Robust stability analysis of Inverse LQ regulator for linear systems with input delay

Takao Fujii and Osamu Kaneko

Abstract — We analyze the robust stability of Inverse LQ regulators for single-input linear systems with uncertain input delay. Unlike the usual LQ regulator, the Inverse LQ regulator has a gain tuning parameter that can be chosen freely to some extent without losing its LQ optimality. Utilizing this freedom we seek the range of tuning parameter that ensures the robust stability against the uncertain delay time, as well as the robust stability condition of the Inverse LQ regulator for some gain tuning parameter. The result is based on the quadratic stabilization problem.

I. INTRODUCTION

The well known good stability robustness properties of LQ regulator are no longer guaranteed for systems with time delay such as input delay [1]. The same is true with the Inverse LQ (abbreviated as ILQ) regulator proposed by the first author [2]. The robustness of LQ regulators for linear systems with input delay have been investigated in [3]. Similar researches have been conducted also for ILQ regulators of linear systems with delays in the "state", for example in [4]. In view of this, we analyze in this paper the robust stability of ILQ regulator for a single-input linear system with input delay against the uncertainty of delay time. In doing so, we focus on the tuning parameter of ILQ regulator, and search for its allowable range to ensure robust stability against a given uncertainty of delay time.

II. PROBLEM FOMLATIONS

Consider the single-input linear time-invariant input delayed system with a delay time L :

$$\boldsymbol{S} : \dot{x} = Ax + Bu(t-L) \tag{1}$$

It is known [5] that the design of LQ regulator for this system is essentially reduced to that for the fictitious system:

$$\boldsymbol{S_p} : \dot{x}_p = Ax_p + \tilde{B}u, \quad \tilde{B} := e^{-AL}B \tag{2}$$

The resulting LQ regulator for $\boldsymbol{S}$ is configured as in Fig. 1, using the gain K in the LQ regulator $u = -Kx_p$ for $\boldsymbol{S_p}$.

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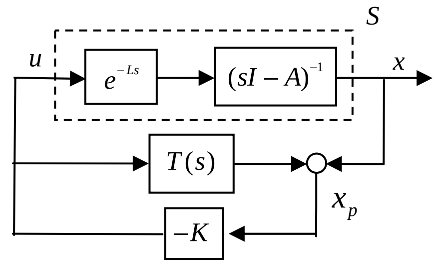

Fig.1 Configuration of LQ regulator for S

where

$$T(s) = (sI-A)^{-1}\tilde{B} - (sI-A)^{-1}Be^{-Ls} \tag{3}$$

In this paper, the LQ gain K in Fig.1 is replaced by the ILQ gain K for $\boldsymbol{S_p}$ given in the form:

$$K = \Sigma F,\ K\tilde{B} = \Sigma > 0 \tag{4}$$

Here F is a fundamental feedback gain designed by pole assignment for the system $\boldsymbol{S_p}$,while Σ is a gain tuning parameter to be chosen any value more than a certain value so as to ensure the LQ optimality of the ILQ regulator. In addition we consider the situation where the single time delay element e^{-Ls} in Fig.1 has a multiplicative uncertainty [6]:

$$e^{-\tilde{L}s} = e^{-Ls}(1+\Delta W(s)),\ \|\Delta\|_\infty \le 1 \tag{5a}$$

where

$$\tilde{L} = L(1+\delta),\ -0.1 \le \delta \le 0.1,\ W(s) = \frac{2.1\times 0.1Ls}{0.1Ls+1} \tag{5b}$$

Replacing e^{-Ls} in Fig.1 with $e^{-\tilde{L}s}$ given by (5) and transforming the block diagram yields the following equivalent configuration with an uncertain element $\tilde{\Delta} = \Delta e^{-Ls}$ of magnitude less than one.

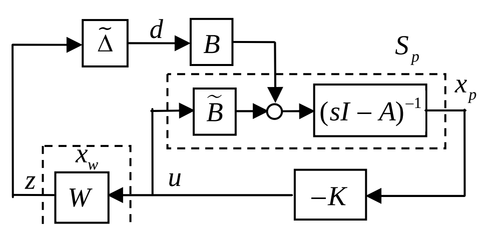

Fig.2 ILQ regulator for $\boldsymbol{S}$ with uncertain delay time

With this setting we analyze the robust stability of ILQ regu-

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lator for a linear input delayed system $\boldsymbol{S}$ against the uncertainty of delay time. Specifically, we solve the following two problems:

Problem 1: Obtain a necessary and sufficient condition for robust stability of the ILQ regulator shown in Fig.2 under the transmission uncertainty $\tilde{\Delta}$ from z to d.

Problem 2: Obtain a range of tuning parameter Σ such that the ILQ regulator is robustly stable under the same uncertainty.

III. QUADRATIC STABILIZATION

In this section we reduce the robust stability problem stated above to a quadratic stabilization problem by the small gain theorem. Let x_w be the state of $W(s)$ with the following realization:

$$W(s)=c_w(s-a_w)^{-1}b_w+d_w,\quad b_w=-a_w=(0.1L)^{-1},\ c_w=-d_w=-2.1 \qquad (6)$$

and $x_e=\begin{bmatrix} x_p^T & x_w^T \end{bmatrix}^T$ be the state of the system $\boldsymbol{S_p}$ augmented with the frequency weight W. Then we have the following state space equation for the augmented system in Fig. 2:

$$\dot{x}_e = A_e x_e + B_e u_w + D_e d \qquad (7a)$$

$$z = C_e x_e + d_w u \qquad (7b)$$

$$A_e := \begin{bmatrix} A & 0 \\ 0 & a_w \end{bmatrix}, B_e := \begin{bmatrix} \tilde{B} \\ b_w \end{bmatrix}, C_e := \begin{bmatrix} 0 & c_w \end{bmatrix}, D_e := \begin{bmatrix} B \\ 0 \end{bmatrix}$$

Combining this with $d=\tilde{\Delta}z$ yields the following uncertain system $\boldsymbol{S_e}$ with uncertainty entering both in the state and in the input matrices.

$$\boldsymbol{S_e}: \dot{x}_e = (A_e + D_e\tilde{\Delta}C_e)x_e + (B_e + D_e\tilde{\Delta}d_w)u \qquad (8)$$

Hence the robust stabilization of $\boldsymbol{S_e}$ by the linear state feedback

$$u=-K_e x_e,\quad K_e=\begin{bmatrix} K & 0 \end{bmatrix} \qquad (9)$$

can be reduced to quadratic stabilization by the linear control (9).

This observation follows from the following fundamental result in quadratic stabilization problem [7] and the small gain theorem.

Lemma 1: The uncertain system $\boldsymbol{S_e}$ with the state feedback (9) is quadratically stable if and only if it is nominally stable and the transfer function from d to z in Fig. 1 has an infinity norm less than one.

The following result plays a key role to solve the problems.

Proposition 1: The uncertain system $\boldsymbol{S_e}$ is quadratically stabilizable via linear control only if the following Riccati inequality has a solution $P_e>0$ for $R=d_w^2$

$$P_e(A_e-B_eR^{-1}d_wC_e)+(A_e-B_eR^{-1}d_wC_e)^TP_e-P_eB_eR^{-1}B_e^TP_e + P_eD_eD_e^TP_e+C_e^T(I-d_wR^{-1}d_w)C_e<0 \qquad (10)$$

Conversely, if (10) has a solution $P_e>0$ for $R\ge d_w^2$, then the uncertain system S_e is quadratically stabilized by a state feedback control given by

$$u_e=-K_ex_e,\quad K_e=R^{-1}(B_e^TP_e+d_wC_e) \qquad (11)$$

Proof: The first half is due to [7] and the latter half is obtained easily by substituting (11) into (10) followed by some rearrangement and then applying the bounded real lemma.

Based on this result we seek a condition under which the state feedback control (9) with K given by (4) is a quadratically stabilizing control defined by (11). A key role for it is played by a similarity transformation given by

$$T_e=\begin{bmatrix} I & \tilde{B} \\ 0 & b_w \end{bmatrix}\begin{bmatrix} I & 0 \\ -F_\alpha & 1 \end{bmatrix}, F_\alpha=\alpha^{-1}F, \alpha=1-a_wd_w^2>0 \qquad (12)$$

Applying this transformation to (11) with setting $R^{-1}=\Sigma$ as well as (10), and using a general expression of P_e satisfying (11) for a given K_e in the inverse LQ design [2, Lemma 2.3] yields:

$$\bar{P}_e=T_e^TP_eT_e=\begin{bmatrix} Y & 0 \\ 0 & \alpha \end{bmatrix}, Y>0$$

as well as other transformed matrices in (10), for example , $\bar{B}_e=T_e^{-1}B_e=\begin{bmatrix} 0 & 1 \end{bmatrix}^T$. As a result, the Riccati inequality (10) can be expressed easily in the form of linear matrix inequality:

$$M(Y)+(1-d_w^2\Sigma)N<0 \qquad (13)$$

where

$$M(Y)=\begin{bmatrix} Y\hat{A}+A^TY & Y\hat{B}+\hat{A}^TF^T & YB \\ \hat{B}^TY+F\hat{A} & 2F\hat{B}-\alpha^2d_w^{-2} & FB \\ B^TY & B^TF^T & -I \end{bmatrix}, \quad \begin{matrix} \hat{A}:=A-\hat{B}F_\alpha \\ \hat{B}:=(A-a_wI)\tilde{B} \end{matrix}$$

$$N=\begin{bmatrix} a_w^2d_w^2F_\alpha{}^TF_\alpha & -a_wF_\alpha{}^T \\ -a_wF_\alpha & d_w^{-2} \end{bmatrix}=\begin{bmatrix} a_wd_wF_\alpha{}^T \\ -d_w^{-1} \end{bmatrix}\begin{bmatrix} a_wd_wF_\alpha{}^T \\ -d_w^{-1} \end{bmatrix}^T \ge 0$$

IV. MAIN RESULTS

Based on the above preliminary results on quadratic stabilization, a solution to Problems 1 and 2 can be derived, respectively from the first half and latter half of Proposition 1.

Theorem 1 (Solution to Problem 1): The ILQ regulator shown in Fig. 2 is robustly stable for some tuning parameter Σ only if the linear matrix inequality $L(Y) < 0$ has a solution $Y > 0$.

Proof: This is obvious since quadratic stabilizability via ILQ control of $\boldsymbol{S_e}$ implies the existence of a solution $P_e > 0$ to (10) for $R = d_w^2$ by Proposition 1, and hence that of a solution $Y > 0$ to (13) for $\Sigma = d_w^{-2}$, namely, $L(Y) < 0$.

Theorem 2 (Solution to Problem 2): Assume that there exists a solution $Y > 0$ to $L(Y) < 0$, or equivalently, (13) has a solution $Y > 0$ for $\Sigma = d_w^{-2}$. Let $\underline{\Sigma}$ be the minimum value of Σ such that there exists a solution $Y > 0$ to (13). Then the ILQ regulator in Fig. 2 is robustly stable for all Σ with $\underline{\Sigma} \le \Sigma \le d_w^{-2}$.

Proof: By Proposition 1 robust stability of the ILQ regulator is guaranteed by the existence of a positive definite solution to (10) and hence that to (13) .Moreover, it follows from the form of (13) that if there exists a solution to (13) for some $\Sigma = \Sigma_1$, then so does for all $\Sigma > \Sigma_1$. Therefore it is enough to seek the minimum value of Σ ensuring the existence of solution to (13).

Finally we state an interesting relation between the quadratic stabilizability of the state feedback (9) for the uncertain augmented system $\boldsymbol{S_e}$ and that of the state feedback control $u = -Kx_p$ for the following uncertain system:

$$\tilde{\boldsymbol{S}}_{\boldsymbol{p}} : \dot{\tilde{x}}_p = A\tilde{x}_p + (\tilde{B} + B\Delta d_w)u \tag{14}$$

Theorem 3: If the state feedback for the fictitious system (2) is a quadratically stabilizing control for the system $\tilde{\boldsymbol{S}}_{\boldsymbol{p}}$, then the corresponding state feedback control (9) is also a quadratically stabilizing control for the augmented system $\boldsymbol{S_e}$.

Proof: By quadratic stability theorem the quadratic stabilizability of $\tilde{\boldsymbol{S}}_{\boldsymbol{p}}$ (14) by $u = -K\tilde{x}_p$ implies the existence of $P > 0$ such that

$$PA + A^T P - P\tilde{B}R^{-1}\tilde{B}^T P + PBB^T P < 0$$

We can then show that the Riccati inequality (10) has a positive definite solution

$$P_e = diag\{P, -a_w\}$$

which yields the state feedback control (9) as a quadratically stabilizing control for $\boldsymbol{S_e}$ via (11).

V. CONCLUSIONS

The robust stability of Inverse LQ regulator for a single-input system with uncertain input delay was analyzed by reducing it to quadratic stabilization via linear control of a system with uncertain system matrices. We have obtained a robust stability condition on the fundamental gain parameter of ILQ regulator in terms of solvability of LMI, together with an allowable range of the gain tuning parameter ensuring robust stability.

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