Stability Analysis of Discrete-Time Systems with Time-Varying Delays via Integral Quadratic Constraints

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***Abstract*— This manuscript presents certain l_2-gain properties of and the integral quadratic constraint characterizations derived from these properties for the discrete-time time-varying operator. These IQC characterizations are crucial for the IQC analysis to be applied to study robustness of discrete-time systems in the presence of time-varying delays. One new contribution of this manuscript is to utilize the information of the variation of the delay parameter to derive less conservative IQCs. The effectiveness of the proposed IQC analysis is verified by numerical experiments, the results of which are compared with those recently published in the literature.**

I. Introduction

The content of this manuscript concerns robust stability analysis of discrete-time systems with time-varying delays in the following forms:

$$\begin{aligned} x[k+1] &= Ax[k] + A_d x[k-\tau[k]] \\ &\quad + \Delta(x[k], x[k-\tau[k]]) + e[k] \end{aligned} \tag{1}$$

where n-dimensional signal x is the signal of interest, e is a finite-energy disturbance, A and $A_d \in \mathbb{R}^{n\times n}$ are constant matrices, and $\Delta(\cdot,\cdot)$ is a causal and bounded operator on the space of all finite-energy signals (the l_2 space). The delay sequence τ is an unknown function where only the bounds on its value and/or variation are available to us. The robust stability problem in question is to verify whether, under any finite-energy disturbance input e, the energy of signal x remains finite for all admissible delay sequences satisfying the given conditions.

Most existing results in the literature for time-varying delay robustness were developed in time domain based on the Lyapunov stability theorem – in which certain form of Lyapunov-Krasovskii functional candidates are used to derive stability conditions (see, for example, [1], [2], [3], [4], [7], [9], [10], [11], [12]). The form of Lyapunov functions is often tied to the formulation of systems under consideration. As such, it is often non-trivial to generalize the result for other systems with similar but slightly different forms because the generalization involves modification of the form of the Lyapunov function, which might not be easy to come up with.

In contrast to the Lyapunov approach, in [6] and [5] we proposed to tackle time-varying delay robustness problems via a frequency-domain approach called Integral Quadratic Constraint (IQC) analysis. The crucial step of applying the IQC analysis to analyze robust stability of time-varying delay systems is to characterize the time-varying delay operator in terms of integral quadratic constraints. With the IQC characterization, stability conditions can be straightforwardly obtained following the IQC stability theorem [8]. In [5] several IQCs were derived for discrete-time time-varying delay operators. It is found that, in terms of energy amplification, the discrete-time time-varying delay operator has some distinct features compared to its continuous-time counterpart. In particular, for the discrete-time time-varying operator, as long as the delay sequence is upper bounded, the energy amplification ratio (i.e.; the l_2-gain) is always bounded regardless of the variation of the delay sequence. This is distinctly different from its continuous-time counterpart in that the $\mathcal{L}_2$-gain depends only on the variation of the delay parameter and the gain becomes unbounded when the variation of the delay parameter exceeds one. For the discrete-time case, it was unclear whether and how the variation of the delay sequence affects the l_2-gain of the discrete-time time-varying delay operator, and how to effectively utilize the information on the variation of the delay sequence in robustness analysis of discrete-time systems with time-varying delays. In this manuscript, we continue our endeavor in [5]. We will focus on making the link between the variation of the delay sequence and the l_2-gain of the time-varying delay operator, which hopefully will lead to better IQC characterizations of the operator and less conservative criteria for verifying varying-time-delay robustness of discrete-time systems. Numerical experiments will be conducted to verify the proposed stability criteria.

The remaining part of the manuscript evolves along the following line. The next section introduces the main notation and the IQC analysis applied to checking very-time-delay robustness of linear time-invariant discrete-time systems. Section III presents the IQCs for the time-varying delay operator we so far discovered. Stability criteria resulting from these IQC characterizations are presented in Section IV. The results of the numerical experiments conducted to verify the effectiveness of the proposed stability criteria are presented in Section V. Finally, we make a concluding remark summarizing the current state of our work and what to be investigated in the future.

II. Notations and Preliminaries

Symbol I_n is used to denote n-dimensional identity matrix. The subscript n is dropped when the dimension is evident from the text. Given a matrix M, the transposition and the conjugate transposition are denoted by M' and M^*,

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respectively. The notation $M > 0$ ("$\geq$","$<$", and "$\leq$", respectively) is used to denote positive definiteness (positive semi-definiteness, negative definiteness, and negative semi-definiteness, respectively). Symbol l_2^m denotes the space of $\mathbf{R}^m$-valued, square summable functions defined on time interval $(-\infty, \infty)$, and l_{2e}^m denotes the extension of the space l_2^m, which consists of functions whose time truncation lies in l_2^m. Notation $\mathbf{Rl}_\infty^{l\times m}$ is used to denote the space of proper rational transfer matrices (of dimension $l \times m$) with no poles on the unit circle, while $\mathbf{Rh}_\infty^{l\times m}$ denotes the subspace of $\mathbf{Rl}_\infty^{l\times m}$ consisting of functions which have no poles outside the open unit disk. Every $H \in \mathbf{Rl}_\infty^{l\times m}$ defines a convolution operator on l_2: let h be the inverse Laplace transform of H. Then for any $u \in l_2$,

$$(Hu)[k] := \sum_{l=-\infty}^{\infty} h[k-l]u[l].$$

Given a signal f in the l_2 space, we use $\|f\|_{l_2}$ to denote the l_2 norm of f. Given a bounded operator G on the l_2 space, we use $\|G\|_{l_2}$ to denote the l_2 induced norm of G.

Let Π be a bounded LTI self-adjoint operator on l_2 space. Then Π defines a quadratic form on l_2

$$\sigma_\Pi(v,w) := \left\langle \begin{bmatrix} v \\ w \end{bmatrix}, \Pi \begin{bmatrix} v \\ w \end{bmatrix} \right\rangle = \sum_{k=-\infty}^{\infty} \begin{bmatrix} v[k] \\ w[k] \end{bmatrix}' \left(\Pi \begin{bmatrix} v \\ w \end{bmatrix} \right)[k]$$
$$= \int_{-\pi}^{\pi} \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix}^* \Pi(\mathrm{e}^{j\omega}) \begin{bmatrix} \hat{v}(j\omega) \\ \hat{w}(j\omega) \end{bmatrix} d\omega$$

where $\hat{v}$ and $\hat{w}$ are Fourier transforms of v and w, respectively. The operator Π is referred to as the multiplier of the quadratic form σ_Π. The multiplier Π is often block partitioned into the form

$$\begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$$

where the dimensions of Π_{ij} are consistent with those of v and w.

Given an operator $\mathcal{H}$ and a quadratic form $\sigma_\Pi(v,w)$ defined on l_2 space, we said that $\mathcal{H}$ satisfies the integral quadratic constraint defined by σ_Π, or more often "$\mathcal{H}$ satisfies IQC defined by Π" to emphasize the multiplier involved, if $\sigma_\Pi(v, \mathcal{H}(v)) \geq 0$ for all $v \in l_2$.

Let $\mathcal{D}_\tau$ denote the time-delay operator $\mathcal{D}_\tau(v) := v[k - \tau[k]]$, and $\mathcal{S}_\tau$ be the "delay-difference" operator $(I - \mathcal{D}_\tau)$; i.e., $\mathcal{S}_\tau(v) := v[k] - v[k-\tau[k]]$. To simplify the notation, in the rest of the paper we will suppress the time dependency on $\tau[k]$ and simply write it as τ.

In order for the readers who are not familiar with IQC analysis to appreciate the technical contents of the next section, in the following we state a stability theorem obtained by applying the IQC analysis to linear time-invariant (LTI) discrete-time (DT) systems with time-varying delays. Consider LTI DT systems with time-varying delays governed by the following equation:

$$x[k+1] = Ax[k] + A_d(x[k-\tau] + f) \qquad (2)$$

where the time-varying delay sequence τ is upper bounded by $\mathcal{T}$ but otherwise unknown. We assume that $A + A_d$ is stable (i.e., all eigenvalues of $A + A_d$ are strictly inside the unit circle), which is a necessary condition for stability. The system can be equivalently expressed as the feedback interconnection

$$x = Gw + e, \quad w = \mathcal{S}_\tau x \qquad (3)$$

where G is a LTI DT stable system with transfer function representation $G(z) = -(zI - (A + A_d))^{-1}A_d$ and $e = -Gf \in l_2$. We have the following stability theorem for (2), which follows straightforwardly the general IQC theory stated in [8].

Theorem 1. *Consider system (2) and the equivalent transformation (3). Suppose*

(i) $\mathcal{S}_\tau$ *satisfies IQC defined by* $\Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$*;*

(ii) $\Pi_{11} \geq 0$ *and* $\Pi_{22} \leq 0$*;*

(iii) *there exists* $\epsilon > 0$ *such that*

$$\begin{bmatrix} G(\mathrm{e}^{j\omega}) \\ I \end{bmatrix}^* \Pi(\mathrm{e}^{j\omega}) \begin{bmatrix} G(\mathrm{e}^{j\omega}) \\ I \end{bmatrix} \leq -\epsilon I, \ \forall\ |\omega| \leq \pi. \quad (4)$$

Then the feedback interconnection (3) is stable, and so is (2).

Condition (4) is a frequency dependent, infinite dimensional Linear Matrix Inequality (LMI). Suppose that $\Pi \in \mathbf{Rl}_\infty$. Then this matrix inequality can be converted into a frequency independent finite dimensional LMI using the Kalman-Yakubovich- Popov (KYP) Lemma.

Note that any IQC for $\mathcal{D}_\tau$ immediately leads to an IQC for $\mathcal{S}_\tau$. For example, let $w = \mathcal{S}_\tau v := v - \mathcal{D}_\tau v$. That $\mathcal{D}_\tau$ satisfies IQC defined by $\Pi := \begin{bmatrix} \Pi_{11} & \Pi_{12} \\ \Pi_{12}^* & \Pi_{22} \end{bmatrix}$ implies $\mathcal{S}_\tau$ satisfies IQC defined by the following multiplier

$$\begin{bmatrix} \Pi_{11} + \Pi_{12} + \Pi_{12}^* + \Pi_{22} & -(\Pi_{12} + \Pi_{22}) \\ -(\Pi_{12}^* + \Pi_{22}) & \Pi_{22} \end{bmatrix}$$

III. Integral Quadratic Constraints for $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$

In this section, conically parameterized integral quadratic constraint characterizations for operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$ are derived, which are crucial for applying IQC analysis to systems with time-varying delays. To this end, we will first present some l_2-gain properties of operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$. To facilitate the development, let us consider the following sets of discrete-time sequences

$$\Upsilon_1 := \{s : s[k] \in \{\mathcal{T}_m, \mathcal{T}_m + 1, \cdots, \mathcal{T}_M\},\ \forall k\}$$
$$\Upsilon_2 := \left\{s : s[k] \in \{\mathcal{T}_m, \mathcal{T}_m + 1, \cdots, \mathcal{T}_M\}, \sum_{m=0}^{M-1} |s[k+1+m] - s[k+m]| \leq d,\ \forall k \right\}.$$

where M is a positive integer, and $\mathcal{T}_m$, T_M, and d are non-negative integers satisfying $d \leq \mathcal{T}_M - \mathcal{T}_m + 1$. In the followings, let h be $\mathcal{T}_M - \mathcal{T}_m + 1$.

Proposition 1. *Consider the time-varying delay operator $\mathcal{D}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the following characterization holds for $\mathcal{D}_\tau$:*

$$\sup_{\tau\in\Upsilon_1} \|\mathcal{D}_\tau\|_{l_2} = \sqrt{h}. \tag{5}$$

Furthermore, it can be shown that a unit impulse function and the following delay sequence

$$\tau[k] = \begin{cases} \mathcal{T}_m & \text{if } k < \mathcal{T}_m \\ k & \text{if } \mathcal{T}_m \le k \le \mathcal{T}_M \\ \mathcal{T}_M & \text{if } k > \mathcal{T}_M \end{cases} \tag{6}$$

realize the worst-case l_2-gain.

Proposition 2. *Consider the time-varying delay operator $\mathcal{D}_\tau$ where the delay parameter τ could be any sequence from Υ_2, in which it is further assumed that $d < M$. Then the following characterization holds for $\mathcal{D}_\tau$:*

$$\sup_{\tau\in\Upsilon_2} \|\mathcal{D}_\tau\|_{l_2} = \sqrt{d+1}. \tag{7}$$

Furthermore, it can be shown that a unit impulse function and the following delay sequence

$$\tau[k] = \begin{cases} \mathcal{T}_m & \text{if } k < \mathcal{T}_m \\ k & \text{if } \mathcal{T}_m \le k \le \mathcal{T}_m + d \\ \mathcal{T}_m + d & \text{if } k \ge T_m + d \end{cases} \tag{8}$$

realize the worst-case l_2-gain.

Remark 1. *Given any $M > 1$, let us define the M-step total variation, and M-step average variation of τ as*

$$\delta_\tau^M[k] := \sum_{m=0}^{M-1} |\tau[k+1+m] - \tau[k+m]|,$$

and

$$\bar{\delta}_\tau^M[k] := \frac{1}{M}\sum_{m=0}^{M-1} |\tau[k+1+m] - \tau[k+m]|,$$

Then Proposition 2 can be interpreted as follows: if one may find a constant $M > 1$ and establish that the M-step average variation of the delay sequence $\bar{\delta}_\tau^M[k]$ is strictly less than one for all k, then the upper bound of the M-step total variation provides a tighter upper bound of the l_2-gain of the delay operator, assuming that the upper bound of the M-step total variation d is strictly less than $\mathcal{T}_M - \mathcal{T}_m + 1$. For the upper bound d of $\delta_\tau^M[k]$ to be useful, it is crucial that $d < \mathcal{T}_M - \mathcal{T}_m + 1$ and $\bar{\delta}_\tau^M[k]$ is strictly less than one for all k. Also notice that the actual value of the upper bound of $\bar{\delta}_\tau^M[k]$ does not offer useful information here. The only crucial information is whether it is strictly less than one.

Proposition 3. *Consider the time-varying "delay-difference" operator $\mathcal{S}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the following characterization holds for $\mathcal{S}_\tau$: for any l_2 signal v,*

$$\|\mathcal{S}_\tau v\|_{l_2}^2 \le \sum_{k=-\infty}^{\infty}\sum_{i=1}^{\mathcal{T}} (v[k] - v[k-i])^2. \tag{9}$$

Proposition 4. *Consider the time-varying "delay-difference" operator $\mathcal{S}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the following characterization holds for $\mathcal{S}_\tau$:*

$$\left\|\mathcal{S}_\tau \circ \frac{z}{z-1}\right\|_{l_2} \le \mathcal{T}_M, \tag{10}$$

where z represent the forward shifting operator and $\frac{z}{z-1}$ the discrete-time integrator.

Propositions 1 to 4 give rise to the following integral quadratic constraints for $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$.

Proposition 5. *Consider the time-varying delay operator $\mathcal{D}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the operator $\mathcal{D}_\tau$ satisfies any integral quadratic constraint defined by*

$$\Pi_1 = \begin{bmatrix} hX_1 & 0 \\ 0 & -X_1 \end{bmatrix} \tag{11}$$

where $X_1 = X_1' \ge 0$ is any positive semi-definite matrix.

Proposition 6. *Consider the time-varying delay operator $\mathcal{D}_\tau$ where the delay parameter τ could be any sequence from Υ_2, in which $d < M$ – in other words, the M-step average variation of τ is strictly less than one for all k. Then the operator $\mathcal{D}_\tau$ satisfies any integral quadratic constraint defined by*

$$\Pi_2 = \begin{bmatrix} (d+1)X_2 & 0 \\ 0 & -X_2 \end{bmatrix} \tag{12}$$

where $X_2 = X_2' \ge 0$ is any positive semi-definite matrix.

Proposition 7. *Consider the time-varying "delay-difference" operator $\mathcal{S}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the operator $\mathcal{S}_\tau$ satisfies any integral quadratic constraint defined by*

$$\Pi_3 = \begin{bmatrix} |\phi(e^{j\omega})|^2 X_3 & 0 \\ 0 & -X_3 \end{bmatrix} \tag{13}$$

where $\phi(z) \in \mathbf{Rl}_\infty$ satisfies

$$|\phi(e^{j\omega})|^2 = \sum_{\kappa=1}^{\mathcal{T}} |1 - e^{-j\kappa\omega}|^2 \tag{14}$$

and $X_3 = X_3' \ge 0$ is any positive semi-definite matrix.

Proposition 8. *Consider the time-varying "delay-difference" operator $\mathcal{S}_\tau$ where the delay parameter τ could be any sequence from Υ_1. Then the operator $\mathcal{S}_\tau$ satisfies any integral quadratic constraint defined by*

$$\Pi_4 = \begin{bmatrix} \mathcal{T}_M^2 |\psi(e^{j\omega})|^2 X_4 & 0 \\ 0 & -X_4 \end{bmatrix} \tag{15}$$

where $\psi(z) = \frac{z-1}{z}$ and $X_4 = X_4' \ge 0$ is any positive semi-definite matrix.

IV. Stability Criteria for Discrete-Time LTI Systems with Time-Varying Delays

To further illustrate the IQC analysis of varying-time-delay robustness presented in Section II, let us consider IQCs defined by Π_2 and Π_3 (equations (12) and (13)) for $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$. Then $\mathcal{S}_\tau$ satisfies IQC defined by

$$\Pi_{\text{comb}} := \begin{bmatrix} (d+1)X_2 + |\phi(\mathrm{e}^{j\omega})|^2 X_3 & X_2 \\ X_2 & -X_2 - X_3 \end{bmatrix}$$

With this IQC, Theorem 1 leads to the following stability criteria: the system is stable if there exists symmetric matrices $X_2 \geq 0$, $X_3 \geq 0$, and $\epsilon > 0$ such that

$$G(\mathrm{e}^{j\omega})^*((d+1)X_2 + |\phi(\mathrm{e}^{j\omega})|^2 X_3)G(\mathrm{e}^{j\omega}) + G(\mathrm{e}^{j\omega})^* X_2 + X_2 G(\mathrm{e}^{j\omega}) - X_2 - X_3 \leq -\epsilon I, \quad \forall \omega \in [-\pi, \pi]. \tag{16}$$

where $G(z) := -(zI - (A + A_d))^{-1} A_d$. Let $(A_\phi, B_\phi, C_\phi, D_\phi)$ be the minimum state space realization of $\phi(z) \cdot I_n$. Define

$$A_t = \begin{bmatrix} A + A_d & 0 \\ B_\phi & A_\phi \end{bmatrix}, \quad B_t = \begin{bmatrix} A_d \\ 0 \end{bmatrix}, \quad C_t = \begin{bmatrix} I_n & 0 \\ D_\phi & C_\phi \end{bmatrix}.$$

and $M_{22} = -X_2 - X_3$,

$$M_{11} = \begin{bmatrix} (d+1)X_2 & 0 \\ 0 & X_3 \end{bmatrix}, \; M_{12} = \begin{bmatrix} -X_2 \\ 0 \end{bmatrix}.$$

A finite dimensional formulation of stability criterion (16) can be obtained by the KYP lemma: the system is stable if there exist a symmetric matrices P, $X_2 \geq 0$, and $X_3 \geq 0$ such that

$$\begin{bmatrix} A_t' P A_t - P & A_t' P B_t \\ B_t' P A_t & B_t' P B_t \end{bmatrix} + \begin{bmatrix} C_t' M_{11} C_t & C_t' M_{12} \\ M_{12}' C_t & M_{22} \end{bmatrix} < 0.$$

V. Numerical Experiments

Consider the following discrete-time system with a time-varying delay

$$x[k+1] = \begin{bmatrix} 0.8 & 0 \\ 0.05 & 0.9 \end{bmatrix} x[k] + \begin{bmatrix} -0.1 & 0 \\ -0.2 & -0.1 \end{bmatrix} x[k - \tau]$$

The example is taken from [4], and is also considered in [3]. Assuming the rate of variation is arbitrarily, the results presented in Table I are obtained by applying the proposed IQC analysis, where IQCs utilized are defined by Π_1, Π_3, and Π_4. The results are compared against those given in [4] and [3]. The IQC analysis gives better stability boundaries than the criterion in [4] does, but is apparently more conservative than the criterion in [3]. This clearly indicates that there are more integral quadratic constraints for operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$ to be explored.

	Assume the variation of τ is arbitrarily. Listed below are the maximal $\mathcal{T}_M$ for given $\mathcal{T}_m$.				
	$\mathcal{T}_m = 2$	$\mathcal{T}_m = 4$	$\mathcal{T}_m = 6$	$\mathcal{T}_m = 10$	$\mathcal{T}_m = 12$
IQC	10	11	11	13	14
by [4]	7	8	9	12	13
by [3]	13	13	14	15	17

TABLE I

	Assume the "average" variation of τ is less than 1; that is, $\frac{1}{M}\sum_{i=1}^{M-1} \lvert\tau[k+i+1] - \tau[k+i]\rvert \leq \frac{d}{M} < 1$. Listed below are the maximal $\mathcal{T}_M$ for given $\mathcal{T}_m$.				
	$\mathcal{T}_m = 2$	$\mathcal{T}_m = 4$	$\mathcal{T}_m = 6$	$\mathcal{T}_m = 10$	$\mathcal{T}_m = 12$
$d/M \leq 1/2$	20	20	20	20	20
$d/M \leq 2/3$	16	16	16	16	16
$d/M \leq 3/4$	14	14	14	14	–
$d/M \leq 4/5$	13	13	13	–	–

TABLE II

If we further assume that there exists an M such that the M-step average variation of τ is strictly less than 1, then the IQC defined by Π_2 becomes applicable. Utilizing this and the IQCs defined by Π_3 and Π_4, we apply the proposed IQC analysis and obtain the stability boundaries as listed in Table II. First of all, we observe that when the average variation of τ is restricted to be strictly less than 1, the stability boundaries predicted by IQC analysis apparently become bigger. Furthermore, the smaller the average variation is, the bigger the upper bounds $\mathcal{T}_M$. Secondly, we observe that for the case $d/M \leq 1/2$, the predicted stability boundary $\mathcal{T}_M = 20$ is better than those predicted by [3] and [4]. Since the notation "M-step average variation" was not introduced in [3] and [4], this is not a fair comparison; nevertheless, it shows that the notation is a useful concept which may reduce conservatism of time-delay robustness analysis. Finally, we observe that for a given upper bound of d/M, the predicted stability boundary is independent of the lower bound of τ, $\mathcal{T}_m$. The information on $\mathcal{T}_m$ somehow becomes useless in this case. Whether it just happens for this particular example or there is more to it requires further investigation.

VI. Concluding Remarks

In this manuscript we present several l_2-gain properties of the discrete-time time-varying delay operator $\mathcal{D}_\tau$ and "delay-difference" operator $\mathcal{S}_\tau := I - \mathcal{D}_\tau$, as well as the resulting IQC characterizations of these two operators. One of the new contributions is to show what role the variation of the delay sequence plays in regard to the energy amplification of $\mathcal{D}_\tau$, which allows us to better understand the behavior of $\mathcal{D}_\tau$ and to derive less conservative stability criteria for verifying varying-time-delay robustness. Numerical experiments are conducted to verify the effectiveness of the proposed stability criteria. The results indicate that one can reduce conservatism of stability analysis by exploiting the information on the variation of the delay sequence. They also show that there are potentially more IQCs to be discovered for the operators $\mathcal{D}_\tau$ and $\mathcal{S}_\tau$.

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