Treatment of Systems Nonlinearities by a Multiplier Method

Éva Gyurkovics and Tibor Takács

Abstract— **This paper investigates the conditions under which an abstract matrix multiplier method can be applied to determine guaranteeing cost controls for systems containing nonlinear/uncertain elements via linear matrix inequalities (LMIs). Quadratically constrained uncertainties and nonlinearities are considered which comprehend the cases of norm-bounded, positive-real and sector-bounded uncertainties/nonlinearities. Both the discrete-time and the continuous-time cases are discussed. Necessary and sufficient conditions are formulated in case of unstructured uncertainty. The conditions are sufficient in the structured case. The cost guaranteeing controls can be determined by solving LMIs. The proposed method provides a guideline to treat system nonlinearities, if the system dynamics can be formulated as considered in the paper by an approriate choice of system parameters.**

I. Introduction

The dynamics of most systems cannot adequately be described by linear time-invariant models because of the presence of some unknown or unmodelled time-varying and/or nonlinear elements in the real system. In the past decades, a huge amount of works has been devoted to the investigation of different aspects of this problem. This paper does not aim at giving a comprehensive review of results in this field, only those are mentioned which have influence on the present work. Among them, [2], [5], [8], [10], [14], [18], [28], [29], [33] deal with robust and quadratic stability of uncertain systems, [4], [6], [7], [15], [23], [39] are devoted to the design of guaranteed cost and H_∞ control, while [11]-[13], [17], [19], [21], [24], [30]-[32], [34]-[37] and [40] deal with positive realness and dissipativity of systems with positive real, sector bounded and dissipative uncertainties. This paper considers control systems, where nonlinearities and/or uncertainties may effect on the system dynamics, and these nonlinearities/uncertainties are allowed to be time-varying and/or unknown. The only information about the nonlinearities/uncertainties is that they are restricted by a given set, on which no a priori algebraic or topological requirements are imposed. Because of the unknown nonlinearities/uncertainties, it is not possible to minimize any objective (or cost) function assigned to this type of systems, therefore only suboptimal solutions can be expected. This paper focuses on the determination of cost guaranteeing controls for the above type of systems formulated either in continuous- or in discrete-time.

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This problem requires the solution of an inequality over the set constraining the nonlinearities/uncertainties. In general, this is numerically a non-tractable problem. However, a generalized multiplier method analogous to the well-known Lagrange-multiplier method provides an opportunity to get rid of the constraining set, if one can add a new term to the original inequality in such a way that the satisfaction of this augmented inequality on the whole space should be equivalent to the original one over the constraining set. Paper [26] presents the so-called full-block S-procedure, which is a matrix multiplier method of this type. It can be applied in cases, when the set of uncertainties/nonlinearities is formulated as a parametrized family of subspaces. Papers [1] and [2] also use matrix multipliers for several special problems without imposing any restriction on the constraint set. A further important contribution of the latter work is the notion of sufficiently rich set of multipliers that allows to use a subset of the multipliers without loss of the necessity part of the conditions. The authors of the present paper developed an abstract multiplier method (see [16]) that is the generalization of the above mentioned approaches. This method can be applied, when the nonlinearities/uncertainties can be formulated as parametrized family of cones instead of subspaces.

Systems with quadratically constrained uncertainties will be investigated in this paper. Necessary and sufficient conditions will be formulated for the existence of cost guaranteeing controls. If there exist cost guaranteeing controls, they can be derived from the solution of LMIs that can easily be solved by standard methods and by appropriate software. The results are based on the notion of sufficiently rich set of multipliers introduced by [2]. On the one hand, the paper is also related with the results of [32] and [36], that formulated only sufficient conditions for the robust dissipative control problem. On the other hand, the presented results can serve as a guideline for the treatment of systems nonlinearities/uncertainties in robustness problems by the appropriate choice of the parameters of nonlinear/uncertain inputs and outputs.

The transpose of matrix A is denoted by A^T, I_n is the identity matrix of dimension n, and $P > 0$ (≥ 0) denotes the positive (semi-)definiteness of P. Symbol ∇V stands for the gradient of the multivariable function V, and symbol $\otimes$ is used for Kronecker-product. The notation of time-dependence is omitted, if it does not cause any confusion. For the sake of brevity, asterisks replace the blocks in hypermatrices, and matrices in expressions that are inferred readily by symmetry.

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II. Problem statement

Consider system

$$\delta x = Ax + Bu + Ew + \sum_{i=1}^{s} H_i p_i \quad (1)$$

$$q_i = C_{q_i} x + E_{q_i} w + D_{q_i} p_i, \qquad i = 1, ..., s, \quad (2)$$

where $x \in \mathbf{R}^{n_x}$ is the state, $u \in \mathbf{R}^{n_u}$ is the input, $w \in \mathbf{R}^{n_w}$ is the exogenous disturbance, δx stands for $\dot{x}$ in the continuous-time and x^+ in the discrete-time case. Further on, p_i and q_i may depend on t, x, and w, and they have values of dimensions l_{p_i} and l_{q_i}, respectively. Functions p_i incorporate all the nonlinear and uncertain elements of the system dynamics, while functions q_i represent the uncertain outputs. The only information that can be used about $\left(p^T, q^T\right)^T$, where $p^T = \left(p_1^T, ..., p_s^T\right)$, $q^T = \left(q_1^T, ..., q_s^T\right)$ is that its values are restricted to a given set $\Omega \subset \mathbf{R}^{l_p + l_q}$, where $l_p = l_{p_1} + ... + l_{p_s}$, $l_q = l_{q_1} + ... + l_{q_s}$. In this paper both the case of unstructured uncertainties (when $s = 1$) and the case of structured uncertainties (when $s > 1$) will be investigated. The set Ω is assumed to be quadratically constrained, i.e. given as

$$\Omega = \left\{ \begin{pmatrix} p \\ q \end{pmatrix} \in \mathbf{R}^{l_p + l_q}, \right.$$
$$\left. \begin{pmatrix} p_i \\ q_i \end{pmatrix}^T \begin{pmatrix} Q_{0i} & S_{0i} \\ S_{0i}^T & R_{0i} \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix} \geq 0, \; i = 1, ..., s \right\}, \quad (3)$$

where $Q_{0i} = Q_{0i}^T$, $R_{0i} = R_{0i}^T \geq 0$ and S_{0i} are matrices of appropriate dimension. We shall use the notation $Q_0 = \text{diag}\{Q_{01}, ..., Q_{0s}\}$, $R_0 = \text{diag}\{R_{01}, ..., R_{0s}\}$ and $S_0 = \text{diag}\{S_{01}, ..., S_{0s}\}$. A system given by (1)-(3) is said to be well-posed, if for any (x, u, w) there is a p so that $\left(p^T, q^T\right)^T \in \Omega$, where q is the vector defined by (2). Because of the assumption $R_0 \geq 0$, the well-posedness of the system (1)-(3) immediately follows.

Let the performance index to be minimized

$$J(x_0, u, w) = \begin{cases} \int_0^\infty L(x(t), u(t), w(t))dt & \text{if } t \in \mathbf{R}, \\ \sum_{t=1}^\infty L(x(t), u(t), w(t)) & \text{if } t \in \mathbf{Z} \end{cases} \quad (4)$$

with

$$L(x, u, w) = x^T Q_L x + u^T R_L u - w^T S_L w \quad (5)$$

be assigned to system (1)-(2), where matrices Q_L, R_L and S_L are positive definite and symmetric.

Since unknown uncertainties/nonlinearities are present in the system dynamics, it is not possible to find an optimum of (4)-(5). Instead, our aim is to find a guaranteed cost and a guaranteeing cost control in the sense of the definition below.

Consider function $\mathcal{V} : \mathbf{R}^{n_x} \to \mathbf{R}^+$, and for any function $f(x, u, w, p)$ introduce the following notation:

$$\mathcal{V}_f^*(x, u, w, p) = \begin{cases} \nabla \mathcal{V}^T(x) \, f(x, u, w, p), & \text{if } t \in \mathbf{R}, \\ \mathcal{V}(f(x, u, w, p)) - \mathcal{V}(x), & \text{if } t \in \mathbf{Z}. \end{cases} \quad (6)$$

Definition 1 Consider the nonlinear/uncertain system

$$\delta x = f(x, u, w, p), \quad q = g(p)$$

with cost function (4)-(5) and with a given set of nonlinearities/uncertainties Ω. The state-feedback $u = k(x)$ is a guaranteeing cost robust minimax strategy if there exists a function $\mathcal{V} : \mathbf{R}^n \to \mathbf{R}^+$ such that

$$\sup_{\begin{pmatrix} p \\ q \end{pmatrix} \in \Omega} \{\mathcal{V}_f^*(f(x, k(x), w, p) + L(x, k(x), w)\} < 0 \quad (7)$$

holds for all x and w, $\left(x^T, w^T\right) \neq \left(0^T, 0^T\right)$. In this case $\mathcal{V}(x_0)$ is called a guaranteed cost.

The paper deals with the determination of a guaranteeing cost linear feedback for system (1)-(2).

Remark 1 Similar definitions of guaranteed cost are frequently used in the literature. (See e.g. [15], [23], [39] and the references therein). If the external disturbances are of class $L_2(0, \infty)$ (or $l_2(0, \infty)$), then one can show that $\mathcal{V}(x_0)$ of Definition 1 yields an upper bound of the cost function for all admissible uncertainties. This way is accepted for example in [4]. Paper [15] also investigates the relation of the two ways of defining the guaranteeing cost control. We note that - by supplementing the system formulation with a performance output - a slight modification of Definition 1 may serve as a starting point for the investigation of dissipativeness of systems, as well (see definition e.g. in [31] and [37]). However, our aim is to show how the abstract multiplier method can be applied to a control problem, therefore we disregard the details.

Remark 2 Even for systems having linear nominal part subject to quadratic cost criterion, it is very difficult to analyze the fulfillment of (7), and it is even more difficult to find an appropriate function $\mathcal{V}$ and feedback $k(x)$ because of the need of maximization over Ω. In order to get rid of this constraining set, one can apply multipliers similarly to the standard Lagrange-multiplier method. In this case, an additional term is supplemented to the expression in the left-hand side of (7) within the curly braces in such a way that the satisfaction of the 'new' inequality *over a whole space* will be equivalent to the original one *over the set* Ω. In this way, the investigation of the inequality and of the uncertainty bounding set is separated.

This idea has been widely used in the literature, among the others, in analysis and design problems of robust control. Several approaches are based on the well-known $S-$procedure of Yakubovich [38]. A generalization called full-block $S-$procedure is developed by Scherer [26]. It can be applied for different kinds of robustness problems provided that the nonlinear/uncertain part of the system dynamics can be represented by a parametrized family of *subspaces*. Such a representation is immediate, if the uncertain input and output are connected through the multiplication by an uncertain matrix. In contrast, paper [2] considers the stability problem of continuous-time linear time-invariant systems with *arbitrary* uncertainty set Ω. In this latter respect, it is the most general case, that one can imagine. Besides, the authors derive appropriate sets of multiplier matrices for many different kinds of uncertainty sets. However, the result of the paper [2] are limited to the above mentioned LTI problems.

In [15], an abstract multiplier method has been established that can be applied for a wide variety of robustness problems, presumed that the uncertainties/nonlinearities can be restricted to a parametrized family of closed cones. It has been shown, as well, that this restriction does not cause any loss of generality, if the question is how an inequality for a quadratic function over a constrained set can equivalently be substituted by a similar one but over a whole linear space. The aim of the present paper is to show, how the abstract multiplier method of [16] can be applied for the solution of the formulated problem.

III. Preliminaries

We recall in this section a special case of the results presented in [16], and we will show in the next section how these results can be applied to the solution of the problem formulated above.

Suppose that $\mathcal{B} \subset \mathbf{R}^N$ is a subspace and matrices $U \in \mathbf{R}^{j\times N}$ and $V \in \mathbf{R}^{l\times N}$ are fixed, where V has maximum row rank. Consider a symmetric matrix $\Psi \in \mathbf{R}^{j\times j}$. Let $\mathcal{Q} \subset \mathbf{R}^l$ be given. Assume that

$$V\mathcal{B} \cap \mathcal{Q} \neq \emptyset.$$

Definition 2 ([2]) A symmetric matrix M is called a *multiplier matrix* for $\mathcal{Q}$ if $q^T Mq \geq 0$ for all $q \in \mathcal{Q}$. If this inequality is strict, then M is called a positive multiplier matrix for $\mathcal{Q}$.

Definition 3 ([16]) The set $\mathcal{M}^+$ of positive multiplier matrices for $\mathcal{Q}$ is called a *sufficiently rich set of positive multipliers* for $\mathcal{Q}$, if for any positive multiplier $\overline{M}$ for $\mathcal{Q}$ there exists an element $M \in \mathcal{M}^+$ such that $M \leq \overline{M}$.

Introduce the following set

$$\mathcal{B}_{\mathcal{Q}} = \{y \in \mathcal{B} : \ Vy \in \mathcal{Q}\}.$$

Suppose that the fulfillment of the inequality

$$y^T U^T \Psi U y < 0 \quad \forall y \in \mathcal{B}_{\mathcal{Q}} \tag{8}$$

has to be investigated. Let $\mathcal{B}_0$ be a subspace of maximum dimension in $\mathcal{B}$, for which $U^T\Psi U \geq 0$. If the dimension of $\mathcal{B}_0$ is equal to zero, then $U^T\Psi U < 0$ is satisfied on $\mathcal{B}$, thus there is an $\varepsilon > 0$ such that (8) equivalent to

$$y^T(U^T\Psi U + \varepsilon V^T V)y < 0 \quad \forall y \in \mathcal{B},\ y \neq 0.$$

Evidently, $\varepsilon I_l \in \mathcal{M}^+$.

In [16], it was proved that, if $\dim \mathcal{B}_0 \geq 1$ for some $\mathcal{B}_0$ and

$$\mathcal{B}_0 \cap \mathcal{B}_{\overline{\text{cone}\mathcal{Q}}} \neq \{0\},$$

then the strict inequality

$$y^T(U^T\Psi U + V^T MV)y < 0$$

is not satisfied for all $0 \neq y \in \mathcal{B}$ whatever multiplier matrix for $\mathcal{Q}$ is considered.

Therefore, we may assume without substantial restriction of generality that the following conditions hold true.

Condition 1

1.) $\mathcal{Q}$ is a cone,

2.) $\mathcal{B}_0 \cap \mathcal{B}_{\overline{\mathcal{Q}}} = \{0\} \quad \forall \mathcal{B}_0$,

and

3.) either $\mathcal{Q} \subset V\mathcal{B}$ or $\mathcal{Q}$ is closed.

Note that, in the investigation of (8), item 1 of Condition 1 does not restrict the generality.

Now we cite the main result of [16].

Theorem 1 ([16]) Assume that Condition 1 holds true, and $\mathcal{M}^+$ is a sufficiently rich set of positive multipliers for $\mathcal{Q}$. Then the following statements are equivalent.

1. Inequality

$$y^T U^T \Psi U y < 0 \tag{9}$$

holds true for all $0 \neq y \in \mathcal{B}_{\mathcal{Q}}$.

2. There exists a $M \in \mathcal{M}^+$ such that

$$y^T(U^T\Psi U + V^T MV)y < 0, \quad y \in \mathcal{B},\ y \neq 0. \quad \square \tag{10}$$

We will need also the following strict version of the S-procedure lemma.

Lemma 1 Suppose that $\Phi_1, \Phi_2 \in \mathbf{R}^{N\times N}$ are symmetric, then the following statements are equivalent.

1. $x^T\Phi_2 x > 0$ for all $x \neq 0$, for which $x^T\Phi_1 x \geq 0$ holds.

2. There exists a positive constant τ such that $x^T(\Phi_2 - \tau\Phi_1)x > 0$ for all $x \neq 0$. $\square$

Remark 3 The original version of Lemma 1 is related to Yakubovich [38]. Since then several generalizations have been established to it, and comprehensive reviews are given e.g. in [3], [9], [20], [22], [25].

IV. Application of the abstract multiplier method

In this section, a necessary and sufficient condition will be given for a linear feedback $u = Kx$ to be guaranteeing cost control for system (1)-(2). The condition will be given by a linear matrix inequality (LMI), and the feedback matrix K can be determined via the solution of this LMI. The key element of solving the problem is to find a sufficiently rich set of positive multipliers.

A. The sufficiently rich set of positive multipliers

Consider positive constants τ_i and ε_i, $i = 1, ..., s$ and set

$$\underline{\tau} = \text{diag}\left\{\tau_1 I_{l_{p_1}}, ..., \tau_s I_{l_{p_s}}\right\},\ \underline{\underline{\tau}} = \text{diag}\left\{\tau_1 I_{l_{q_1}}, ..., \tau_s I_{l_{q_s}}\right\},$$
$$\underline{\varepsilon} = \text{diag}\left\{\varepsilon_1 I_{l_{p_1}}, ..., \varepsilon_s I_{l_{p_s}}\right\},\ \underline{\underline{\varepsilon}} = \text{diag}\left\{\varepsilon_1 I_{l_{q_1}}, ..., \varepsilon_s I_{l_{q_s}}\right\}.$$

We note that, if $s = 1$, matrices $\underline{\tau}$, $\underline{\underline{\tau}}$, $\underline{\varepsilon}$ and $\underline{\underline{\varepsilon}}$ consist of a single block, thus two scalar parameters can be used instead. In order to avoid the repetition of big formulas, we shall use the matrix notations in the special case of $s = 1$, as well.

Proposition The set

$$\mathcal{M}^+ = \left\{ M:\ M = \begin{pmatrix} \underline{\tau} Q_0 + \underline{\varepsilon} & \underline{\tau} S_0 \\ S_0^T \underline{\tau} & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \end{pmatrix}, \right. \left. \tau_i, \varepsilon_i > 0,\ i = 1, ..., s \right\} \quad (11)$$

consists of positive multiplier matrices for Ω. If $s = 1$, then $\mathcal{M}^+$ is sufficiently rich.

Proof Since for all $\left(p^T, q^T\right) \neq \left(0^T, 0^T\right)$ from Ω we have

$$\begin{pmatrix} p \\ q \end{pmatrix}^T M \begin{pmatrix} p \\ q \end{pmatrix} = \sum_{i=1}^{s} \left\{ \tau_i \begin{pmatrix} p_i \\ q_i \end{pmatrix}^T \begin{pmatrix} Q_{0i} & S_{0i} \\ S_{0i}^T & R_{0i} \end{pmatrix} \begin{pmatrix} p_i \\ q_i \end{pmatrix} + \varepsilon_i (p_i^T p_i + q_i^T q_i) \right\} > 0, \quad (12)$$

the first statement is obvious.

Suppose now that $s = 1$ and M is an arbitrary positive multiplier for Ω. Then the application of Lemma 1 gives the existence of a positive constant τ_1 such that

$$\begin{pmatrix} p \\ q \end{pmatrix}^T M \begin{pmatrix} p \\ q \end{pmatrix} - \tau_1 \begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} > 0 \quad (13)$$

for all $\left(p^T, q^T\right)^T \in \mathbf{R}^{l_p + l_q} \setminus \{0\}$. Because of continuity, the left-hand side of (13) takes on its positive minimum on the unit sphere, therefore there exists an $\varepsilon_1 > 0$ such that

$$\begin{pmatrix} p \\ q \end{pmatrix}^T \begin{pmatrix} \tau_1 Q_0 + \varepsilon_1 I & \tau_1 S_0 \\ S_0^T \tau_1 & \tau_1 R_0 + \varepsilon_1 I \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} < \begin{pmatrix} p \\ q \end{pmatrix}^T M \begin{pmatrix} p \\ q \end{pmatrix}. \quad (14)$$

This means that $\mathcal{M}^+$ is a sufficiently rich set of positive multipliers with positive constants τ_1 and ε_1 if $s = 1$. □

B. Main results

In order to derive the main results, we need the following assumption

Assumption 1 Inequality

$$Q_0 + D_q^T S_0^T + S_0 D_q + D_q^T R_0 D_q < 0 \quad (15)$$

holds, where $D_q = \text{diag}\{D_{q_1}, ..., D_{q_s}\} \in \mathbf{R}^{l_q \times l_p}$.

Remark 4 We shall see that this inequality is necessary and sufficient for the satisfaction of item 2 in Condition 1. In this sense Assumption 1 is required not only as a technical tool for the proof, because, if it is not satisfied, then it is not possible to get rid of the constraints (see considerations before Condition 1).

A condition of type (15) with $R_0 \geq 0$ is employed also e.g. in [31], [36] and [37] for the matrices characterizing system uncertainties. This assumption is less strict than that of [35], where $Q_0 = 0$, $S_0 = I$ and D_q must be quadratic and invertible. If $Q_0 = 0$, $S_0 = I$ and $R_0 = I$, then (15) is the inequality characterizing the so called positive real uncertainty (see e.g. [17]). By appropriate choice of Q_0, S_0 and R_0, one can obtain norm-bounded and sector-bounded uncertainties, as well.

Introduce the $2n_x \times 2n_x$ matrix

$$\Phi = \begin{cases} \Phi_c = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, & \text{if } t \in \mathbf{R}, \\ \Phi_d = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}, & \text{if } t \in \mathbf{Z}, \end{cases}$$

and for any given $K \in \mathbf{R}^{n_u \times n_x}$, set

$$\mathcal{A} = A + BK, \quad \overline{Q} = Q_L + K^T R_L K. \quad (16)$$

Moreover, set

$$C_q^T = \left(C_{q_1}^T, ..., C_{q_s}^T\right), \quad H = (H_1, ..., H_s).$$

Theorem 2 Suppose that $s = 1$. Then the state-feedback $u = Kx$ is a guaranteeing cost robust minimax strategy and the guaranteed cost is $\mathcal{V}(x_0)$ with $\mathcal{V}(x) = x^T P x$ if and only if there exist such positive constants τ_i and ε_i $(i = 1)$ that

$$(*)\, \text{diag} \left\{ \Phi \otimes P;\ \begin{pmatrix} \overline{Q} & 0 \\ 0 & -S_L \end{pmatrix}; \begin{pmatrix} \underline{\tau} Q_0 + \underline{\varepsilon} & \underline{\tau} S_0 \\ S_0^T \underline{\tau} & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \end{pmatrix} \right\} \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & E & H \\ I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ C_q & E_q & D_q \end{pmatrix} < 0. \quad (17)$$

If $s > 1$, then inequality (17) is sufficient for $u = Kx$ and $\mathcal{V}(x_0)$ to be a cost guaranteeing robust control and a guaranteed cost, respectively.

Proof If $u = Kx$, $\mathcal{V}(x) = x^T P x$, then (7) is equivalent to

$$\sup_{\begin{pmatrix} p \\ q \end{pmatrix} \in \Omega} F(x, w, p) < 0 \quad (18)$$

with

$$F(x, w, p) = (*)\, (\Phi \otimes P) \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & E & H \end{pmatrix} \begin{pmatrix} x \\ w \\ p \end{pmatrix} + (*) \begin{pmatrix} \overline{Q} & 0 \\ 0 & -S_L \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$$

for all x and w, $\begin{pmatrix} x^T & w^T \end{pmatrix} \neq \begin{pmatrix} 0^T & 0^T \end{pmatrix}$. Suppose first that $s = 1$ and apply the abstract multiplier method as follows. In this case $\mathcal{Q} = \Omega$, thus item 1 of Condition 1 is obviously satisfied. Set

$$N = 3n_x + n_w + l_p + l_q, \quad U = I_N,$$

$$\Psi = \text{diag} \left\{ \Phi \otimes P; \begin{pmatrix} \overline{Q} & 0 \\ 0 & -S_L \end{pmatrix}; \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\},$$

$$\mathcal{L}_1 = \begin{pmatrix} I & 0 \\ \mathcal{A} & E \\ I & 0 \\ 0 & I \\ 0 & 0 \\ C & E_q \end{pmatrix}, \quad \mathcal{L}_0 = \begin{pmatrix} 0 \\ H \\ 0 \\ 0 \\ I \\ D_q \end{pmatrix}$$

and

$$\mathcal{B}_0 = \text{im}\mathcal{L}_0, \quad \mathcal{B}_1 = \text{im}\mathcal{L}_1, \quad \mathcal{B} = \text{im}\left(\mathcal{L}_1, \mathcal{L}_0\right).$$

Obviously, $\dim \mathcal{B} = n_x + n_w + l_p$, $\dim \mathcal{B}_1 = n_x + n_w$, $\dim \mathcal{B}_0 = l_p$ and

$$F(x, w, p) = y^T \Psi y, \quad \text{if} \quad y = (\mathcal{L}_1, \mathcal{L}_0) \begin{pmatrix} x \\ w \\ p \end{pmatrix}.$$

Set

$$V = \begin{pmatrix} 0\,0\,0\,0\,I\,0 \\ 0\,0\,0\,0\,0\,I \end{pmatrix} \in \mathbf{R}^{(l_p + l_q) \times N},$$

then

$$\mathcal{B}_\Omega = \left\{ y \in \mathcal{B} \subset \mathbf{R}^N : \quad Vy \in \Omega \right\}. \tag{19}$$

Set (19) is not empty, since the problem is well posed. The fulfillment of (18) is equivalent to the negative definiteness of Ψ on $\mathcal{B}_\Omega$. We have to show that the conditions of Theorem 1 are satisfied. It is enough to prove that items 2 and 3 of Condition 2 hold true.

If $y \in \mathcal{B}_0$, i.e.

$$y^T = (0^T; (Hp)^T; 0^T; 0^T; p^T; (Dp)^T),$$

then on the one hand

$$y^T \Psi y = (*) \left(\Phi \otimes P\right) \begin{pmatrix} 0 \\ Hp \end{pmatrix} \geq 0.$$

On the other hand, if $y \in \mathcal{B}_0$, then

$$Vy = \begin{pmatrix} p \\ D_q p \end{pmatrix} \in \Omega$$

if and only if

$$\begin{pmatrix} p \\ D_q p \end{pmatrix}^T \begin{pmatrix} Q_0 & S_0 \\ S_0^T & R_0 \end{pmatrix} \begin{pmatrix} p \\ D_q p \end{pmatrix} \geq 0,$$

i.e.

$$p^T \left(Q_0 + D_q^T S_0^T + S_0 D_q + D_q^T R_0 D_q \right) p \geq 0,$$

which holds only for $p = 0$ because of Assumption 1. This means that only $p = 0$ is an admissible nonlinearity/uncertainty at $x = 0$. Thus $\mathcal{B}_0 \cap \mathcal{B}_\Omega = \{0\}$ for $\mathcal{B}_0 = \text{im}\mathcal{L}_0$.

Let $\widetilde{\mathcal{B}}_0$ any (other) subspace of $\mathcal{B}$ such that for any $\widetilde{y} \in \widetilde{\mathcal{B}}_0$, inequality $\widetilde{y}^T \Psi \widetilde{y} \geq 0$ holds. Let $\widetilde{y} \in \widetilde{\mathcal{B}}_0$, $\widetilde{y} \notin \mathcal{B}_0$ be given as

$$\widetilde{y} = \mathcal{L}_1 \begin{pmatrix} \widetilde{x} \\ \widetilde{w} \end{pmatrix} + \mathcal{L}_0 \widetilde{p} = \widetilde{y}_1 + \widetilde{y}_2,$$

i.e.

$$\widetilde{y}_1 = \begin{pmatrix} \widetilde{x} \\ \mathcal{A}\widetilde{x} + E\widetilde{w} \\ \widetilde{x} \\ \widetilde{w} \\ 0 \\ C_q \widetilde{x} \end{pmatrix} \in \mathcal{B}_1, \quad \widetilde{y}_2 = \begin{pmatrix} 0 \\ H\widetilde{p} \\ 0 \\ 0 \\ \widetilde{p} \\ D_q \widetilde{p} \end{pmatrix} \in \mathcal{B}_0.$$

One may assume that $\widetilde{y}_1 \neq 0$, i.e. $\left(\widetilde{x}^T, \ \widetilde{w}^T\right) \neq \left(0^T, \ 0^T\right)$, since otherwise $\widetilde{y} = \widetilde{y}_2 \in \mathcal{B}_0$ would hold. We obtain that

$$V\widetilde{y} = \begin{pmatrix} \widetilde{p} \\ C\widetilde{x} + D\widetilde{p} \end{pmatrix} \notin \Omega,$$

since otherwise $\widetilde{y}^T \Psi \widetilde{y} < 0$ would hold. Thus $\widetilde{y} \notin \mathcal{B}_\Omega$, which means that $\mathcal{B}_\Omega \cap \widetilde{\mathcal{B}}_0 = \{0\}$, consequently the second item of Condition 2 is satisfied. Moreover,

$$V\mathcal{B} = \Big\{ \left(p^T, \, q^T\right)^T : \, q = C_q x + D_q p, \\ p \in \mathbf{R}^{l_p}, \ x \in \mathbf{R}^{n_x} \},$$

thus $V\mathcal{B} \supset \Omega$. Therefore, all requirements of Condition 1 are satisfied. Moreover, $\mathcal{M}^+$ is sufficiently rich, if $s = 1$. Since the assumptions of Theorem 2 are satisfied, (18) is equivalent to the existence of $M \in \mathcal{M}^+$ such that

$$\Psi + V^T M V < 0 \tag{20}$$

on $\mathcal{B}$. Using the definition of $\mathcal{B}$, this gives the first statement of the theorem.

To show the second statement, we note that the existence of a positive multiplier $M \in \mathcal{M}^+$ such that (20) holds true is sufficient for (18), thus the second statement of the theorem is true as well. □

We note that, if $s > 1$, then $\mathcal{M}^+$ may not be sufficiently rich, therefore the equivalence of (20) and (18) does not follow from Theorem 1.

In what follows we shall derive LMIs for the computation of matrices P, K and the parameters τ_i, ε_i, $i = 1, ..., s$. Introduce the notations $\rho_i = \tau_i^{-1}$, $\mu_i = \varepsilon_i^{-1}$ and

$$\begin{aligned} \overline{\rho} &= \text{diag}\left\{ \rho_1 I_{l_{p_1}}, ..., \rho_s I_{l_{p_s}} \right\}, \\ \overline{\overline{\rho}} &= \text{diag}\left\{ \rho_1 I_{l_{q_1}}, ..., \rho_s I_{l_{q_s}} \right\}, \\ \overline{\mu} &= \text{diag}\left\{ \mu_1 I_{l_{p_1}}, ..., \mu_s I_{l_{p_s}} \right\}, \\ \overline{\overline{\mu}} &= \text{diag}\left\{ \mu_1 I_{l_{q_1}}, ..., \mu_s I_{l_{q_s}} \right\}, \end{aligned} \tag{21}$$

where τ_i, and ε_i are positive constants, $i = 1, ..., s$.

Theorem 3 Let system (1)-(3) be considered in continuous time. Inequality (17) with (16) holds true for $P = P^T > 0$, K, $\tau_i > 0$, $\varepsilon_i > 0$, $i = 1, ..., s$ if and only if $W = P^{-1}$, $Y = KP^{-1}$, $\overline{\rho}$, $\overline{\overline{\rho}}$, $\overline{\mu}$ and $\overline{\overline{\mu}}$ satisfy the following LMI:

$$\begin{pmatrix} \psi_{11} & * & * & * & * & * & * & * \\ E^T & -S_L & * & * & * & * & * & * \\ \psi_{31} & S_0 E_q & \psi_{33} & * & * & * & * & * \\ 0 & 0 & \overline{\rho} & -\overline{\mu} & * & * & * & * \\ C_q W & E_q & D_q \overline{\rho} & 0 & -\overline{\overline{\mu}} & * & * & * \\ \psi_{61} & \psi_{62} & \psi_{63} & 0 & 0 & -\overline{\overline{\rho}} & * & * \\ Y & 0 & 0 & 0 & 0 & 0 & -R_L^{-1} & * \\ W & 0 & 0 & 0 & 0 & 0 & 0 & -\overline{Q}_L \end{pmatrix} < 0, \tag{22}$$

with

$$\begin{aligned} \psi_{11} &= WA^T + AW^T + Y^T B^T + BY, \\ \psi_{31} &= \overline{\rho} H^T + S_0 C_q W, \\ \psi_{33} &= Q_0 \overline{\rho} + S_0 D_q \overline{\rho} + \overline{\rho} D_q^T S_0^T, \\ \psi_{61} &= R_0^{1/2} C_q W, \ \ \psi_{62} = R_0^{1/2} E_q, \ \ \psi_{63} = R_0^{1/2} D_q \overline{\rho}. \end{aligned}$$

Proof Consider inequality (17) and multiply from left and right the middle blockdiagonal matrix by $LL = I$, where

$$L = \begin{pmatrix} I & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \end{pmatrix}.$$

We obtain that

$$(*) \begin{pmatrix} 0 & P & 0 & 0 & 0 & 0 \\ P & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \underline{\tau} Q_0 + \underline{\varepsilon} & 0 & 0 & \underline{\tau} S_0 \\ 0 & 0 & 0 & -S_L & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{Q} & 0 \\ 0 & 0 & S_0^T \underline{\tau} & 0 & 0 & \underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \end{pmatrix} \begin{pmatrix} I & 0 & 0 \\ \mathcal{A} & E & H \\ 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \\ C_q & E_q & D_q \end{pmatrix} < 0. \qquad (23)$$

Applying the linearization lemma (see [27]) for (23) one arrives at

$$\begin{pmatrix} \phi_{11} & * & * & * & * \\ E^T P & -S_L & * & * & * \\ \phi_{31} & \underline{\tau} S_0 E_q & \phi_{33} + \underline{\varepsilon} & * & * \\ I & 0 & 0 & -\overline{Q}^{-1} & * \\ C_q & E_q & D_q & 0 & -(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}})^{-1} \end{pmatrix} < 0, \qquad (24)$$

where

$$\begin{aligned} \phi_{11} &= \mathcal{A}^T P + P\mathcal{A}, \\ \phi_{31} &= H^T P + \underline{\tau} S_0 C_q, \\ \phi_{33} &= \underline{\tau} Q_0 + \underline{\tau} S_0 D_q + D_q^T S_0^T \underline{\tau}. \end{aligned}$$

By Schur complement, one obtains from (24) the following inequality:

$$\begin{pmatrix} \phi_{11} + \overline{Q} & * & * & * \\ E^T P & -S_L & * & * \\ \phi_{31} & \underline{\tau} S_0 E_q & \phi_{33} + \underline{\varepsilon} & * \\ C_q & E_q & D_q & -(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}})^{-1} \end{pmatrix} < 0. \qquad (25)$$

Let us apply the Schur complement again for (25):

$$\begin{pmatrix} \phi_{11} + \overline{Q} & * & * \\ E^T P & -S_L & * \\ \phi_{31} & \underline{\tau} S_0 E_q & \phi_{33} \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ I \end{pmatrix} \underline{\varepsilon} \, (*) + (*) \left(\underline{\underline{\tau}} R_0 + \underline{\underline{\varepsilon}} \right) \begin{pmatrix} C_q & E_q & D_q \end{pmatrix} < 0. \qquad (26)$$

Inequality (26) is equivalent to

$$\begin{pmatrix} \phi_{11} + Q_L & * & * \\ E^T P & -S_L & * \\ \phi_{31} & \underline{\tau} S_0 E_q & \phi_{33} \end{pmatrix} + \begin{pmatrix} K^T \\ 0 \\ 0 \end{pmatrix} R_L \, (*) + (*) \, \mathrm{diag}\{\underline{\varepsilon}; \underline{\underline{\varepsilon}}; \underline{\underline{\tau}}\} \begin{pmatrix} 0 & 0 & I \\ C_q & E_q & D_q \\ R_0^{1/2} C_q & R_0^{1/2} E_q & R_0^{1/2} D_q \end{pmatrix} + < 0. \qquad (27)$$

By Schur complement again, one obtains that

$$\begin{pmatrix} \phi_{11} + Q_L & * & * & * & * & * & * \\ E^T P & -S_L & * & * & * & * & * \\ \phi_{31} & \underline{\tau} S_0 E_q & \phi_{33} & * & * & * & * \\ 0 & 0 & I & -\underline{\varepsilon}^{-1} & * & * & * \\ C_q & E_q & D_q & 0 & -\underline{\underline{\varepsilon}}^{-1} & * & * \\ R_0^{1/2} C_q & R_0^{1/2} E_q & R_0^{1/2} D_q & 0 & 0 & -\underline{\underline{\tau}}^{-1} & * \\ K & 0 & 0 & 0 & 0 & 0 & -R_L^{-1} \end{pmatrix} < 0. \qquad (28)$$

Finally, inequality (22) is obtained from (28) by the congruence transformation $\mathrm{diag}\{P^{-1}; I; \underline{\tau}^{-1} I; I; I; I\}$. □

Analogous considerations provide the results for the discrete-time case, therefore we present the next theorem without proof for the lack of space.

Theorem 4 Let system (1)-(3) be considered in discrete-time. Inequality (17) with (16) holds true for $P = P^T > 0$, K, τ_i, ε_i, $i = 1, ..., s$ if and only if matrices $W = P^{-1}$, $Y = KP^{-1}$, $\overline{\rho}$, $\overline{\overline{\rho}}$, $\overline{\mu}$ and $\overline{\overline{\mu}}$ satisfy the following LMI:

$$\begin{pmatrix} -W & * & * & * & * & * & * & * & * \\ \varphi_{21} & -W & * & * & * & * & * & * & * \\ 0 & E^T & -S_L & * & * & * & * & * & * \\ \varphi_{41} & \overline{\rho} H^T & S_0 E_q & \varphi_{44} & * & * & * & * & * \\ 0 & 0 & 0 & \overline{\rho} & -\overline{\mu} & * & * & * & * \\ C_q W & 0 & E_q & D_q \overline{\rho} & 0 & -\overline{\overline{\mu}} & * & * & * \\ \varphi_{71} & 0 & \varphi_{73} & \varphi_{74} & 0 & 0 & -\overline{\overline{\rho}} & * & * \\ Y & 0 & 0 & 0 & 0 & 0 & 0 & -R_L^{-1} & * \\ W & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_L \end{pmatrix} < 0, \qquad (29)$$

with

$$\begin{aligned} &\varphi_{21} = AW + BY, \\ &\varphi_{41} = S_0 C_q W, \quad \varphi_{44} = Q_0 \overline{\rho} + S_0 D_q \overline{\rho} + \overline{\rho} D_q^T S_0^T, \\ &\varphi_{71} = R_0^{1/2} C_q W, \quad \varphi_{73} = R_0^{1/2} E_q, \quad \varphi_{74} = R_0^{1/2} D_q \overline{\rho}. \end{aligned}$$

Remark 4 If LMIs (22) and (29) have solutions with respect to the unknowns W, Y, $\overline{\rho}$, $\overline{\overline{\rho}}$, $\overline{\mu}$ and $\overline{\overline{\mu}}$, then from a feasible solution, the matrices P, K and the constants τ_i and ε_i $i = 1, ..., s$ can be determined, thus the guaranteed cost control problem is reduced to the solution of LMIs both in the continuous and in the discrete-time case. If $s = 1$, and these LMIs have no solution, one can conclude that the formulated guaranteed cost control problem has no solution at all.

Remark 5 Theorem 2 together with Theorem 3 or Theorem 4 reduces the guaranteed cost control problem for system (1)-(3) to the solution of an LMI. A similar methodology can be used in different problems when an inequality for a quadratic function over a constrained set has to be satisfied. These results show that the solution of guaranteed cost (or H_∞ or dissipative) control problems with nonlinear system dynamics can be reduced to LMIs, if a suitable uncertain output and a suitable constraint set Ω can be given. The key point in this respect is that Condition 1 has to be satisfied for the 'abstract' formulation of the problem.

Remark 6 Each feasible solution of (22) and (29) determines an appropriate guaranteeing cost control with the appropriate guaranteed cost. However, one should achieve as low upper bound on the guaranteed cost as possible. This can be obtained by choosing an appropriate objective function for the LMIs (22) and (29). Introduce a new scalar variable ω, add a new inequality

$$\omega I < W \tag{30}$$

to the LMIs (22) and (29) and find the maximum of ω under the constraints given by the inequalities. This assures that the maximum eigenvalue of P, therefore the maximum of the guaranteed cost over the unit sphere is minimal.

V. A NUMERICAL EXAMPLE

To illustrate the results, consider system (1) in continuous time with

$$A = \begin{pmatrix} -1 & 0.5 \\ -0.5 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 1 \end{pmatrix},$$
$$H = \begin{pmatrix} 2 \\ 2 \end{pmatrix}, \qquad C_q = (2;1),$$
$$D_q = -1, \qquad E = 0.01 I_2.$$

Set furthermore $Q_L = I_2$, $R_L = 1$, $S_L = I_2$ in (5), and consider $Q_0 = 0$, $R_0 = 0$, $S_0 = 1$ in the definition of Ω, for which (15) is satisfied. For each pair p and $q = C_q x + D_q p$,

$$\begin{pmatrix} p \\ q \end{pmatrix} \in \Omega,$$

i.e. the inequality

$$q^T p \geq 0 \tag{31}$$

holds if and only if $(C_q x + D_q p)p \geq 0$, i.e. $C_q x p \geq p^2$. By completing the square we obtain that

$$-\left|\frac{C_q x}{2}\right| + \frac{C_q x}{2} \leq p \leq \left|\frac{C_q x}{2}\right| + \frac{C_q x}{2}.$$

For simulation, a nonlinearity of the form

$$p = \begin{cases} \frac{1}{2} C_q x + \frac{1}{2}\left|C_q x\right| \sin(C_q x)^{-1}, & \text{if} \quad C_q x \neq 0, \\ 0, & \text{if} \quad C_q x = 0 \end{cases}$$

satisfying this condition has been taken. (Note that this nonlinearity is non-Lipschitzian.) The solution of the LMI system (22) and (30) provides the guaranteeing cost feedback

$$K = (-1.7445; -6.4793),$$

where ω was maximized as proposed in Remark 6. Suppose that $w^T(t) = (t+10)^{-2}(1;1)$ $(t = 0, 1, \ldots)$. Simulation results with several initial points are given in Figure 1, where the solid lines represent the corresponding trajectories. For the sake of comparison, another simulation is also presented in the figure, where the feedback $K = (0; -2)$ rendering the nominal (linear) system to be exponentially stable was applied starting from the same initial points. These trajectories represented by dashed lines blow up very rapidly. This does not contradict to Theorem 4 of [14], because this nonlinearity cannot be given in the form of (7) of [14].

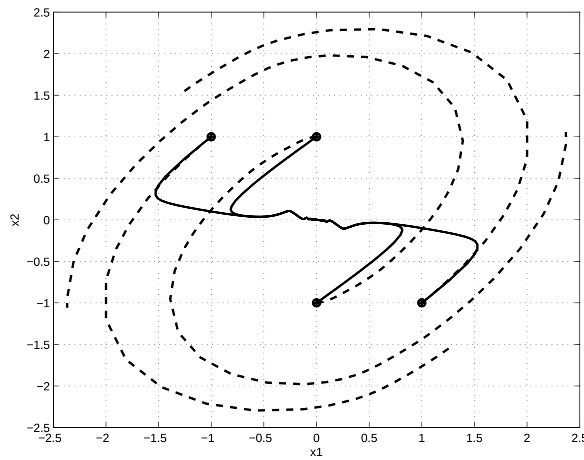

Fig. 1. Behavior of the trajectories.

VI. CONCLUSIONS

The paper deals with the determination of guaranteeing cost controls for systems with quadratically constrained uncertainties/nonlinearities. The problem is solved by an abstract matrix multiplier method. The matrix multiplier methods developed earlier cannot be applied to the considered type of systems. The so called full-block S-procedure technique of [26] cannot be applied, because the set of system uncertainty/nonlinearity is not a subspace. The results of [2] are not applicable, either, because the inequalities to be investigated are different from those, for which the results of [2] are valid. The proposed method requires only that the uncertainties should be described as a family of closed cones parametrized by the elements of a compact set. Both structured and unstructured uncertainties/nonlinearities are considered. In the unstructured case, necessary and sufficient conditions for controls to be cost guaranteeing are formulated via LMIs, and cost guaranteeing linear feedback controls can be determined by solving these LMIs, if the set of solutions is feasible. The results are based on the concept of the so-called sufficiently rich set of positive multipliers. This concept is analogous to that given by [2]. The presented conditions are sufficient also in the structured case. The main results of the paper can be applied to all robustness problems, where the systems can be formulated as considered in the paper by an appropriate choice of the system parameters. For example, the results generalize those developed for systems with positive real uncertainty. A numerical example is presented, where the system nonlinearity is not Lipschitzian.

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