dissipation phenomena. Under the assumption of the Markovian dynamics, the evolution is governed by *the Lindblad-Kossakowski* master equation. Including Hamiltonian control terms finally leads to the following master equation

$$\dot\rho=-\mathrm{i}\Big[H_0+\sum_{k=1}^m u_k(t)H_k,\rho\Big]+\mathscr{L}_D(\rho),\quad \rho(0)=\rho_0\in\mathscr{P},\tag{5}$$

where $\mathrm{i}H_0\in\mathfrak{su}(N)$ and $\mathrm{i}H_1\ldots,\mathrm{i}H_m\in\mathfrak{su}(N)$ denote the internal and control Hamiltonians (commonly referred to as *drift* and *control direction*), respectively. The $u_k(t)$ are admissible control signals, e.g. piecewise constant inputs, which may take arbitrary values in $\mathbb{R}$. The term $\mathscr{L}_D$ modelling interactions with the environment can be expressed as a linear operator originally derived by Gorini, Kossakowski and Sudarshan (GKS) [13] for finite dimensional $N$-level systems

$$\mathscr{L}_D(\rho)=\frac{1}{2}\sum_{j,k=1}^d a_{jk}\left([B_j,\rho B_k^\dagger]+[B_j\rho,B_k^\dagger]\right),\tag{6}$$

where $(B_1,\ldots,B_d)$, with $d:=N^2-1$, is any orthonormal basis of $\mathfrak{sl}(N,\mathbb{C})$, the set of all complex $N\times N$ matrices with trace zero. Moreover, $A:=(a_{jk})_{j,k=1,\ldots,d}$, with $A=A^\dagger\geq 0$, is called the associated GKS matrix. Positive semidefiniteness of the GKS matrix $A$ is required to ensure complete positivity of the Lindblad-Kossakowski master equation. The definition of complete positivity and issues related to its physical interpretations in open quantum systems can be found e.g. in [2]. For further sophisticated issues on completely positive maps and their relations to Lie semigroups, Lie wedges and reachable sets of open quantum systems, see e.g. [12], [20] and references therein. By unitarily diagonalizing $A=WDW^\dagger$, $D=\mathrm{diag}(\lambda_1,\ldots,\lambda_d)$, the linear operator $\mathscr{L}_D$ can be rewritten equivalently as

$$\mathscr{L}_D(\rho)=\frac{1}{2}\sum_{k=1}^d\left[F_k\rho,F_k^\dagger\right]+\left[F_k,\rho F_k^\dagger\right],\tag{7}$$

with $F_k=\sqrt{\lambda_k}\sum_{j=1}^d w_{jk}B_j$, $(w_{jk})_{j,k=1,\ldots,d}:=W$, which obviously has trace zero. Observe that $\mathscr{L}_D(\rho)$ in (3) and (7) looks the same except that $V_j\in\mathbb{C}^{N\times N}$ and $N_c$ in (3) are arbitrary, i.e. $V_j$ are *not necessarily* an orthonormal basis of $\mathfrak{sl}(N,\mathbb{C})$. Indeed, it is straightforward to show that $\mathscr{L}_D(\rho)$ in (3) and (7) are really equivalent [20]. The form (3) was independently derived by Lindblad [21] for the infinite dimensional case. Throughout this paper, we assume without loss of generality that $B_j$ are Hermitian matrices such that the *real* span of $(\mathrm{iB}_1,\ldots,\mathrm{iB}_\mathrm{d})$ is $\mathfrak{su}(N)$.

The Lindblad-Kossakowski master equation (5) is called *unital* if it leaves the completely mixed state $\rho=I_N/N\in\mathscr{P}$ unchanged, i.e. $\mathscr{L}_D(I_N)=0$. Otherwise, when $\mathscr{L}_D(I_N)\neq 0$, it is called *non-unital*. Next, we summarize some basic properties of $\mathscr{L}_D$. Some proofs are omitted and can be found in [20].

*Lemma 1:* The Lindblad-Kossakowski master equation (5) is unital if and only if $\sum_{k=1}^d[F_k,F_k^\dagger]=0$, or equivalently, $\sum_{j,k=1}^d a_{jk}[B_j,B_k]=0$. In particular, a real GKS matrix $A$ guarantees unitality. However, only for $N=2$ (two-level systems) a real GKS matrix is equivalent to unitality.

Consider the real vector space of Hermitian matrices $\mathfrak{her}(N)\subset\mathbb{C}^{N\times N}$ equipped with the Hilbert-Schmidt inner product $\langle A,B\rangle:=\mathrm{Tr}(AB)$, for all $A,B\in\mathfrak{her}(N)$.

*Lemma 2:* Let $\mathscr{L}_D:\mathfrak{her}(N)\to\mathfrak{her}(N)$ be the linear operator defined by (6). Then, for $\mathscr{L}_D\neq 0$, the strict inequality $\mathrm{Tr}(\mathscr{L}_D)<0$ is satisfied.

*Lemma 3:* For $\mathscr{L}_D\neq 0$, the followings are equivalent.

(a) $\mathscr{L}_D$ is self-adjoint.
(b) $\mathscr{L}_D$ is negative semidefinite.
(c) $\mathscr{L}_D$ has real GKS matrix $A$.
(d) $\mathscr{L}_D$ can be represented in a "double-bracket" form $\mathscr{L}_D(\rho)=-\sum_{j=1}^d\big([C_j,[C_j,\rho]]\big)$, with $C_j\in\mathfrak{her}(N)$.

*Proof:* It is shown in [20] that the double-bracket form is negative semidefinite and thus, by definition, self-adjoint. Now consider the decomposition of $\mathscr{L}_D$ in (3) into its self-adjoint and skew-adjoint part, cf. Proposition 2.21 of [20]. Setting the skew-adjoint part of $\mathscr{L}_D$ to zero for all density operators $\rho$ (including the completely mixed state $I_N/N$) implies the prescribed double-bracket form. This proves the equivalence of (a),(b) and (d). Similarly, consider the decomposition of $\mathscr{L}_D$ in (6) with respect to the matrix elements of $A$, cf. Section 2.4.2 of [20]. Real GKS matrix $A$ implies that $\mathscr{L}_D$ must be self-adjoint. The converse holds by forcing the skew-adjoint part of $\mathscr{L}_D$ to zero. Hence, (a) and (c) are equivalent. ■

By identifying the set of Hermitian matrices with an appropriate real Euclidean space via the affine map $\Phi:\rho\mapsto v=[v_1,\ldots,v_d]^\top$, $d=N^2-1$, where $v_j:=\mathrm{Tr}(\rho B_j)$, we can identify $\mathscr{P}$ with the subset $\Phi(\mathscr{P})$ of the ball $\mathscr{B}\subset\mathbb{R}^d$ of radius $r=\left(1-\frac{1}{N}\right)^{\frac{1}{2}}$ centered at the origin. We call $v\in\Phi(\mathscr{P})\subset\mathscr{B}$ the *vector of coherence* representation of the density operator $\rho$ [2]. The Lindblad-Kossakowski master equation (5) then is equivalent to the affine control system on $\mathbb{R}^d$

$$\dot v=\Big(\mathscr{A}_0+\sum_{k=1}^m u_k(t)\mathscr{A}_k\Big)v+q_0,\quad v(0)=v_0\in\mathbb{R}^d,\tag{8}$$

which, by construction, leaves $\Phi(\mathscr{P})$ invariant. Here, $\mathscr{A}_k \in \mathfrak{so}(d)$ and $\mathscr{A}_0 \in \mathfrak{gl}(d,\mathbb{R})$, where $\mathfrak{so}(d)$ and $\mathfrak{gl}(d,\mathbb{R})$ denote the set of all real skew-symmetric and $d\times d$ matrices, respectively, are the matrix representations of the linear operators

$$\begin{aligned}\mathscr{A}_k \cong -\mathrm{iad}_{H_k} \quad &: \quad \sigma \longmapsto -\mathrm{i}[H_k,\sigma]\\ \mathscr{A}_0 \cong -\mathrm{iad}_{H_0}+\mathscr{L}_D(\cdot) \quad &: \quad \sigma \longmapsto -\mathrm{i}[H_0,\sigma]+\mathscr{L}_D(\sigma),\end{aligned}$$

acting on $\mathfrak{her}_0(N)$, the real vector space of $N\times N$ Hermitian matrices with trace zero. The vector $q_0 \in \mathbb{R}^d$ is the vector of coherence representation of

$$\mathscr{L}_D(I_N/N) = \frac{1}{N}\sum_{j,k=1}^{d} a_{jk}[B_j,B_k] \in \mathfrak{her}_0(N),$$

i.e. $\Phi^{-1}(q_0) = \mathscr{L}_D(I_N/N)$. By Lemma 1, it is clear that the system (8) is unital if and only if $q_0 = 0$. In this case, the Lindblad-Kossakowski equation (8) reduces to a standard bilinear control systems on $\mathbb{R}^d$.

*Remark 1:* As a matter of convenience, treating the Lindblad-Kossakowski master equation in the vector of coherence formalism provides a direct connection to classical transitivity results on $\mathbb{R}^d\setminus\{0\}$. One should bear in mind that all results of this paper dealing with accessibility of open quantum systems, cf. Section IV, can be straightforwardly reformulated in a coordinate-free way with respect to the real vector space $\mathfrak{her}_0(N)$.

## III. CONTROLLABILITY OF CLOSED QUANTUM SYSTEMS

We first focus on closed quantum systems where interaction with the environment is not present. By setting $\mathscr{L}_D = 0$ in (5), we arrive at the so-called *Liouville-von Neumann* master equation

$$(\Sigma_i) \qquad \dot\rho(t) = -\mathrm{i}\Big[H_0+\sum_{k=1}^{m}u_k(t)H_k,\rho(t)\Big], \quad \rho(0)=\rho_0\in\mathscr{P}. \tag{9}$$

Clearly, system (9) "lifts" to the special unitary group $SU(N)$

$$(\Sigma) \qquad \dot U(t) = -\mathrm{i}\Big(H_0+\sum_{k=1}^{m}u_k(t)H_k\Big)U(t), \quad U(0)=I, \tag{10}$$

such that (9) can be regarded as the induced system of (10) via the group action $\alpha(U,\rho) := U(t)\rho U(t)^\dagger$, $U\in SU(N)$. This immediately implies that any solution $\rho(t)$ of (9) evolves on the *unitary orbit* $\mathscr{O}(\rho_0) := \{U\rho_0U^\dagger \mid U\in SU(N)\}$ of $\rho_0$. Thus, controllability issues for (9) should be stated with respect to the unitary orbit $\mathscr{O}(\rho_0)$. For instance, system (9) is called controllable if the reachable set $\mathscr{R}(\rho_0)$ coincides with the entire unitary orbit $\mathscr{O}(\rho_0)$.

Recall the *system Lie algebra* $\mathfrak{s}_\Sigma$ of (10), which is the Lie algebra generated by $\mathrm{i}H_0,\mathrm{i}H_1,\ldots\mathrm{i}H_m$. Then, the *system group* of (10) is defined as

$$\mathscr{G}_\Sigma := \langle\exp(\mathfrak{s}_\Sigma)\rangle_G, \tag{11}$$

which is also a Lie group [15]. As an intermediate step, we consider the case where the Liouville equation (9) evolves on the *complex Grassmannian*, which is defined as the set of all orthogonal projections of rank $k$, i.e.

$$\mathbf{P}(k,N) := \{P\in\mathbb{C}^{N\times N} \mid P^\dagger = P,\ P^2=P,\ \mathrm{Tr}(P)=k\}.$$

Since the Lie group $SU(N)$ is compact, controllability of the Liouville equation on $\mathbf{P}(k,N)$ is equivalent to the transitivity of the system group action $\mathscr{G}_\Sigma$ of (10) on $\mathbf{P}(k,N)$, e.g. [10], [15]. Clearly, $\mathbf{P}(k,N)$ can be identified with the manifold of $k$-dimensional complex subspaces of $\mathbb{C}^N$ called *Grassmann manifold* [14]. Therefore, the classification of all matrix Lie groups acting transitively on Grassmann manifold (see [23] for a complete list) leads to the following necessary and sufficient controllability criterion for the Liouville equation on $\mathbf{P}(k,N)$.

*Theorem 1:* Let $\mathfrak{s}_\Sigma$ be the system Lie algebra generated by $\mathrm{i}H_0,\mathrm{i}H_1,\ldots\mathrm{i}H_m$. The Liouville equation (9) is controllable on the Grassmannian $\mathbf{P}(k,N)$ if and only if

(a) $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{su}(N)$ or conjugate to $\mathfrak{sp}(N/2)$, for $N$ even and $k=1$ or $k=N-1$.
(b) $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{su}(N)$, for $N$ odd or $1<k<N-1$.

This immediately implies the following controllability results for the Liouville equation on the unitary orbit $\mathscr{O}(\rho_0)$.

*Corollary 1:* Let $\rho_0$ be a fixed density operator.

(a) If $N$ is even and $\rho_0$ has only two distinct eigenvalues $\lambda_1$ and $\lambda_2$ with algebraic multiplicity 1 and $N-1$, respectively, then the Liouville equation (9) is controllable on the unitary orbit $\mathscr{O}(\rho_0)$ if and only if $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{su}(N)$ or conjugate to $\mathfrak{sp}(N/2)$.
(b) If $N$ is odd or $\rho_0$ has eigenvalue configurations other than in (a), then the Liouville equation (9) is controllable on the unitary orbit $\mathscr{O}(\rho_0)$ if and only if $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{su}(N)$.

*Proof:* First, consider the case that $\rho_0$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$ with algebraic multiplicity $k$ and $N-k$. Indeed, the unitary orbit $\mathscr{O}(\rho_0)$ then is diffeomorphic to $\mathbf{P}(k,N) = \{UPU^\dagger | U\in SU(N),\ P = \mathrm{diag}(1,\ldots,1,0,\ldots,0), \mathrm{Tr}(P)=k\}$. Hence, the system group acts transitively on $\mathscr{O}(\rho_0)$ if and only if it also acts transitively on $\mathbf{P}(k,N)$. By Theorem 1, we conclude statement (a) and part of statement (b) when $N$ is odd.

In the case where $\rho_0$ has more than two distinct eigenvalues, transitivity on $\mathscr{O}(\rho_0)$ can be shown to imply transitivity on a Grassmannian $\mathbf{P}(k,N)$ with $1<k<N-1$ suitably chosen, see [20]. So again Theorem 1 yields the desired result for the remaining part of statement (b). ■

We remark that the term *density operator controllability* in quantum control literature [1], [3], [22] was introduced to mean controllability of the Liouville equation on *each* unitary orbit $\mathscr{O}(\rho_0)$, $\rho_0\in\mathscr{P}$. Hence, by Corollary 1, a necessary and sufficient condition for density operator controllability is just $\mathfrak{s}_\Sigma = \mathfrak{su}(N)$. Therefore, density operator controllability is completely equivalent to controllability of the lifted Liouville equation (10) on $SU(N)$, often termed as *operator controllability* [1], [15].

*Theorem 2:* Let $\mathfrak{s}_\Sigma$ be the system Lie algebra generated by $\mathrm{i}H_0,\mathrm{i}H_1,\ldots\mathrm{i}H_m$. Then the followings are equivalent.

(a) The algebra $\mathfrak{s}_\Sigma$ equals $\mathfrak{su}(N)$.
(b) The Liouville master equation (9) is density operator controllable.
(c) The lifted Liouville equation (10) is controllable on $SU(N)$, i.e. operator controllable.

In the special case of $k=1$ with $\lambda_1=1$ and $\lambda_2=0$, the set $\mathbf{P}(1,N)$ is obviously known as the complex projective space $\mathbb{C}\mathbf{P}^{N-1}$. The classification of Lie groups acting transitively on complex projective spaces has been well-known [18], [23]. In this case, Corollary 1(a) characterizes what is commonly known in quantum control literature as *projective or pure state controllability*, e.g. [1], [3]. The same applies to the case of $k=1$ with $\lambda_1\neq 0$, $\lambda_2\neq 0$, where Corollary 1(a) reduces to the notion of *pure state like controllability* [22].

## IV. ACCESSIBILITY OF OPEN QUANTUM SYSTEMS

In contrast to closed quantum systems, the Lindblad-Kossakowski master equation (5) modelling open quantum systems does no longer admit a group lift to $SU(N)$. This can be seen from the fact that the evolution of $\rho(t)$ is no longer confined to the unitary orbit $\mathscr{O}(\rho_0)$ due to the presence of dissipation. Instead, a suitable group lift to a non-compact Lie group is needed to derive accessibility results for open quantum systems.

### A. Unital Case

For the unital case, the Lindblad-Kossakowski master equation (8) with $q_0=0$ admits the structure of a bilinear control systems on $\mathbb{R}^d$

$$(\Sigma_i)\quad \dot v=\Big(\mathscr{A}_0+\sum_{k=1}^m u_k(t)\mathscr{A}_k\Big)v,\quad v(0)=v_0\in\mathbb{R}^d, \qquad (12)$$

which leaves $\Phi(\mathscr{P})$ invariant. Therefore, it is trivial to “lift” the system (12) to a bilinear control system on the identity component of $GL(d,\mathbb{R})$

$$(\Sigma)\qquad \dot X=\left(\mathscr{A}_0+\sum_{k=1}^m u_k(t)\mathscr{A}_k\right)X,\quad X(0)=I, \qquad (13)$$

where $\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_m$ are as in (12). In this case, (12) is the induced system of (13) via the group action $\alpha(X,v):=Xv$, $v\in\mathbb{R}^d$. The system Lie algebra $\mathfrak{s}_\Sigma$ of the Lindblad-Kossakowski equation then is the Lie algebra generated by $\mathscr{A}_0,\mathscr{A}_1,\ldots,\mathscr{A}_m$. The corresponding system group $\mathscr{G}_\Sigma$ is defined similarly as in (11).

Due to the presence of dissipation, the Lindblad-Kossakowski master equation (5) is never controllable [10]. Hence, a meaningful question to ask for is that of accessibility of the Lindblad-Kossakowski master equation, i.e. whether the reachable sets $\mathscr{R}(\rho_0)$ of the unital equation (5) have non-empty interior in $\mathscr{P}$ for any initial state $\rho_0\in\mathscr{P}\setminus\{I_N/N\}$. This is equivalent to accessibility of (12) on $\Phi(\mathscr{P})\setminus\{0\}$. Note that the completely mixed state $I_N/N\in\mathscr{P}$, and correspondingly the origin $0\in\Phi(\mathscr{P})$, is excluded because it is a common equilibrium point of the unital Lindblad-Kossakowski master equation.

*Lemma 4:* [20] The unital Lindblad-Kossakowski master equation (12) is accessible on $\Phi(\mathscr{P})\setminus\{0\}$ if and only if it is accessible on $\mathbb{R}^d\setminus\{0\}$.

Based on Lemma 4 and the group lift (13), it is immediate that accessibility of the unital Lindblad-Kossakowski master equation is equivalent to the transitivity of the system group action $\mathscr{G}_\Sigma$ of (13) on $\mathbb{R}^d\setminus\{0\}$, e.g. see [10], [11]. Indeed, the classification of all connected Lie subgroups of $GL(d,\mathbb{R})$ acting transitively on $\mathbb{R}^d\setminus\{0\}$ is a well-known result, cf. [7], [18]. See also the list in [8] written in terms of Lie subalgebras of $\mathfrak{gl}(d,\mathbb{R})$. However, note that the lists in [7], [8] are incomplete and the complete one is provided in [18].

Now we are in the position to state our main results. Proofs can be seen in [20] and will appear elsewhere as a subsequent journal version of this paper.

*Theorem 3:* The unital $N$-level Lindblad-Kossakowski master equation (5) is accessible if and only if the system Lie algebra $\mathfrak{s}_\Sigma\subseteq\mathfrak{gl}(d,\mathbb{R})$ of (13) is conjugate to one of the following types.

(a) For $N$ even: $\mathfrak{gl}(d,\mathbb{R})$, $\mathfrak{so}(d)\oplus\mathbb{R}$
(b) For $N$ odd:
 - $\mathfrak{so}(d)\oplus\mathbb{R}$, $\mathfrak{su}(d/2)\oplus e^{i\alpha}\mathbb{R}$, $\mathfrak{su}(d/2)\oplus\mathbb{C}$,
 - $\mathfrak{sp}(d/4)\oplus e^{i\alpha}\mathbb{R}$, $\mathfrak{sp}(d/4)\oplus\mathbb{C}$, $\mathfrak{sp}(d/4)\oplus\mathbb{H}$,
 - $\mathfrak{gl}(d,\mathbb{R})$, $\mathfrak{gl}(d/2,\mathbb{C})$, $\mathfrak{sl}(d/2,\mathbb{C})\oplus e^{i\beta}\mathbb{R}$,
 - $\mathfrak{sl}(d/4,\mathbb{H})\oplus e^{i\beta}\mathbb{R}$, $\mathfrak{sl}(d/4,\mathbb{H})\oplus\mathbb{C}$, $\mathfrak{sl}(d/4,\mathbb{H})\oplus\mathbb{H}$,
 - $\mathfrak{sp}(d/2,\mathbb{R})\oplus\mathbb{R}$, $\mathfrak{sp}(d/4,\mathbb{C})\oplus e^{i\beta}\mathbb{R}$, $\mathfrak{sp}(d/4,\mathbb{C})\oplus\mathbb{C}$,

where $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\beta\in[-\frac{\pi}{2},\frac{\pi}{2}]$.

For the canonical representations of the above Lie algebras as subalgebras of $\mathfrak{gl}(d,\mathbb{R})$, see [7], [8]. Moreover, for a *particular class* of unital master equations, we can improve the above accessibility condition by excluding more Lie algebras, provided that an additional assumption is satisfied.

*Theorem 4:* Assume that the system Lie algebra $\mathfrak{s}_\Sigma\subseteq\mathfrak{gl}(d,\mathbb{R})$ of (13) contains a Lie subalgebra $\mathfrak{k}\subset\mathfrak{so}(d)$ which acts irreducibly on $\mathbb{R}^d$. Then, the unital $N$-level Lindblad-Kossakowski master equation (5) with a real GKS matrix is accessible if and only if $\mathfrak{s}_\Sigma$ is conjugate to one of the following types.

(a) For $N>2$ even: $\mathfrak{gl}(d,\mathbb{R})$
(b) For $N$ odd:
 - $\mathfrak{gl}(d,\mathbb{R})$, $\mathfrak{gl}(d/2,\mathbb{C})$, $\mathfrak{sl}(d/2,\mathbb{C})\oplus e^{i\beta}\mathbb{R}$,
 - $\mathfrak{sl}(d/4,\mathbb{H})\oplus e^{i\beta}\mathbb{R}$, $\mathfrak{sl}(d/4,\mathbb{H})\oplus\mathbb{C}$, $\mathfrak{sl}(d/4,\mathbb{H})\oplus\mathbb{H}$,
 - $\mathfrak{sp}(d/2,\mathbb{R})\oplus\mathbb{R}$, $\mathfrak{sp}(d/4,\mathbb{C})\oplus e^{i\beta}\mathbb{R}$, $\mathfrak{sp}(d/4,\mathbb{C})\oplus\mathbb{C}$,

 where $\alpha\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\beta\in[-\frac{\pi}{2},\frac{\pi}{2}]$.
(c) For $N=2$: $\mathfrak{gl}(3,\mathbb{R})$, $\mathfrak{so}(3)\oplus\mathbb{R}$.

*Remark 2:* The assumption in Theorem 4, which requires that $\mathfrak{s}_\Sigma$ contains a subalgebra $\mathfrak{k}\subset\mathfrak{so}(d)$ acting irreducibly on $\mathbb{R}^d$, can be fulfilled for example when the Lie algebra generated by the control Hamiltonian $iH_1,iH_2,\ldots,iH_m$ is $\mathfrak{su}(N)$. This corresponds to the fact that the corresponding Lie algebra $\mathfrak{k}_c\subset\mathfrak{s}_\Sigma$ generated by $\mathscr{A}_1,\mathscr{A}_2,\ldots,\mathscr{A}_m$ in (13) indeed satisfies $\mathfrak{su}(N)\cong\mathfrak{k}_c\subset\mathfrak{so}(d)$ and acts irreducibly on $\mathbb{R}^d$, see [20]. Other subalgebras $\mathfrak{k}$ acting irreducibly on $\mathbb{R}^d$ are e.g. the embeddings of $\mathfrak{su}(d/2)$ and $\mathfrak{sp}(d/4)$ in $\mathfrak{so}(d)$.

The following corollary shows that for $n$-coupled spin-$\frac{1}{2}$ systems (e.g. $n$-qubit systems, as frequently appearing in NMR spectroscopy and quantum computing applications [17]), the characterization of accessiblity becomes surprisingly simple in the sense that most of the Lie algebras on the classification list can be excluded. This remarkable result is of course due to the fact that $n$-coupled spin-$\frac{1}{2}$ systems are $N = 2^n$-level systems, and thus $N$ is even.

*Corollary 2:* Consider the unital Lindblad-Kossakowski master equation of $n$-coupled spin-$\frac{1}{2}$ systems.

(a) For $n \geq 1$, the spin system is accessible if and only if the system Lie algebra $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{gl}(2^{2n}-1,\mathbb{R})$ or conjugate to $\mathfrak{so}(2^{2n}-1)\oplus\mathbb{R}$.

(b) For $n > 1$, assume that the GKS matrix is real and $\mathfrak{s}_\Sigma$ contains an irreducible subalgebra of $\mathfrak{so}(2^{2n}-1)$. Then the spin system is accessible if and only if $\mathfrak{s}_\Sigma$ is equal to $\mathfrak{gl}(2^{2n}-1,\mathbb{R})$.

We briefly remark that Theorem 3 and 4 correct the accessibility results previously stated in [4], [5], where the listed classification of transitive Lie-algebras is incomplete. Moreover in [4], it was incorrectly stated that $\mathfrak{s}_\Sigma = \mathfrak{gl}(d,\mathbb{R})$ is necessary and sufficient condition for accessibility of arbitrary $N$-level systems. Our theorems also slightly revise our previous statement in [19].

### B. Non-Unital Case

We consider the non-unital Lindblad-Kossakowski master equation

$$(\Sigma_i) \quad \dot{v} = \Big(\mathscr{A}_0 + \sum_{k=1}^{m} u_k(t)\mathscr{A}_k\Big)v + q_0, \quad v(0) = v_0 \in \mathbb{R}^d, \tag{14}$$

with $q_0 \neq 0$. Note that the origin of $\mathbb{R}^d$ is no longer a common fixed point due to the constant drift $q_0$. The system (14) then lifts to a bilinear control system on the semidirect product $G' = GL(d,\mathbb{R}) \rtimes \mathbb{R}^d$

$$(\Sigma') \quad \dot{X}' = \Big(\mathscr{A}'_0 + \sum_{k=1}^{m} u_k(t)\mathscr{A}'_k\Big)X', \quad X'(0) = (I, v_0) \in G', \tag{15}$$

with $\mathscr{A}'_0 = (\mathscr{A}_0, q_0) \in \mathfrak{g}'$ and $\mathscr{A}'_k = (\mathscr{A}_k, 0) \in \mathfrak{g}'$, where $\mathfrak{g}' = \mathfrak{gl}(d,\mathbb{R}) \rtimes \mathbb{R}^d$ is the Lie algebra of $G'$. For a standard matrix representation of the semidirect product $G'$ and $\mathfrak{g}'$, see [15]. Note that $\mathscr{A}_0$, $\mathscr{A}_k$ and $q_0$ are as in (14). Therefore, the non-unital Lindblad-Kossakowski master equation (14) can be regarded as an induced system of (15) with respect to the Lie group action $\alpha : G' \times \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$\alpha(X', v) = \alpha((X,x), v) := Xv + x. \tag{16}$$

More explicitly, we have for $(\Sigma')$

$$(\dot{X},\dot{x}) = \Big(\big(\mathscr{A}_0 + \sum_{k=1}^{m} u_k(t)\mathscr{A}_k\big)X\,,\, \big(\mathscr{A}_0 + \sum_{k=1}^{m} u_k(t)\mathscr{A}_k\big)x + q_0\Big),$$

for $X(0) = I \in GL(d,\mathbb{R})$ and $x(0) = v_0 \in \mathbb{R}^d$. Thus, the non-unital Lindblad-Kossakowski master equation (14) can also be viewed as a projection of the system $(\Sigma')$ in (15) onto the second factor, while the projection of $(\Sigma')$ onto the first factor yields the bilinear control system on $GL(d,\mathbb{R})$,

$$(\Sigma) \quad \dot{X} = \Big(\mathscr{A}_0 + \sum_{k=1}^{m} u_k(t)\mathscr{A}_k\Big)X, \quad X(0) = I. \tag{17}$$

Moreover, it follows that the system Lie algebra $\mathfrak{s}_{\Sigma'}$ and the system Lie group $\mathscr{G}_{\Sigma'}$ of (15) satisfy

$$\mathfrak{s}_{\Sigma'} \subset \mathfrak{s}_\Sigma \rtimes \mathbb{R}^d \quad , \quad \mathscr{G}_{\Sigma'} \subset \mathscr{G}_\Sigma \rtimes \mathbb{R}^d,$$

with $D\pi(\mathfrak{s}_{\Sigma'}) = \mathfrak{s}_\Sigma$ and $\pi(\mathscr{G}_{\Sigma'}) = \mathscr{G}_\Sigma$, where $\pi$ is the canonical projection $\pi : G' \to G, (X,x) \mapsto X$ with the differential $D\pi$. Note that $\mathfrak{s}_\Sigma$ and $\mathscr{G}_\Sigma$ are, respectively, the system Lie algebra and the system Lie group of the corresponding projected system $(\Sigma)$ in (17).

As far as the question of accessibility is concerned, transitive Lie group action again plays a crucial role [10], [11]. Accessibility then is equivalent to determining when the system group $\mathscr{G}_{\Sigma'}$ of (15) acts transitively on $\mathbb{R}^d$ with respect to the group action (16).

*Lemma 5:* Let $\mathscr{G}$ be a Lie subgroup of the semidirect product $GL(d,\mathbb{R}) \rtimes \mathbb{R}^d$, with the corresponding Lie group action on $\mathbb{R}^d$, cf. (16). Then, $\mathscr{G}$ acts transitively on $\mathbb{R}^d$ if the following two conditions hold.

(a) The group $\pi(\mathscr{G})$ acts transitively on $\mathbb{R}^d \setminus \{0\}$,

(b) There exists at least one element $B'$ in the Lie algebra $\mathfrak{g}'$ of $\mathscr{G}$, which corresponds to a "pure translation", i.e. $B'$ is of the form $(0,b)$, $b \neq 0$.

*Proof:* The second condition implies that there exists $T := (I,b) \in \mathscr{G}$. Take an element $X' = (X,x) \in \mathscr{G}$. Since $\mathscr{G}$ is a group we have $X'T(X')^{-1} = (I, Xb) \in \mathscr{G}$. As $\pi(\mathscr{G})$ acts transitively on $\mathbb{R}^d \setminus \{0\}$, we have $\pi(\mathscr{G})b = \mathbb{R}^d \setminus \{0\}$ and $(I,w) \in \mathscr{G}$ for all $w \in \mathbb{R}^d \setminus \{0\}$. Thus, we have *all* pure translations in $\mathscr{G}$ which implies transitivity of $\mathscr{G}$ on $\mathbb{R}^d$. ∎

Lemma 5 immediately provides the following sufficient condition for accessibility of the non-unital Lindblad-Kossakowski master equation.

*Theorem 5:* Let $(\Sigma_i)$ be the non-unital $N$-level Lindblad-Kossakowski master equation (14) and let $(\Sigma')$ be the corresponding lifted system (15). Then, $(\Sigma_i)$ is accessible if the following two conditions hold.

(a) The system Lie algebra $\mathfrak{s}_\Sigma = D\pi(\mathfrak{s}_{\Sigma'})$ of the projected system $(\Sigma)$ of (17) is conjugate to one of the transitive Lie algebras listed in Theorem 3.

(b) There exists at least a "pure translation" in the Lie algebra $\mathfrak{s}_{\Sigma'}$, i.e. $B' = (0,b) \in \mathfrak{s}_{\Sigma'}$, with $b \neq 0$.

However, it is not so straightforward to check whether a pure translation $(0,b)$, $b \neq 0$, is really contained in $\mathfrak{s}_{\Sigma'}$. By assuming the subsequennt "no-common fixed point" condition which might be easier to check, it is possible to conclude that the transitivity of $\pi(\mathscr{G}_{\Sigma'}) = \mathscr{G}_\Sigma$ on $\mathbb{R}^d \setminus \{0\}$ even implies the existence of *all* pure translations in $\mathfrak{s}_{\Sigma'}$. The system $(\Sigma_i)$ of the non-unital Lindblad-Kossakowski master equation (14) is said to have *no-common fixed point* in $\mathbb{R}^d$ if, for all $v \in \mathbb{R}^d$, there exist $u_k \in \mathbb{R}$ such that

$$\big(\mathscr{A}_0 + \sum_{k=1}^{m} u_k\mathscr{A}_k\big)v + q_0 \neq 0.$$

Thus, we arrive at the following theorem which is basically an adaptation of the results by Jurdjevic, Sallet and Kupka [6], [16]. For a more detailed and elaborated version of the original proof, see [20].

*Theorem 6:* Let $(\Sigma_i)$ be the non-unital $N$-level Lindblad-Kossakowski master equation (14), and let $(\Sigma')$ be the corresponding lifted system (15). Assume that $(\Sigma_i)$ has no common fixed point in $\mathbb{R}^d$. Then, $(\Sigma_i)$ is accessible if the system Lie algebra $\mathfrak{s}_\Sigma = D\pi(\mathfrak{s}_{\Sigma'})$ of the projected system $(\Sigma)$ in (17) is conjugate to one of the transitive Lie algebras listed in Theorem 3.

The proof in [20] shows that, when the conditions of Theorem 6 are satisfied, all pure translations are contained in the system Lie algebra $\mathfrak{s}_{\Sigma'}$ of (15). Then we conclusively have $\mathfrak{s}_{\Sigma'} = \mathfrak{s}_\Sigma \rtimes \mathbb{R}^d$, and the transitivity of the system group $\mathscr{G}_{\Sigma'}$ on $\mathbb{R}^d$ follows. On the other hand, we note that there are solvable Lie subalgebras of $\mathfrak{gl}(d,\mathbb{R}) \rtimes \mathbb{R}^d$ without any pure translation which do act transitively on $\mathbb{R}^d$, see e.g.[9].

### C. Genericity Results

Finally, we address the problem whether accessibility of the general $N$-level Lindblad-Kossakowski master equation is a generic property. We call a set generic if it contains an open and dense subset.

*Theorem 7:* (a) The unital Lindblad-Kossakowski master equation (12) with $m = 1$ is generically accessible, i.e. the set of all $(\mathscr{A}_0, \mathscr{A}_1)$ such that unital accessibility holds is open and dense in the set of all admissible unital Lindblad-Kossakowski generators.

(b) The non-unital Lindblad-Kossakowski master equation (14) with $m = 1$ is generically accessible, i.e. the set of all $(\mathscr{A}_0, \mathscr{A}_1)$ such that non-unital accessibility holds contains an open and dense subset of the set of all admissible non-unital Lindblad-Kossakowski generators.

Therefore, our results assert that "almost all" Lindblad-Kossakowski master equation are accessible even with only a *single control* no matter how large $N$ might be. Obviously, the results remain valid when more than one controls are available. A similar statement can also be found in [5] with an incomplete argument invoking Theorem 12, Chapter 6 of [15] from the theory of semisimple Lie algebras. However, the result in [15] can not be directly applied here as one is restricted to the class of non-trivial Lindblad-Kossakowski generators. Thus, more arguments are involved to prove the statements. The detailed versions of Theorem 7 and complete proofs can be consulted in [20] and will appear in a subsequent full paper.

## V. CONCLUSIONS

A Lie theoretical approach has been applied to analyze controllability aspects of finite dimensional (N-level) bilinear quantum control systems. In particular, the well-known classical results on the classifications of Lie groups which act transitively on Grassmann manifolds, and respectively on $\mathbb{R}^d \setminus \{0\}$, have been exploited to derive controllability results for closed quantum systems (the Liouville master equation) and accessibility results for open quantum systems (the Lindblad-Kossakowski master equation). Specifically, within this unifying framework of transitive Lie groups action, we are able to simplify characterization for different notions of controllability of closed quantum systems and derive conditions for accessibility of open quantum systems, both for the unital and non-unital case.

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