Periodically Time-Varying Controller Synthesis for Multiobjective H_2/H_∞ Control of Discrete-Time Systems and Analysis of Achievable Performance

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***Abstract*— In this paper, we propose a linear periodically time-varying (LPTV) controller synthesis approach for the multiobjective H_2/H_∞ control problem of discrete-time linear time-invariant (LTI) systems. By artificially regarding the LTI plant as N-periodic and applying the discrete-time system lifting, we first derive an SDP for the synthesis of suboptimal multiobjective LPTV controllers. Furthermore, we show that we can reduce the conservatism and improve the control performance gradually by simply increasing the controller period. On the other hand, in the latter part of the paper, we propose another SDP for the computation of a lower bound of the control performance that is achievable via LPTV controllers of any period and order. Similarly to the LPTV controller synthesis, the SDP is derived based on the lifting-based treatment of the LTI plant, and it is shown that we can improve the lower bound gradually by increasing the fictitious period N. We validate all of these theoretical results through an illustrative example.**

I. INTRODUCTION

When designing control systems, we are often required to satisfy multiple (and mostly conflicting) design specifications. Achieving desirable disturbance rejection while ensuring adequate robustness against uncertainties of the plant would be a typical example. This design problem is naturally formulated as the multiobjective H_2/H_∞ problem in which we seek for a controller that satisfies prescribed H_2 and H_∞ norm specifications imposed on each input-to-output channel. General multiobjective control problems including other performance criteria have been studied intensively since the late 90's and several striking results are obtained to this date. Among them, the works [9], [8] are pioneering and first provide semidefinite programming (SDP) formulations for the multiobjective control problems. Since the SDPs there are derived by forcing a common Lyapunov variable for the set of linear matrix inequalities (LMIs) representing multiple design specifications, the approaches in [9], [8] are conservative. Since then, the reduction of the conservatism has been recognized as an important issue, and the "extended" or "dilated" LMI approaches [4], [5] succeeded in part along this direction. The LMI approach with finite-dimensional Q parametrization [7], [10] is also promising, where we can reduce the conservatism gradually at the expense of increased controller order. In spite of this crucial achievement, we cannot still draw any definite conclusion on the quality of the designed controllers quantitatively. Apart from these LMI approaches, another important contribution is the "compensator blending" [3], which enables us to design genuine optimal multiobjective controllers without any conservatism. This is in sharp contrast with the known LMI approaches. Unfortunately, however, we can apply this method only to the generalized plant that meets some strong structural assumptions.

In view of the current state briefly sketched above, and in particular most existing studies focus on LTI controller synthesis without definite theoretical grounds, we propose in this paper a linear periodically time-varying (LPTV) controller synthesis approach for the multiobjective H_2/H_∞ control problem of discrete-time linear time-invariant (LTI) systems. By artificially regarding the LTI plant as N-periodic and applying the discrete-time system lifting [2], we first derive an SDP for the suboptimal multiobjective H_2/H_∞ LPTV controller synthesis. This SDP readily follows from [4]. The impact of this lifting-based approach lies in the fact that we can reduce the conservatism and improve the control performance gradually by simply increasing the controller period. On the other hand, in the latter part of the paper, we propose another SDP for the computation of a lower bound of the control performance that is achievable via LPTV controllers of any period and order. Similarly to the LPTV controller synthesis, the SDP is derived based on the lifting-based treatment of the LTI plant, and it is again shown that we can improve the lower bound by increasing the fictitious period N. To the best of the author's knowledge, such lower bound computation is completely missing in the literature. Obviously, this lower bound is useful for evaluating the quality of a given controller quantitatively, irrespective of how the controller is designed. We validate all of these theoretical results through an illustrative example.

We use the following notations in this paper. The set of positive-definite symmetric matrices of the size n are denoted by $\mathbf{P}_n$. For $A \in \mathbf{R}^{n\times n}$ and $B \in \mathbf{R}^{n\times m}$, we define $\mathrm{He}\{A\} := A + A^T$ and $\mathrm{Sq}\{B\} = BB^T$.

II. MULTIOBJECTIVE H_2/H_∞ CONTROL PROBLEM AND SCOPE OF THE PAPER

Let us consider the discrete-time linear time-invariant (LTI) plant P described by

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$$\begin{bmatrix} x_{k+1} \\ z_{2,k} \\ z_{\infty,k} \\ y_k \end{bmatrix} = \begin{bmatrix} A & B_{w_2} & B_{w_\infty} & B_u \\ C_{z_2} & D_{z_2 w_2} & 0 & D_{z_2 u} \\ C_{z_\infty} & 0 & D_{z_\infty w_\infty} & D_{z_\infty u} \\ C_y & D_{y w_2} & D_{y w_\infty} & 0 \end{bmatrix} \begin{bmatrix} x_k \\ w_{2,k} \\ w_{\infty,k} \\ u_k \end{bmatrix}. \quad (1)$$

Here, $x \in \mathbf{R}^n$ is the state, $u \in \mathbf{R}^m$ the control input, $y \in \mathbf{R}^l$ the measured output, $w_j \in \mathbf{R}^{m_j}$ $(j = 2, \infty)$ the disturbance input and $z_j \in \mathbf{R}^{l_j}$ $(j = 2, \infty)$ the performance output, respectively. For the LTI plant P and a given controller K, not necessarily restricted to LTI, we denote by $T(P,K)_{z_j w_j}$ the closed-loop system with respect to the input w_j and the output z_j $(j = 2, \infty)$. Under these settings, the multiobjective (or mixed) H_2/H_∞ control problem discussed in [4], [8], [9], [10] can be stated formally as follows.

Problem: For given $\gamma_\infty > 0$, design a stabilizing LTI controller K such that

(i) the H_∞ norm constraint $\|T(P,K)_{z_\infty w_\infty}\|_\infty < \gamma_\infty$ is satisfied;

(ii) the H_2 norm $\|T(P,K)_{z_2 w_2}\|_2$ is minimized.

In contrast with [4], [8], [9], [10], in this paper, we explore linear periodically time-varying (LPTV) controller synthesis so that we can achieve better H_2 performance under the H_∞ performance constraint. Namely, we are interested in designing an N-periodic controller K_N of the form

$$\begin{bmatrix} \xi_{k+1} \\ u_k \end{bmatrix} = \begin{bmatrix} \mathsf{A}_k & \mathsf{B}_k \\ \mathsf{C}_k & \mathsf{D}_k \end{bmatrix} \begin{bmatrix} \xi_k \\ y_k \end{bmatrix} \quad (2)$$

where $\mathsf{A}_k \in \mathbf{R}^{p\times p}$, $\mathsf{B}_k \in \mathbf{R}^{p\times l}$, $\mathsf{C}_k \in \mathbf{R}^{m\times p}$ and $\mathsf{D}_k \in \mathbf{R}^{m\times l}$ are all N-periodic, i.e., $\mathsf{A}_{k+N} = \mathsf{A}_k$ and so on. Moreover, to evaluate the quality of the designed LPTV controllers quantitatively, we propose an efficient method to compute a lower bound of the H_2 performance that is achievable via LPTV controllers of *any order and period*, under the H_∞ performance constraint. If designed LTI or LPTV controllers achieve the H_2 performance that is close to this lower bound, we can conclude that we have achieved almost the best performance within LPTV controllers of any order and period.

III. Preliminary Results

In this section we quickly review basic results around lifting-based treatment of discrete-time periodic systems [2]. We also assemble basic LMI formulas for the analysis of discrete-time LPTV systems and clarify their interesting properties in terms of system lifting.

A. Basics of System Lifting

Let us consider the discrete-time linear N-periodic system G_N described by

$$\begin{bmatrix} x_{k+1} \\ z_k \end{bmatrix} = \begin{bmatrix} A_k & B_k \\ C_k & D_k \end{bmatrix} \begin{bmatrix} x_k \\ w_k \end{bmatrix} \quad (3)$$

where $x \in \mathbf{R}^n$, $w \in \mathbf{R}^{m_k}$ and $z \in \mathbf{R}^{l_k}$. Here we allow the size of B_k, C_k and D_k is also periodically time-varying. As is well-known, the behavior of the periodic system of the form (3) can be fully grasped via an LTI system $\widehat{G}_N^{[N]}$ that is obtained by the discrete-time system lifting (i.e., lifted reformulation in [2]). This fictitious LTI system $\widehat{G}_N^{[N]}$ is explicitly described by

$$\begin{bmatrix} \widehat{x}_{k+1} \\ \widehat{z}_k \end{bmatrix} = \begin{bmatrix} \widehat{A}^{[N]} & \widehat{B}^{[N]} \\ \widehat{C}^{[N]} & \widehat{D}^{[N]} \end{bmatrix} \begin{bmatrix} \widehat{x}_k \\ \widehat{w}_k \end{bmatrix} \quad (4)$$

where $\widehat{x}_k = x_{kN}$ and

$$\widehat{w}_k = \begin{bmatrix} w_{kN}^T & \cdots & w_{(k+1)N-1}^T \end{bmatrix} \in \mathbf{R}^{N_m},$$
$$\widehat{z}_k = \begin{bmatrix} z_{kN}^T & \cdots & z_{(k+1)N-1}^T \end{bmatrix} \in \mathbf{R}^{N_l},$$
$$N_m := \sum_{k=0}^{N-1} m_k, \; N_l := \sum_{k=0}^{N-1} l_k.$$

Explicit expressions of the coefficient matrices in (4) are given in (5) at the top of the next page.

In the following, we often regard (5) as a map from given $M_k := (A_k, B_k, C_k, D_k)$ $(k = 0, \cdots, N-1)$ to $\widehat{M}^{[N]} := (\widehat{A}^{[N]}, \widehat{B}^{[N]}, \widehat{C}^{[N]}, \widehat{D}^{[N]})$. It is straightforward to see that the map (5) satisfies the following property.

Fact 1: Given $M_k = (A_k, B_k, C_k, D_k)$ $(k = 0, \cdots, N-1)$, we have

$$\begin{bmatrix} \widehat{A}^{[N']} & \widehat{B}^{[N']} \\ \widehat{C}^{[N']} & \widehat{D}^{[N']} \end{bmatrix} = \left[\begin{array}{c|cc} A_{N'-1}\widehat{A}^{[N'-1]} & A_{N'-1}\widehat{B}^{[N'-1]} & \widehat{B}_{N'-1} \\ \hline \widehat{C}^{[N'-1]} & \widehat{D}^{[N'-1]} & 0 \\ C_{N'-1}\widehat{A}^{[N'-1]} & C_{N'-1}\widehat{B}^{[N'-1]} & D_{N'-1} \end{array}\right]$$

for $N' = 2, \cdots, N$.

It is known that the original periodic system G_N is stable if and only if the fictitious LTI system $\widehat{G}_N^{[N]}$ is stable, i.e., the matrix $\widehat{A}^{[N]}$ in (4) is Schur stable. Moreover, as shown in [1], [12], we can define the square of the generalized H_2 norm of the N-periodic system (3) as $\|\widehat{G}_N^{[N]}\|_2^2/N$, where $\|\widehat{G}_N^{[N]}\|_2$ stands for the H_2 norm of the LTI system (4). The generalized H_2 norm, denoted by $\|G_N\|_2$, corresponds to the mean of all the responses corresponding to impulsive inputs applied to each of the m_k input channels at each time k in the N-period. We can also naturally define the H_∞ norm of (3) by $\|\widehat{G}_N^{[N]}\|_\infty$, which is exactly the same as the H_∞ norm of the LTI system (4). In the time-domain, this norm can be interpreted as the input-to-output l_2 induced norm. In view of this fact, with a little abuse of notation, we often write $\|G_N\|_\infty$ and call it as the H_∞ norm of the periodic system G_N, which precisely means $\|\widehat{G}_N^{[N]}\|_\infty$. Similarly to the LTI case, these two norms are reasonable measures to assess the performance of the periodic system (3).

To design an LPTV controller K_N for the LTI plant P, we consider applying the lifting to the closed-loop system $T(P, K_N)_{z_j w_j}$ $(j = \infty, 2)$. If this closed-loop system is (well-posed) and stable, then we have

$$\begin{aligned} \|\widehat{T}^{[N]}(P,K_N)_{z_\infty w_\infty}\|_\infty &= \|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\|_\infty, \\ \|\widehat{T}^{[N]}(P,K_N)_{z_2 w_2}\|_2 &= \|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_2 \widehat{w}_2}\|_2. \end{aligned} \quad (6)$$

This readily follows if we note that input-to-output behaviour of the system is preserved under lifting. It follows that

$$\begin{aligned} \|T(P,K_N)_{z_\infty w_\infty}\|_\infty &= \|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\|_\infty, \\ \|T(P,K_N)_{z_2 w_2}\|_2^2 &= \frac{1}{N}\|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_2 \widehat{w}_2}\|_2^2. \end{aligned} \quad (7)$$

Based on this fact, we will obtain a multiobjective LPTV controller by designing a multiobjective LTI controller $K^{[N]}$ for the fictitious LTI plant $\widehat{P}^{[N]}$. It should be noted that $K^{[N]}$ is of course N-periodic for the original plant P. If the coefficient matrices of $K^{[N]}$ $(N \geq 2)$ are given as

$$\begin{bmatrix} \widehat{A}^{[N]} & \widehat{B}^{[N]} \\ \widehat{C}^{[N]} & \widehat{D}^{[N]} \end{bmatrix} := \left[\begin{array}{c:ccccc} \prod_{k=0}^{N-1} A_k & \left(\prod_{k=1}^{N-1} A_k\right) B_0 & \cdots & A_{N-1}B_{N-2} & B_{N-1} \\ \hdashline C_0 & D_0 & 0 & \cdots & 0 \\ C_1 A_0 & C_1 B_0 & D_1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ C_{N-1}\prod_{k=0}^{N-2} A_k & C_{N-1}\left(\prod_{k=1}^{N-2} A_k\right) B_0 & \cdots & C_{N-1}B_{N-2} & D_{N-1} \end{array}\right], \quad \prod_{k=p}^{q} A_k := A_q \cdots A_{p+1} A_p. \tag{5}$$

$$\left[\begin{array}{c:c} \mathsf{A}^{[N]} & \mathsf{B}^{[N]} \\ \hline \mathsf{C}^{[N]} & \mathsf{D}^{[N]} \end{array}\right] = \left[\begin{array}{c:cccc} \mathsf{A}^{[N]} & \mathsf{B}_0^{[N]} & \mathsf{B}_1^{[N]} & \cdots & \mathsf{B}_{N-1}^{[N]} \\ \hdashline \mathsf{C}_0^{[N]} & \mathsf{D}_{0,0}^{[N]} & 0 & \cdots & 0 \\ \mathsf{C}_1^{[N]} & \mathsf{D}_{1,0}^{[N]} & \mathsf{D}_{1,1}^{[N]} & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \mathsf{C}_{N-1}^{[N]} & \mathsf{D}_{N-1,0}^{[N]} & \cdots & \mathsf{D}_{N-1,N-2}^{[N]} & \mathsf{D}_{N-1,N-1}^{[N]} \end{array}\right], \tag{8}$$

then this controller can be represented as the standard periodic form (2) where

$$\begin{aligned} &\mathsf{A}_k = I_{n+(N-1)l} \ (k=0,\cdots,N-2), \\ &\mathsf{A}_{N-1} = \begin{bmatrix} \mathsf{A}^{[N]} & \mathsf{B}_0^{[N]} & \cdots & \mathsf{B}_{N-2}^{[N]} \\ 0_{(N-1)l,n} & 0_{(N-1)l,l} & \cdots & 0_{(N-1)l,l} \end{bmatrix}, \\ &\mathsf{B}_k = \begin{bmatrix} 0_{n,l} \\ e_{k+1} \otimes I_l \end{bmatrix} \ (k=0,\cdots,N-2), \\ &\mathsf{B}_{N-1} = \begin{bmatrix} \mathsf{B}_{N-1}^{[N]} \\ 0_{(N-1)l,l} \end{bmatrix}, \quad \mathsf{C}_0 = \begin{bmatrix} \mathsf{C}_0^{[N]} & 0_{m,(N-1-k)l} \end{bmatrix}, \\ &\mathsf{C}_k = \begin{bmatrix} \mathsf{C}_k^{[N]} \mathsf{D}_{k,0}^{[N]} \cdots \mathsf{D}_{k,k-1}^{[N]} 0_{m,(N-1-k)l} \end{bmatrix} \ (k=1,\cdots,N-1), \\ &\mathsf{D}_k = \mathsf{D}_{k,k}^{[N]}. \end{aligned} \tag{9}$$

Here, e_k stands for the k-th column of I_{N-1}. We emphasize that $K^{[N]}$ becomes causal if and only if $\mathsf{D}^{[N]}$ is block lower-triangular as in (8). To denote this property compactly, let us make the following definition.

Definition 1: We define the set $\mathcal{D}^{[N]}$, which consists of block lower-triangular matrices of the form $\mathsf{D}^{[N]}$ in (8). Moreover, we define $\mathcal{D}_{\text{toeplitz}}^{[N]} \subset \mathcal{D}^{[N]}$, where $\mathsf{D}^{[N]} \in \mathcal{D}_{\text{Toeplitz}}^{[N]}$ is block lower-triangular and block Toeplitz.

B. Basics of LMI-based LPTV System Analysis

Let us consider the LPTV system G_N given by (3). For the analysis of this LPTV system, the following two LMI results are fundamental.

Proposition 1: [2] The LPTV system (3) is stable and $\|G_N\|_\infty < \gamma_\infty$ holds if and only if there exist $X_{\infty,k} \in \mathbf{P}_n$, $G_{\infty,k} \in \mathbf{R}^{n\times n}$ $(k=0,\cdots,N-1)$ such that

$$\begin{bmatrix} -X_{\infty,k+1} & 0 & A_k G_{\infty,k} \\ * & -\gamma_\infty^2 I & C_k G_{\infty,k} \\ * & * & X_{\infty,k} - \mathrm{He}\{G_{\infty,k}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} B_k \\ D_k \\ 0 \end{bmatrix} \right\} \prec 0 \tag{10}$$
$$(k=0,\cdots,N-1)$$

where $X_{\infty,N} = X_{\infty,0}$.

Proposition 2: [6] The LPTV system (3) is stable and $\|G_N\|_2 < \gamma_2$ holds if and only if there exist $X_{2,k} \in \mathbf{P}_n$, $G_{2,k} \in \mathbf{R}^{n\times n}$, $W_k \in \mathbf{P}_{l_k}$ and $\gamma_{2,K}$ $(k=0,\cdots,N-1)$ such that

$$\mathrm{trace}(W_k) - \gamma_{2,k}^2 < 0 \tag{11a}$$

$$\begin{bmatrix} -W_k & C_k G_{2,k} \\ * & X_{2,k} - \mathrm{He}\{G_{2,k}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} D_k \\ 0 \end{bmatrix} \right\} \prec 0, \tag{11b}$$

$$\begin{bmatrix} -X_{2,k+1} & A_k G_{2,k} \\ * & X_{2,k} - \mathrm{He}\{G_{2,k}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} B_k \\ 0 \end{bmatrix} \right\} \prec 0 \tag{11c}$$
$$(k=0,\cdots,N-1),$$

$$\frac{1}{N}\sum_{k=0}^{N-1} \gamma_{2,k}^2 - \gamma_2^2 < 0 \tag{12}$$

where $X_{2,N} = X_{2,0}$.

For ease of description, let us denote the left-hand side of the LMI (10) by $\Phi_\infty(M_k, X_{\infty,k}, X_{\infty,k+1}, G_{\infty,k}, \gamma_\infty)$. Similarly, we denote by $\Phi_2(M_k, X_{2,k}, X_{2,k+1}, G_{2,k}, W_k, \gamma_{2,k})$ the block-diagonal augmentation of the left-hand side of the three LMIs in (11). Namely, we imply

$$\Phi_2(M_k, X_{2,k}, X_{2,k+1}, G_{2,k}, W_k, \gamma_{2,k}) \prec 0 \ \Leftrightarrow (11).$$

Under these notations, we can establish the following result that relates the LMIs (10) and (11) with those of fictitious LTI system $\widehat{G}^{[N]}$, respectively.

Lemma 1: For the LPTV system G_N given by (3), the following results hold true.

(i) For given $\gamma_\infty > 0$ and $X_{\infty,k} \in \mathbf{P}_n$, $G_{\infty,k} \in \mathbf{R}^{n\times n}$ $(k=0,\cdots,N-1)$, suppose (10) holds. Then,
$$\Phi_\infty(\widehat{M}^{[N]}, X_{\infty,0}, X_{\infty,0}, G_{\infty,0}, \gamma_\infty) \prec 0. \tag{13}$$

(ii) For given $\gamma_{2,k} > 0$, $X_{2,k} \in \mathbf{P}_n$, $G_{2,k} \in \mathbf{R}^{n\times n}$, and $W_k \in \mathbf{P}_{l_k}$ $(k=0,\cdots,N-1)$, suppose (11) holds. Then, there exist $W \in \mathbf{P}_{N_l}$ such that
$$\Phi_2(\widehat{M}^{[N]}, X_{2,0}, X_{2,0}, G_{2,0}, W, \bar{\gamma}_2) \prec 0, \quad \bar{\gamma}_2 := \sqrt{\sum_{k=0}^{N-1} \gamma_{2,k}^2}. \tag{14}$$

In the following we state the proof of (i). The proof for (ii) is given in the appendix section.

Proof of (i) of Lemma 1: From the inequalities for $k=0,1$ in (10), we can easily confirm that

$$\begin{bmatrix} -X_{\infty,2} & 0 & A_1 G_{\infty,1} & 0 & 0 \\ * & -\gamma_\infty^2 I & C_1 G_{\infty,1} & 0 & 0 \\ * & * & -\mathrm{He}\{G_{\infty,1}\} & 0 & A_0 G_{\infty,0} \\ * & * & * & -\gamma_\infty^2 I & C_0 G_{\infty,0} \\ * & * & * & * & X_{\infty,0} - \mathrm{He}\{G_{\infty,0}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} B_1 & 0 \\ D_1 & 0 \\ 0 & B_0 \\ 0 & D_0 \\ 0 & 0 \end{bmatrix} \right\} \prec 0.$$

Multiplying the above inequality by

$$\begin{bmatrix} I & 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & I & C_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$

from left and its transpose from right, we obtain

$$\left[\begin{array}{c|cc|c} -X_{\infty,2} & 0 & 0 & A_1A_0G_{\infty,0} \\ \hline * & -\gamma_\infty^2 I & 0 & C_0G_{\infty,0} \\ * & * & -\gamma_\infty^2 I & C_1A_0G_{\infty,0} \\ \hline * & * & * & X_{\infty,0}-\mathrm{He}\{G_{\infty,0}\} \end{array}\right] + \mathrm{Sq}\left\{\left[\begin{array}{cc} A_1B_0 & B_1 \\ \hline D_0 & 0 \\ C_1B_0 & D_1 \\ \hline 0 & 0 \end{array}\right]\right\} \prec 0.$$

From Fact 1, we see that this is nothing but

$$\Phi_\infty(\widehat{M}^{[2]}, X_{\infty,0}, X_{\infty,2}, G_{\infty,0}, \gamma_\infty) \prec 0. \quad (15)$$

By applying similar procedure to (15) and the inequality for $k = 2$ in (10) and noting Fact 1, we can readily obtain

$$\Phi_\infty(\widehat{M}^{[3]}, X_{\infty,0}, X_{\infty,3}, G_{\infty,0}, \gamma_\infty) \prec 0.$$

By repeating this procedure recursively, we arrive at (13). This completes the proof. Q.E.D.

In the next section, we will use this lemma to prove our main result, Theorem 1.

Remark 1: When we analyze the H_∞ and H_2 performance of an LTI system G with the coefficient matrices $M = (A, B, C, D)$, the corresponding LMI conditions for (10) and (11) are given, respectively, in the following form:

$$\Phi_\infty(M, X_\infty, X_\infty, G_\infty, \gamma_\infty) \prec 0, \quad (16a)$$

$$\Phi_2(M, X_2, X_2, G_2, W, \gamma_2) \prec 0. \quad (16b)$$

These coincide with the "extended LMI conditions" for LTI system analysis derived in [4].

IV. Multiobjective LPTV Controller Synthesis and Reduction of Conservatism

A. Multiobjective LPTV Controller Synthesis

Now we are ready to state our main results for multiobjective H_2/H_∞ LPTV controller synthesis. To design LPTV controllers of period N, we first apply the discrete-time system lifting of period N to the plant P. Then, we can obtain a fictitious LTI plant $\widehat{P}_N$ given as follows:

$$\begin{bmatrix} \widehat{x}_{k+1} \\ \widehat{z}_{2,k} \\ \widehat{z}_{\infty,k} \\ \widehat{y}_k \end{bmatrix} = \begin{bmatrix} \widehat{A}^{[N]} & \widehat{B}_{w_2}^{[N]} & \widehat{B}_{w_\infty}^{[N]} & \widehat{B}_u^{[N]} \\ \widehat{C}_{z_2}^{[N]} & \widehat{D}_{z_2w_2}^{[N]} & \bullet & \widehat{D}_{z_2u}^{[N]} \\ \widehat{C}_{z_\infty}^{[N]} & \bullet & \widehat{D}_{z_\infty w_\infty}^{[N]} & \widehat{D}_{z_\infty u} \\ \widehat{C}_y^{[N]} & \widehat{D}_{yw_2}^{[N]} & \widehat{D}_{yw_\infty}^{[N]} & \widehat{D}_{yu}^{[N]} \end{bmatrix} \begin{bmatrix} \widehat{x}_k \\ \widehat{w}_{2,k} \\ \widehat{w}_{\infty,k} \\ \widehat{u}_k \end{bmatrix}. \quad (17)$$

Here, $\widehat{x}_k \in \mathbf{R}^n$, $\widehat{u}_k \in \mathbf{R}^{Nm}$, $\widehat{y}_k \in \mathbf{R}^{Nl}$ and $\widehat{w}_{j,k} \in \mathbf{R}^{Nm_j}$, $\widehat{z}_{j,k} \in \mathbf{R}^{Nl_j}$ $(j = 2, \infty)$. Explicit expressions of $\widehat{A}^{[N]}$, etc., can be easily traced from (5). Those terms denoted by • are irrelevant in the subsequent discussion.

For this fictitious LTI plant $\widehat{P}^{[N]}$, let us design a *full-order* LTI controller $K^{[N]} = (\mathsf{A}^{[N]}, \mathsf{B}^{[N]}, \mathsf{C}^{[N]}, \mathsf{D}^{[N]})$ where $\mathsf{A}^{[N]} \in \mathbf{R}^{n\times n}$. As already noted, this controller is N-periodic for the original plant P and becomes causal if and only if $\mathsf{D}^{[N]} \in \mathcal{D}^{[N]}$. We denote by $\mathcal{K}^{[N]}$ the set of these full-order and causal controllers. Then, we can obtain the following two lemmas for the synthesis of $K^{[N]} \in \mathcal{K}^{[N]}$.

Lemma 2: [4] There exists $K^{[N]} \in \mathcal{K}^{[N]}$ that satisfies the H_∞ norm constraint $\|T(\widehat{P}^{[N]}, K^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\|_\infty < \gamma_\infty$ if and only if the LMI of the following form holds:

$$\begin{array}{l} \Phi_\infty^{\mathrm{syn}}(\widehat{M}_{\infty,0}^{[N]}, X_\infty, X_\infty, G_\infty, \gamma_\infty, \mathsf{J}_\infty, \widetilde{\mathsf{D}}_\infty) \prec 0, \\ \widetilde{\mathsf{D}}_\infty \in \mathcal{D}^{[N]}. \end{array} \quad (18)$$

If this LMI is feasible, then the desired H_∞ controller can be constructed via $(\mathsf{A}^{[N]}, \mathsf{B}^{[N]}, \mathsf{C}^{[N]}, \mathsf{D}^{[N]}) = F(G_\infty, \mathsf{J}_\infty, \widetilde{\mathsf{D}}_\infty, D_{yu}^{[N]})$ where $F(\cdot)$ is an appropriately defined nonlinear map.

Lemma 3: [4] There exists $K^{[N]} \in \mathcal{K}^{[N]}$ that satisfies the H_2 norm constraint $\|T(\widehat{P}^{[N]}, K^{[N]})_{\widehat{z}_2\widehat{w}_2}\|_2 < \gamma_2$ if and only if the LMI of the following form holds:

$$\begin{array}{l} \Phi_2^{\mathrm{syn}}(\widehat{M}_{2,0}^{[N]}, X_2, X_2, G_2, W, \gamma_2, \mathsf{J}_2, \widetilde{\mathsf{D}}_2) \prec 0, \\ \widetilde{\mathsf{D}}_2 \in \mathcal{D}^{[N]}. \end{array} \quad (19)$$

If this LMI is feasible, then the desired H_2 controller can be constructed via $(\mathsf{A}^{[N]}, \mathsf{B}^{[N]}, \mathsf{C}^{[N]}, \mathsf{D}^{[N]}) = F(G_2, \mathsf{J}_2, \widetilde{\mathsf{D}}_2, D_{yu}^{[N]})$.

Remark 2: (i) In [4], the LMI (18) is derived by applying a linearizing transformation of variables for the LMI (16a) corresponding to the close-loop system. The variables in (18) are X_∞, the transformed Lyapunov matrix, G_∞, the transformed auxiliary variable, J_∞, the transformed variable corresponding to $(\mathsf{A}^{[N]}, \mathsf{B}^{[N]}, \mathsf{C}^{[N]})$ and $\widetilde{\mathsf{D}}_\infty$ the transformed variable corresponding to $\mathsf{D}^{[N]}$ and is of block lower-triangular. On the other hand, $\widehat{M}_{\infty,0}^{[N]}$ denotes the matrix data in (17) that is relevant to the H_∞ norm specification, where $D_{yu}^{[N]}$ is treated as zero. This treatment is indispensable since the LMI in [4] assumes that the direct feedthrough term from the control input to the measured output is zero. All of the above remarks apply also to the LMI (19) in Lemma 3.

(ii) In the nonlinear map $F(\cdot)$, we note that $\mathsf{D}^{[N]}$ for the H_∞ synthesis case depends only on $\widetilde{\mathsf{D}}_\infty$ and $D_{yu}^{[N]}$ and is given explicitly as

$$\mathsf{D}^{[N]} = \widetilde{\mathsf{D}}_\infty(I + D_{yu}^{[N]}\widetilde{\mathsf{D}}_\infty)^{-1}. \quad (20)$$

This implies

$$\widetilde{\mathsf{D}}_\infty = (I - \mathsf{D}^{[N]}D_{yu}^{[N]})^{-1}\mathsf{D}^{[N]}.$$

Since $D_{yu}^{[N]}$ is lower block-triangular, these two equalities prove that $\mathsf{D}^{[N]} \in \mathcal{D}^{[N]}$ holds if and only if $\widetilde{\mathsf{D}}_\infty \in \mathcal{D}^{[N]}$. The necessity of the condition $\widetilde{\mathsf{D}}_2 \in \mathcal{D}^{[N]}$ in (19) follows similarly.

From Lemmas 2 and 3, we propose the following SDP for the multiobjective H_2/H_∞ LPTV controller synthesis:

$$\begin{array}{rl} \gamma_2^{[N]} := & \inf_{V_\infty, V_2} \gamma_2 \quad \text{subject to } (18), (19) \text{ and} \\ & G_\infty = G_2,\ \mathsf{J}_\infty = \mathsf{J}_2,\ \widetilde{\mathsf{D}}_\infty = \widetilde{\mathsf{D}}_2. \\ V_j := & \left\{X_j, G_j, \mathsf{J}_j, \widetilde{\mathsf{D}}_j\right\}\ (j = \infty, 2). \end{array} \quad (21)$$

Note that the latter three equality constraints in (21) are indispensable to ensure a single controller that satisfies both the H_∞ and H_2 performance specifications. The essential restriction is $G_\infty = G_2$, and due to this constraint, the controller synthesis based on the SDP (21) is conservative in general. However, we can systematically reduce the conservatism by simply increasing the period of the controller. This is a striking feature of the proposed LPTV synthesis method. The result can be stated more rigorously as follows.

Theorem 1: For the SDP (21), we have $\gamma_2^{[N_2]} \le \gamma_2^{[N_1]}$ if N_2 is a multiple of N_1.

Proof: For ease of description, let us denote the coefficient matrices of the closed-loop system $T(\widehat{P}^{[N]}, K^{[N]})_{\widehat{w}_j\widehat{z}_j}$ by $M_{\mathrm{cl},j,[N]}$ $(j = \infty, 2)$. Then, for given γ_∞ and γ_2, we see from Lemma 1 that if

$$\Phi_\infty(M_{\mathrm{cl},\infty,[N_1]}, X_\infty, X_\infty, G, \gamma_\infty) \prec 0,$$
$$\Phi_2(M_{\mathrm{cl},2,[N_1]}, X_2, X_2, G, W, \gamma_2) \prec 0$$

hold, then

$$\Phi_\infty(\widehat{M}^{[q]}_{\mathrm{cl},\infty,[N_1]}, X_\infty, X_\infty, G, \gamma_\infty) \prec 0,$$
$$\Phi_2(\widehat{M}^{[q]}_{\mathrm{cl},2,[N_1]}, X_2, X_2, G, W', \sqrt{q\gamma_2^2}) \prec 0$$

hold for some $W' \succ 0$, where $q := N_2/N_1$. This fact clearly shows that if there exists an N_1-periodic controller $K^{[N_1]}$ that ensures both performance levels γ_∞ and γ_2 via common G in the LMIs (18) and (19) for $\widehat{P}^{[N_1]}$, then the N_2 periodic controller $(\widehat{K^{[N_1]}})^{[q]}$ ensures both the performance levels γ_∞ and γ_2 via exactly the same common G in the LMIs (18) and (19) for $\widehat{P}^{[N_2]}$. This completes the proof. Q.E.D.

If we let $N = 1$, we can confirm that the SDP (21) reduces to the extended-LMI-based multiobjective LTI controller synthesis proposed in [4]. It follows from Theorem 1 that we can always design better (no worse) LPTV controllers by means of (21).

B. Effectiveness of the Lifting-Based Treatment

In the preceding subsection, we have shown that we can design better (no worse) LPTV controllers by applying discrete-time system lifting to the plant and increasing the period N. In this subsection, we discuss possible reasons why we can achieve such conservatism reduction, especially from the view point of "relaxing the restriction to common G." To this end, we focus on the LPTV controller synthesis of the form (2) by directly working on the LMIs (10) and (11) without lifting. This treatment also forms an important basis for the discussion in the next section, where we analyze lower bounds of the H_2 performance that is achievable via LPTV controllers of any order and period under the H_∞ performance constraint.

For the LTI plant P and the LPTV controller K_N given respectively in (1) and (2), let us consider the closed-loop system $T(P, K_N)_{w_j z_j}$ $(j = \infty, 2)$. We denote the coefficient matrices of this closed-loop system by $M_{\mathrm{cl},j,k}$ $(j = \infty, 2,\ k = 0, \cdots, N-1)$. Then, from Propositions 1 and 2, we see that this LPTV controller K_N renders the closed-loop system stable and achieves both the H_∞ performance level γ_∞ and the H_2 performance level γ_2 if and only if the matrix inequalities

$$\Phi_\infty(M_{\mathrm{cl},\infty,k}, X_{\infty,k}, X_{\infty,k+1}, G_{\infty,k}, \gamma_\infty) \prec 0, \quad (k = 0, \cdots, N-1), \tag{22a}$$

$$\Phi_2(M_{\mathrm{cl},2,k}, X_{2,k}, X_{2,k+1}, G_{2,k}, W_k, \gamma_{2,k}) \prec 0, \quad (k = 0, \cdots, N-1), \tag{22b}$$

as well as (12) hold. The inequalities in (22) are non-convex with respect to the controller variables and other variables. However, we can linearize them and obtain LMI conditions for the synthesis of K_N of the form (2) provided that it is full-order, i.e., $p = n$. These LMIs are written in the following form:

$$\widetilde{\Phi}^{\mathrm{syn}}_\infty(M_\infty, X_{\infty,k}, X_{\infty,k+1}, G_{\infty,k}, G_{\infty,k+1}, \gamma_\infty, \mathsf{J}_{\infty,k}, \mathsf{D}_{\infty,k}) \prec 0, \quad (k = 0, \cdots, N-1), \tag{23a}$$

$$\widetilde{\Phi}^{\mathrm{syn}}_2(M_2, X_{2,k}, X_{2,k+1}, G_{2,k}, G_{2,k+1}, W_k, \gamma_{2,k}, \mathsf{J}_{2,k}, \mathsf{D}_{2,k}) \prec 0. \quad (k = 0, \cdots, N-1). \tag{23b}$$

Explicit expressions of these LMIs are given in [11] and omitted here due to limited space. The variables in (23a) are $X_{\infty,k}$ and $X_{\infty,k+1}$, the transformed Lyapunov matrices, $G_{\infty,k}$ and $G_{\infty,k+1}$, the transformed auxiliary variables, $\mathsf{J}_{\infty,k}$, the transformed variable corresponding to $(\mathsf{A}_k, \mathsf{B}_k, \mathsf{C}_k)$ and $\mathsf{D}_{\infty,k}$ that is nothing but D_k. On the other hand, M_∞ denotes the matrix data in (1) that is relevant to the H_∞ norm specification. Similar comments apply also to (23b).

The LMIs in (23) can be regarded as periodic counterparts of the extended LMI conditions for the LTI controller synthesis shown in [4]. Important properties of the LMIs in (23) clarified in [11] are summarized as follows:

Remark 3: (i) In the case of LTI controller synthesis (i.e., $\mathsf{A}_k = \mathsf{A}$ $(k = 0, \cdots, N-1)$, etc.) in (2), the N LMIs in (23a) reduce to an identical LMI and this coincides with the extended LMI condition for the H_∞ LTI controller synthesis [4]. Similarly for (23b).

(ii) The linearizing transformation for the k-th LMI in (22a) depends not only on $G_{\infty,k}$ but also on $G_{\infty,k+1}$ where $G_{\infty,N} = G_{\infty,0}$. This fact validates the description of (23a). Due to this reason, the controller parametrization for $(\mathsf{A}_k, \mathsf{B}_k, \mathsf{C}_k)$ is given as $(\mathsf{A}_k, \mathsf{B}_k, \mathsf{C}_k) = \widetilde{F}(G_{\infty,k}, G_{\infty,k+1}, \mathsf{J}_{\infty,k}, \mathsf{D}_{\infty,k})$ where $\widetilde{F}(\cdot)$ is an appropriately defined nonlinear map. Similarly, the controller parametrization for (23b) is given by $(\mathsf{A}_k, \mathsf{B}_k, \mathsf{C}_k) = \widetilde{F}(G_{2,k}, G_{2,k+1}, \mathsf{J}_{2,k}, \mathsf{D}_{2,k})$.

(iii) If we sum up the N LMIs in (23a) and divide it by N, the resulting condition is essentially equivalent to the extended LMI condition for the H_∞ LTI controller synthesis described by

$$\Phi^{\mathrm{syn}}_\infty(M_\infty, X_\infty, X_\infty, G_\infty, \gamma_\infty, \mathsf{J}_\infty, \mathsf{D}_\infty) \prec 0 \tag{24}$$

where

$$X_\infty = \frac{1}{N}\sum_{k=0}^{N-1} X_{\infty,k}, \quad G_\infty = \frac{1}{N}\sum_{k=0}^{N-1} G_{\infty,k}, \quad \mathsf{J}_\infty = \frac{1}{N}\sum_{k=0}^{N-1} \mathsf{J}_{\infty,k}, \quad \mathsf{D}_\infty = \frac{1}{N}\sum_{k=0}^{N-1} \mathsf{D}_{\infty,k}. \tag{25}$$

This verifies the well-known fact that, for a given LTI plant, we cannot improve the H_∞ performance of the closed-loop system even if we resort to LPTV controllers. Exactly the same comments apply also to the H_2 case (23b).

Under these preparations, let us consider designing multiobjective H_2/H_∞ LPTV controllers by means of (23). In view of the property (ii) in Remark 3, it is inevitable to enforce

$$G_{\infty,k} = G_{2,k},\ \mathsf{J}_{\infty,k} = \mathsf{J}_{2,k},\ \mathsf{D}_{\infty,k} = \mathsf{D}_{2,k} \quad (k = 0, \cdots, N-1) \tag{26}$$

in order to generate a single controller that satisfies both the H_∞ and the H_2 specifications. Therefore we are naturally led to the following SDP:

$$\inf_{V_{\infty,k}, V_{2,k}} \gamma_2 \quad \text{subject to } (18), (19) \text{ and } (26), \quad V_{j,k} := \{X_{j,k}, G_{j,k}, \mathsf{J}_{j,k}, \mathsf{D}_{j,k}\} \quad (k = 0, \cdots, N-1,\ j = \infty, 2). \tag{27}$$

If take the property (iii) into account, however, it is obvious that the above SDP is essentially equivalent to the extended-LMI-based LTI controller synthesis with "common G" (i.e., the case of $N=1$ in the SDP (21)). This fact implies that, if we directly working on the LMIs (10) and (11) without lifting, it is hard to design multiobjective LPTV controllers better than the LTI controllers designed from [4].

Another interpretation of the above result is that the LPTV full-order controllers we can seek for via (27) is such that they ensure the performance levels γ_∞ and γ_2 of the closed-loop system via common auxiliary variables $G_{\infty,k} = G_{2,k}$ $(k = 0,\cdots,N-1)$ in the LMIs (10) and (11). On the other hand, we can easily confirm from Lemma 1 that if there exists a full-order LPTV controller that ensures the performance levels γ_∞ and γ_2 of the closed-loop system via common auxiliary variables $G_{\infty,0} = G_{2,0}$ in the LMIs (10) and (11), then we can always obtain an LPTV controller $K^{[N]}$ that satisfies exactly the same performance levels by solving the SDP (21). Namely, due to the lifting, the restriction $G_{\infty,k} = G_{2,k}$ $(k = 0,\cdots,N-1)$ is relaxed to $G_{\infty,0} = G_{2,0}$ only, and we believe that this is the reason why we can reduce conservatism by increasing the period N in our lifting-based treatment.

V. Analysis of Achievable Performance via LPTV Controllers

By solving the SDP (21) and increasing the period N, we can reduce the conservatism of the design and improve the control performance gradually. To evaluate the performance of the designed controllers quantitatively, it is of prime importance to compute a lower bound of the achievable H_2 performance under the H_∞ performance constraint. In this section, we propose an SDP that enables us to compute such a lower bound efficiently and clarify its several promising properties. More precisely, the goal of this section is to establish the following theorem.

Theorem 2: For a fixed period N, let us consider the following SDP:

$$\underline{\gamma}_2^{[N]} := \inf_{V_\infty, V_2} \gamma_2 \quad \text{subject to } (18), (19) \text{ and } \widetilde{\mathsf{D}}_\infty = \widetilde{\mathsf{D}}_2 \in \mathcal{D}_{\mathrm{Toeplitz}}^{[N]}. \tag{28}$$

Then, for any N, the quantity $\underline{\gamma}_2^{[N]}$ gives a lower bound of the H_2 performance that is achievable via LPTV controllers of *any order and period*, under the H_∞ performance constraint. Moreover,

$$\underline{\gamma}_2^{[N_1]} \le \underline{\gamma}_2^{[N_2]} \tag{29}$$

holds if N_2 is a multiple of N_1.

The SDP (28) can be used in conjunction with (21). If the computed values $\gamma_2^{[N]}$ and $\underline{\gamma}_2^{[N]}$ come close, we can conclude that designed $K^{[N]}$ achieves almost the best performance among the LPTV controllers of any order and period, and therefore we have no need to increase the period N further.

For the proof of Theorem 2, we first establish the following two lemmas. The proofs of these lemmas are given in Subsections B and C in the appendix section, respectively. For ease of notation, in the following, we denote by $\gamma_2^{[N]\mathrm{opt}}$ the best achievable H_2 performance via LPTV controllers of period N and of any order under the H_∞ performance constraint.

Lemma 4: For each fixed N, the quantity $\underline{\gamma}_2^{[N]}$ in (28) satisfies $\underline{\gamma}_2^{[N]} \le \gamma_2^{[N]\mathrm{opt}}$.

Lemma 5: In the SDP (28), the relation (29) holds if N_2 is a multiple of N_1.

Now we are ready to prove Theorem 2.

Proof of Theorem 2: Since (29) is verified via Lemma 5, our remaining task is to prove the first assertion in Theorem 2. For each fixed N, suppose there exists $\widetilde{N}$ $(\neq N)$ such that $\gamma_2^{[\widetilde{N}]\mathrm{opt}} < \underline{\gamma}_2^{[N]}$ for contradiction. Then, from Lemma 5, we have $\gamma_2^{[\widetilde{N}]\mathrm{opt}} < \underline{\gamma}_2^{[N]} \le \underline{\gamma}_2^{[\widetilde{N}N]}$. From Lemma 4, we further obtain

$$\gamma_2^{[\widetilde{N}]\mathrm{opt}} < \underline{\gamma}_2^{[N]} \le \underline{\gamma}_2^{[\widetilde{N}N]} \le \gamma_2^{[\widetilde{N}N]\mathrm{opt}}.$$

This contradicts to the obvious fact that $\gamma_2^{[\widetilde{N}N]\mathrm{opt}} \le \gamma_2^{[\widetilde{N}]\mathrm{opt}}$ does hold. This completes the proof. Q.E.D.

VI. Numerical Example

Let us consider the plant (1) where the coefficient matrices are given by

$$\begin{bmatrix} A & B_{w_2} & B_{w_\infty} & B_u \\ C_{z_2} & D_{z_2 w_2} & 0 & D_{z_2 u} \\ C_{z_\infty} & 0 & D_{z_\infty w_\infty} & D_{z_\infty u} \\ C_y & D_{y w_2} & D_{y w_\infty} & D_{yu} \end{bmatrix} = \left[\begin{array}{ccc:c:c:c} 0.5 & 1 & 1.5 & 1 & 0 & 0 \\ -1 & 1 & 2.1 & 0 & 1 & 0 \\ 1 & -1 & -0.6 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

By setting $\gamma_\infty = 24$, we solved the SDPs (21) and (28) and obtained the results shown in Table I. In this table, the values $\gamma_2^{[N]\star}$ and $\gamma_\infty^{[N]\star}$ denote the genuine H_2 and H_∞ norms of the corresponding closed-loop systems, respectively. When $N=20$, we see that $\gamma_2^{[N]\star}$ and $\underline{\gamma}_2^{[N]}$ are very close. Therefore we can conclude that the designed LPTV controller achieves almost the best performance achievable via LPTV controllers of any order and period.

VII. Conclusion

In this paper, we studied multiobjective H_2/H_∞ controller synthesis problem for discrete-time LTI systems. In stark contrast with the most existing studies that are confined to LTI controller synthesis, we proposed an SDP formulation for LPTV controller synthesis so that better performance can be achieved. Moreover, we also proposed an SDP to compute a lower bound of the multiobjective performance

TABLE I
Computed Upper and Lower Bounds

N	1	2	5	10	20
$\gamma_2^{[N]}$	19.353	16.035	14.372	13.405	12.944
$\gamma_2^{[N]\star}$	15.441	13.811	13.185	12.835	12.651
$\underline{\gamma}_2^{[N]}$	8.233	8.317	11.189	11.928	12.240
$\gamma_\infty^{[N]\star}$	23.933	23.983	24.000	24.000	24.000

achievable via LPTV controllers of any period and order. We illustrated through a numerical example that the proposed LPTV controller synthesis methodology is indeed effective, and by means of the computed lower bounds, we drew a definite result on the quality of the designed LPTV controller.

APPENDIX

A. Proof of (ii) of Lemma 1

Proof of (ii) of Lemma 1: From (11b) for $k = 1$ and (11c) for $k = 0$, we can readily obtain

$$\begin{bmatrix} -W_1 & C_1 G_{2,1} & 0 \\ * & -\mathrm{He}\{G_{2,1}\} & A_0 G_{2,0} \\ * & * & X_{2,0} - \mathrm{He}\{G_{2,0}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} D_1 & 0 \\ 0 & B_0 \\ 0 & 0 \end{bmatrix} \right\} \prec 0.$$

Multiplying the above inequality by

$$\begin{bmatrix} I & C_1 & 0 \\ 0 & 0 & I \end{bmatrix}$$

from left and its transpose from right, we obtain

$$\begin{bmatrix} -W_1 & C_1 A_0 G_{2,0} \\ * & X_{2,0} - \mathrm{He}\{G_{2,0}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} C_1 B_0 & D_1 \\ 0 & 0 \end{bmatrix} \right\} \prec 0. \quad (30)$$

Since

$$X_{2,0} - \mathrm{He}\{G_{2,0}\} \prec 0, \quad (31)$$

the inequality (30) and (11b) for $k = 0$ can be rewritten, respectively, as

$$-W_1 + C_1 B_0 B_0^T C_1^T + D_1 D_1^T \\ -C_1 A_0 G_{2,0} (X_{2,0} - \mathrm{He}\{G_{2,0}\})^{-1} G_{2,0}^T A_0^T C_1^T \prec 0,$$

$$-W_0 + D_0 D_0^T - C_0 G_{2,0} (X_{2,0} - \mathrm{He}\{G_{2,0}\})^{-1} G_{2,0}^T C_0^T \prec 0.$$

These two inequalities imply

$$\begin{bmatrix} -W_0 & -W_{01} \\ * & -W_1 \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} D_0 & 0 \\ C_1 B_0 & D_1 \end{bmatrix} \right\} \\ - \begin{bmatrix} C_0 G_{2,0} \\ C_1 A_0 G_{2,0} \end{bmatrix} (X_{2,0} - \mathrm{He}\{G_{2,0}\})^{-1} \begin{bmatrix} C_0 G_{2,0} \\ C_1 A_0 G_{2,0} \end{bmatrix}^T \prec 0 \quad (32)$$

where

$$W_{01} := D_0 (C_1 B_0)^T \\ -C_0 G_{2,0} (X_{2,0} - \mathrm{He}\{G_{2,0}\})^{-1} (C_1 A_0 G_{2,0})^T.$$

From (31) and (32), we obtain

$$\left[\begin{array}{cc:c} -W_0 & -W_{01} & C_0 G_{2,0} \\ * & -W_1 & C_1 A_0 G_{2,0} \\ \hdashline * & * & X_{2,0} - \mathrm{He}\{G_{2,0}\} \end{array}\right] + \mathrm{Sq}\left\{ \left[\begin{array}{cc} D_0 & 0 \\ C_1 B_0 & D_1 \\ \hdashline 0 & 0 \end{array}\right] \right\} \prec 0. \quad (33)$$

If we define

$$W := \begin{bmatrix} W_0 & W_{01} \\ * & W_1 \end{bmatrix}, \quad (34)$$

it is obvious that

$$\mathrm{trace}(W) - \gamma_{2,0}^2 - \gamma_{2,1}^2 < 0. \quad (35)$$

We next consider (11c) for $k = 0$ and $k = 1$. Then, we can readily obtain

$$\begin{bmatrix} -X_{2,2} & A_1 G_{2,1} & 0 \\ * & -\mathrm{He}\{G_{2,1}\} & A_0 G_{2,0} \\ * & * & X_{2,0} - \mathrm{He}\{G_{2,0}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} B_1 & 0 \\ 0 & B_0 \\ 0 & 0 \end{bmatrix} \right\} \prec 0.$$

Multiplying the above inequality by

$$\begin{bmatrix} I & A_1 & 0 \\ 0 & 0 & I \end{bmatrix}$$

from left and its transpose from right, we are led to

$$\begin{bmatrix} -X_{2,2} & A_1 A_0 G_{2,0} \\ 0 & X_{2,0} - \mathrm{He}\{G_{2,0}\} \end{bmatrix} + \mathrm{Sq}\left\{ \begin{bmatrix} A_1 B_0 & B_1 \\ 0 & 0 \end{bmatrix} \right\} \prec 0. \quad (36)$$

The inequalities (35), (33), (36) together with Fact 1 clearly show that

$$\Phi_2(\widehat{M}^{[2]}, X_{2,0}, X_{2,2}, G_{2,0}, W, \sqrt{\gamma_{2,0}^2 + \gamma_{2,1}^2}) \prec 0.$$

By applying this procedure recursively and noting Fact 1, we are led to the desired conclusion (14). Q.E.D.

B. Proof of (ii) of Lemma 4

Proof of Lemma 4: For the proof, it suffices to show that the following fact holds:

(A) Suppose there exists an N-periodic controller K_N of the form (2) and of arbitrary order p that achieves both performance levels γ_∞ and γ_2. Then, the LMIs (18) and (19) is feasible for the same performance levels γ_∞ and γ_2 with some $\widetilde{\mathsf{D}}_\infty = \widetilde{\mathsf{D}}_2 \in \mathcal{D}_{\mathrm{Toeplitz}}^{[N]}$.

If $p < n$ in (2), we can add stable uncontrollable and unobservable modes to the controller so that the order of the controller can be regarded as n. Therefore we can assume $p \geq n$ without loss of generality.

From the underlying assumption in (A), we have $\|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\| < \gamma_\infty$. As noted, the order p of the controller $\widehat{K}_N^{[N]}$ is larger than (or equal to) n in general. However, the LMI result for LTI controller synthesis in [8] ensures that if $\|T(\widehat{P}^{[N]}, \widehat{K}_N^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\| < \gamma_\infty$ holds, then there exists $K^{[N]}$ that satisfies $\|T(\widehat{P}^{[N]}, K^{[N]})_{\widehat{z}_\infty \widehat{w}_\infty}\| < \gamma_\infty$ where

- the order of $K^{[N]}$ is n,
- $K^{[N]}$ shares identical direct feedthrough term with $\widehat{K}_N^{[N]}$, i.e., the direct feedthrough term of $K^{[N]}$ is $\widehat{\mathsf{D}}_{\langle 0 \rangle}^{[N]}$ given explicitly by

$$\widehat{\mathsf{D}}_{\langle 0 \rangle}^{[N]} = \begin{bmatrix} \mathsf{D}_0 & 0 & \cdots & 0 \\ \mathsf{C}_1 \mathsf{B}_0 & \mathsf{D}_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathsf{C}_{N-1} \left(\prod_{k=1}^{N-2} \mathsf{A}_k \right) \mathsf{B}_0 & \cdots & \mathsf{C}_{N-1} \mathsf{B}_{N-2} & \mathsf{D}_{N-1} \end{bmatrix}.$$

These facts clearly show that

$$\Phi_\infty^{\mathrm{syn}}(\widehat{M}_{\infty,0}^{[N]}, X_{\infty,\langle 0 \rangle}, X_{\infty,\langle 0 \rangle}, \\ G_{\infty,\langle 0 \rangle}, \gamma_\infty, \mathsf{J}_{\infty,\langle 0 \rangle}, \widetilde{\mathsf{D}}_{\langle 0 \rangle}) \prec 0 \quad (37)$$

holds for some $X_{\infty,\langle 0 \rangle}$, $G_{\infty,\langle 0 \rangle}$, $\mathsf{J}_{\infty,\langle 0 \rangle}$ and

$$\widetilde{\mathsf{D}}_{\langle 0 \rangle} := (I - \widehat{\mathsf{D}}_{\langle 0 \rangle}^{[N]} D_{yu}^{[N]})^{-1} \widehat{\mathsf{D}}_{\langle 0 \rangle}^{[N]} \in \mathcal{D}^{[N]}. \quad (38)$$

Similar observations for the H_2 case also ensure the existence of $X_{2,\langle 0 \rangle}$, $G_{2,\langle 0 \rangle}$, $\mathsf{J}_{2,\langle 0 \rangle}$ and $W_{\langle 0 \rangle}$ such that

$$\Phi_2^{\mathrm{syn}}(\widehat{M}_{2,0}^{[N]}, X_{2,\langle 0 \rangle}, X_{2,\langle 0 \rangle}, \\ G_{2,\langle 0 \rangle}, W_{\langle 0 \rangle}, \gamma_2, \mathsf{J}_{2,\langle 0 \rangle}, \widetilde{\mathsf{D}}_{\langle 0 \rangle}) \prec 0. \quad (39)$$

To arrive at the desired conclusion, we next introduce the notion of the timing of the lifting. When deriving (37) and (39), we implicitly assume that we apply the discrete-time system lifting by stacking the signals y and u as follows:

$$\widehat{y}_{\langle 0\rangle,k} = \begin{bmatrix} y_{kN} \\ \vdots \\ y_{(k+1)N-1} \end{bmatrix}, \quad \widehat{u}_{\langle 0\rangle,k} = \begin{bmatrix} u_{kN} \\ \vdots \\ u_{(k+1)N-1} \end{bmatrix}.$$

However, since the plant is LTI, exactly the same results as (37) and (39) will follow if we stack the signals y and u as

$$\widehat{y}_{\langle s\rangle,k} = \begin{bmatrix} y_{kN+s} \\ \vdots \\ y_{(k+1)N+s-1} \end{bmatrix}, \quad \widehat{u}_{\langle s\rangle,k} = \begin{bmatrix} u_{kN+s} \\ \vdots \\ u_{(k+1)N+s-1} \end{bmatrix}.$$

Here, we call the parameter s $(s = 0, \cdots, N-1)$ the timing of lifting. It follows that

$$\begin{aligned} &\Phi_\infty^{\rm syn}(\widehat{M}_{\infty,0}^{[N]}, X_{\infty,\langle s\rangle}, X_{\infty,\langle s\rangle}, \\ &\qquad G_{\infty,\langle s\rangle}, \gamma_\infty, \mathsf{J}_{\infty,\langle s\rangle}, \widetilde{\mathsf{D}}_{\langle s\rangle}) \prec 0, \\ &\Phi_2^{\rm syn}(\widehat{M}_{2,0}^{[N]}, X_{2,\langle s\rangle}, X_{2,\langle s\rangle}, \\ &\qquad G_{2,\langle s\rangle}, W_{\langle s\rangle}, \gamma_2, \mathsf{J}_{2,\langle s\rangle}, \widetilde{\mathsf{D}}_{\langle s\rangle}) \prec 0, \\ &\qquad\qquad (s = 0, \cdots, N-1) \end{aligned} \tag{40}$$

hold for some $X_{j,\langle s\rangle}$, $G_{j,\langle s\rangle}$, $\mathsf{J}_{j,\langle s\rangle}$, $W_{\langle s\rangle}$ $(j = \infty, 2,\ s = 0, \cdots, N-1)$ and

$$\widetilde{\mathsf{D}}_{\langle s\rangle} := (I - \widehat{\mathsf{D}}_{\langle s\rangle}^{[N]} D_{yu}^{[N]})^{-1} \widehat{\mathsf{D}}_{\langle s\rangle}^{[N]} \in \mathcal{D}^{[N]}, \tag{41}$$

$$\widehat{\mathsf{D}}_{\langle s\rangle}^{[N]} = \begin{bmatrix} \mathsf{D}_s & 0 & \cdots & 0 \\ \mathsf{C}_{s+1}\mathsf{B}_s & \mathsf{D}_{s+1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \mathsf{C}_{s+N-1}\left(\prod_{k=s+1}^{s+N-2} \mathsf{A}_k\right)\mathsf{B}_s & \cdots & \mathsf{C}_{s+N-1}\mathsf{B}_{s+N-2} & \mathsf{D}_{s+N-1} \end{bmatrix}.$$

By summing up the N LMIs for the H_∞ and the H_2 specifications in (40) and dividing them by N, we can ensure that the assertion (A) holds since

$$\frac{1}{N}\sum_{s=0}^{N-1} \widetilde{\mathsf{D}}_{\langle s\rangle} \in \mathcal{D}_{\rm Toeplitz}^{[N]}.$$

This completes the proof. Q.E.D.

C. Proof of (ii) of Lemma 5

Proof of Lemma 5: Let $q := N_2/N_1$. For the proof, it suffices to show that the following fact holds:
(B) For given $\gamma_\infty > 0$ and $\gamma_2 > 0$, suppose the LMIs (18) and (19) for $N = N_2$ hold for some $\widetilde{\mathsf{D}}_\infty = \widetilde{\mathsf{D}}_2 = \widetilde{\mathsf{D}}^{[N_2]} \in \mathcal{D}_{\rm Toeplitz}^{[N_2]}$. Let us partition $\widetilde{\mathsf{D}}^{[N_2]}$ as

$$\widetilde{\mathsf{D}}^{[N_2]} = \begin{bmatrix} \widetilde{\mathsf{D}}_0 & 0 & \cdots & 0 \\ \widetilde{\mathsf{D}}_1 & \widetilde{\mathsf{D}}_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ \widetilde{\mathsf{D}}_{q-1} & \cdots & \widetilde{\mathsf{D}}_1 & \widetilde{\mathsf{D}}_0 \end{bmatrix}, \tag{42}$$
$$\widetilde{\mathsf{D}}_\kappa \in \mathbf{R}^{N_1 m \times N_1 l}\ (\kappa = 0, \cdots, q-1).$$

Then, the LMIs (18) and (19) for $N = N_1$ hold for $\widetilde{\mathsf{D}}_\infty = \widetilde{\mathsf{D}}_2 = \widetilde{\mathsf{D}}_0 \in \mathcal{D}_{\rm Toeplitz}^{[N_1]}$.

We first note that an LPTV controller of period $N_2(= qN_1)$ for the LTI plant P can be regarded as q-periodic for the fictitious LTI plant $\widehat{P}^{[N_1]}$. It follows from the underlying assumption in (B) that for the fictitious LTI plant $\widehat{P}^{[N_1]}$, there exists q-periodic stabilizing controllers $K_{\infty,q}$ and $K_{2,q}$ that achieve the performance levels γ_∞ and γ_2, respectively. Moreover, they share a identical direct feedthrough term. Let us represent these controllers in the standard form (2) by means of the formula (9) and denote the resulting state space matrices by $(\mathsf{A}_{j,\kappa}, \mathsf{B}_{j,\kappa}, \mathsf{C}_{j,\kappa}, \mathsf{D}_{j,\kappa})$ $(j = \infty, 2, \kappa = 0, \cdots, q-1)$. Then, from (42), (20) and (9), we can let

$$\mathsf{D}_{j,\kappa} = \widetilde{\mathsf{D}}_0(I + D_{yu}^{[N_1]}\widetilde{\mathsf{D}}_0)^{-1}\ (j = \infty, 2, \kappa = 0, \cdots, q-1). \tag{43}$$

On the other hand, since $\mathsf{A}_{j,\kappa} \in \mathbf{R}^{n+(q-1)N_1 l_j}$ $(j = \infty, 0)$, the orders of $K_{\infty,q}$ and $K_{2,q}$ are larger than n in general. However, similarly to the proof of Lemma 4, the existence of such $K_{\infty,q}$ and $K_{2,q}$ ensures the existence of q-periodic full-order controllers $K_{q,\infty}^{\rm FO}$ and $K_{2,\infty}^{\rm FO}$ where the former achieves the H_∞ performance level γ_∞ while the latter achieves the H_2 performance level γ_2. Moreover, these two controllers share the same feedthrough terms as (43). If we focus on the LMI (23), this fact ensures that the following LMIs hold:

$$\begin{aligned} &\widetilde{\Phi}_\infty^{\rm syn}(\widehat{M}_{\infty,0}^{[N_1]}, X_{\infty,\kappa}, X_{\infty,\kappa+1}, \\ &\qquad G_{\infty,k}, G_{\infty,k+1}, \gamma_\infty, \mathsf{J}_{\infty,k}, \widetilde{\mathsf{D}}_0) \prec 0 \\ &\qquad\qquad (\kappa = 0, \cdots, q-1), \end{aligned} \tag{44a}$$

$$\begin{aligned} &\widetilde{\Phi}_2^{\rm syn}(\widehat{M}_{2,0}^{[N_1]}, X_{2,\kappa}, X_{2,\kappa+1}, \\ &\qquad G_{2,\kappa}, G_{2,\kappa+1}, W_\kappa, \gamma_{2,\kappa}, \mathsf{J}_{2,\kappa}, \widetilde{\mathsf{D}}_0) \prec 0 \\ &\qquad\qquad (\kappa = 0, \cdots, q-1). \end{aligned} \tag{44b}$$

By summing up the q LMIs (44a) for the H_∞ performance and (44b) for the H_2 performance and dividing them by q, we can easily confirm from (iii) in Remark 3 that the assertion (B) holds. This completes the proof. Q.E.D.

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