Convex inner approximations of nonconvex semialgebraic sets applied to fixed-order controller design

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Abstract— We describe an elementary algorithm to build convex inner approximations of nonconvex sets. Both input and output sets are basic semialgebraic sets given as lists of defining multivariate polynomials. Even though no optimality guarantees can be given (e.g. in terms of volume maximization for bounded sets), the algorithm is designed to preserve convex boundaries as much as possible, while removing regions with concave boundaries. In particular, the algorithm leaves invariant a given convex set. The algorithm is based on Gloptipoly 3, a public-domain Matlab package solving nonconvex polynomial optimization problems with the help of convex semidefinite programming (optimization over linear matrix inequalities, or LMIs). We illustrate how the algorithm can be used to design fixed-order controllers for linear systems, following a polynomial approach.

I. Introduction

The set of controllers stabilizing a linear system is generally nonconvex in the parameter space, and this is an essential difficulty faced by numerical algorithms of computer-aided control system design, see e.g. [2] and references therein. It follows from the derivation of the Routh-Hurwitz stability criterion (or its discrete-time counterpart) that the set of stabilizing controllers is real basic semialgebraic, i.e. it is the intersection of sublevel sets of given multivariate polynomials. A convex inner approximation of this nonconvex semialgebraic stability region was obtained in [2] in the form of linear matrix inequalities (LMI) obtained from univariate polynomial positivity conditions. Convex inner approximations make it possible to design stabilizing controllers with the help of convex optimization techniques, at the price of loosing optimality w.r.t. closed-loop performance criteria (H_2 norm, H_∞ norm or alike).

Generally speaking, the technical literature abounds of convex *outer* approximations of nonconvex semialgebraic sets. In particular, such approximations form the basis of many branch-and-bound global optimization algorithms [5]. By construction, Lasserre's hierarchy of LMI relaxations for polynomial programming is a sequence of embedded convex inner approximations which are semidefinite representable, i.e. which are obtained by projecting affine sections of the convex cone of positive semidefinite matrices, at the price of introducing lifting variables [3]. In [1, Section 6] it is conjectured that all convex semialgebraic sets (and, in

particular, convex hulls of nonconvex semialgebraic sets) are semidefinite representable.

After some literature search, we could not locate any systematic constructive procedure to generate convex *inner* approximations of nonconvex semialgebraic sets, contrasting sharply with the many convex outer approximations mentioned above. In the context of fixed-order controller design, inner approximations correspond to a guarantee of stability, at the price of loosing optimality. No such stability guarantee can be ensured with outer approximations.

II. Convex inner approximation

Given a basic closed semialgebraic set

$$S = \{x \in \mathbb{R}^n : p_1(x) \leq 0 \ \ldots \ p_m(x) \leq 0\}$$

where p_i are multivariate polynomials, we are interested in computing another basic closed semialgebraic set

$$\bar{S} = \{x \in \mathbb{R}^n : \bar{p}_1(x) \leq 0 \ \ldots \ \bar{p}_{\bar{m}}(x) \leq 0\}$$

which is a valid inner approximation of S, in the sense that

$$\bar{S} \subset S.$$

Ideally, we would like to find the tightest possible approximation, in the sense that the complement set $S \backslash \bar{S} = \{x \in S : x \notin \bar{S}\}$ is as small as possible. Mathematically we may formulate the problem as the volume minimization problem

$$\inf_{\bar{S}} \int_{S \backslash \bar{S}} dx$$

but since set S is not necessarily bounded we should make sure that this integral makes sense. Moreover, computing the volume of a given semialgebraic set is a difficult task in general, so we expect that optimizing such a quantity is as much as difficult. For these reasons, we have not been able to define a mathematically sound while tractable measure of tightness of the inner approximation. In practice we will content ourselves of an inner approximation that removes the nonconvex parts of the boundary and keeps the convex parts as much as possible.

III. Algorithm

Our main contribution is therefore an elementary algorithm, readily implementable in Matlab, that generates convex inner approximations of nonconvex sets. Both input and output sets are basic semialgebraic sets given as lists of defining multivariate polynomials. Even though no optimality guarantees can be given in terms of volume maximization for bounded sets, the algorithm is designed to preserve convex

boundaries as much as possible, while removing regions with concave boundaries. In particular, the algorithm leaves invariant a given convex set.

The idea behind the algorithm is as follows. We identify a point of minimal curvature along algebraic varieties defining the boundary of S. If the minimal curvature is negative, then we separate the point from the set with a gradient hyperplane, and we iterate this procedure on the resulting semialgebraic set. At the end, we obtain a valid inner approximation.

The algorithm is based on GloptiPoly 3, a public-domain Matlab package solving nonconvex polynomial optimization problems with the help of convex LMIs [4]. We illustrate how the algorithm can be used to design fixed-order controllers for linear systems, following a polynomial approach.

IV. EXAMPLE

Consider the fourth degree polynomial

$$x_2 + x_1 z - (x_1 + x_2)z^3 + z^4.$$

Its roots are in the open unit disk if and only if $x = (x_1, x_2)$ belongs to the interior of stability region $S = \{x \in \mathbb{R}^2 : p_1(x) = 2x_1^2 x_2 + 3x_1 x_2^2 + 2x_1^2 + x_1 x_2 + x_2^2 + x_2 - 1 \leq 0, \; p_2(x) = -2x_2 - 1 \leq 0, \; p_3(x) = -2x_1 + x_2 - 2 \leq 0\}$.

Our GloptiPoly 3 implementation of our algorithm detects that S is non-convex, and generates the additional affine constraint $p_4(x) = g_1^T(x^1)(x - x^1) + \epsilon \leq 0$ where ϵ is a small positive real, say 10^{-3}, $g_1(x)$ is the gradient of $p_1(x)$, and x^1 is a point along the the boundary of S. Running again our algorithm on the new semialgebraic set $\bar{S} = \{x : p_1(x) \leq 0, \ldots p_4(x) \leq 0\}$ returns a numerical certificate of convexity, so we obtain a valid convex inner approximation of nonconvex stability region S, see Figure 1.
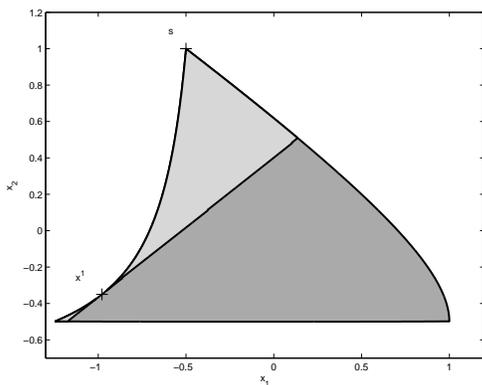

Fig. 1. Convex inner approximation (dark gray) of nonconvex fourth-order discrete-time stability region (light gray).

From Figure 1 we see that the choice of point x^1 is not necessarily optimal in terms of maximizing the surface of $\bar{S}$. A point chosen between x^1 and s along the boundary would be likely to generate a larger convex inner approximation.

V. CONCLUSION

We have presented a general-purpose computational algorithm to generate a convex inner approximation of a given basic semialgebraic set. The inner approximation is not guaranteed to be of maximum volume, but the algorithm has the favorable features of leaving invariant a convex set, and preserving convex boundaries while removing nonconvex regions by enforcing linear constraints at points of minimum curvature.

Generally speaking, one may question the relevance of applying a relatively computationally expensive algorithm to obtain a convex inner approximation in the form of a list of defining polynomials which are not necessary individually convex. Indeed, using standard duality arguments, convexity of the polynomials would allow the use of constant multipliers to certificate optimality in a nonlinear optimization framework. Instead, with no guarantee of convexity of the defining polynomials, the geometric proprety of convexity of the sets is more delicate to exploit efficiently by optimization algorithms. We hope however that recent results on semidefinite representability of convex semialgebraic sets may be exploited to derive an explicit representation of our inner convex approximation as a projection of an affine section of the semidefinite cone, or equivalently, in our target application domain, to use semidefinite programming to find a suboptimal stabilizing fixed-order controller.

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