Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Robust Stabilization with Real Parametric Uncertainty via Linear Programming

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***Abstract*— A numerical method is proposed for optimal robust control synthesis. A dual interpretation of the problem is derived. In the special case, when uncertainty parameter is real-valued, it is shown that the dual problem becomes essentially finite dimensional in the space of variables (semi-infinite convex programming). It makes possible to efficiently obtain a numerical solution of the dual problem and to construct the optimal robustly stabilizing controller via the alignment principle. In order to illustrate the method, several examples on the robust stabilization with real uncertainties are solved (both analytically and numerically).**

I. Introduction

Large number of analysis and synthesis problems in robust control were stated in terms of convex optimization. In particular, in [4] it was shown that the robust stabilization problem under the parametric uncertainties has convex formulation if the characteristic polynomial depends linearly on the uncertainty parameters.

Consider the uncertainty as an artificial feedback loop

$$G_\delta = \left\{ \begin{array}{c} \begin{pmatrix} y \\ z \end{pmatrix} = G \begin{pmatrix} w \\ u \end{pmatrix} \\ w = \nu \delta^\top z, \end{array} \right. ,$$

where G is the nominal plant, w is the scalar input and δ is the uncertain vector. The objective is to robustly stabilize the plant for all δ satisfying the norm bound $|\delta|_{pr} \leq 1$, where $|\cdot|_{pr}$ stands for a vector norm in $\mathbb{R}^m$ (pr for *primary*).

All closed-loop transfer functions from w to z are of the form

$$T_{zw} = G_1 + G_2 Q,$$

where Q is stable and T_1, T_2 are determined by G. The condition for robust stability becomes

$$[1 + \nu\delta^\top (G_1 + G_2 Q)]^{-1} \in \mathbf{RH}^\infty, \forall \delta : |\delta|_{pr} \leq 1.$$

A convex parametrization of all robustly stabilizing controllers was constructed in [4]. We slightly modify the result and state the theorem as follows.

Theorem 1. *Suppose $G_1 \in \mathbf{RH}^\infty_{m\times 1}$, $G_2 \in \mathbf{RH}^\infty_{m\times n}$ and $\nu > 0$ is given. Then the following two conditions on the rational matrix Q are equivalent:*

1) $Q \in \mathbf{RH}^\infty_{n\times 1}$ *and for all* $\delta \in \mathbb{R}^m$ *with* $|\delta|_{pr} \leq 1$
$$[1 + \nu\delta^\top (G_1 + G_2 Q)]^{-1} \in \mathbf{RH}^\infty.$$

2) *There exist* $\alpha \in \mathbf{RH}^\infty$ *and* $\beta \in \mathbf{RH}^\infty$ *such that* $Q = \beta/\alpha$ *and* $\forall \omega \in \mathbb{R} \cup \{\infty\}$
$$|Re[G_1\alpha + G_2\beta](j\omega)|_{du} < \nu^{-1} Re\, \alpha(j\omega).$$

Here the dual norm $|\cdot|_{du}$ *defined as* $|x|_{du} = \sup\{\delta^\top x : \ |\delta|_{pr} \leq 1\}$ *(du for dual).*

This work is supported by the Swedish Research Council, Project 2005-3911

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In [2] it was shown that we can allow the large set $\mathbf{H}^\infty$, i.e. $\alpha,\ \beta \in \mathbf{H}^\infty$. Denote by $F = \begin{pmatrix} 1 & 0 \end{pmatrix}$, by $G = \begin{pmatrix} G_1 & G_2 \end{pmatrix}$ and by $h = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Due to Theorem 1 and the result from [2] the problem of finding a controller is equivalent to the following condition in terms of a function $h \in \mathbf{H}^\infty$,

$$Re\,(F + \nu\delta^\top G(z))h(z) > 0,\ \forall z \in \mathbb{T},\ \forall \delta \in \Delta. \quad (1)$$

In [3] it has been developed the algorithm that solves the problem (1) for the maximal possible ν. The algorithm is the combination of two finite-dimensional approximations of the primal and dual infinite-dimensional problems. In this paper we discuss the case when the uncertainty is real-valued and we obtain the finite dimensional solution of the problem (1).

The paper is organized as follows. Section II introduces all major notation used throughout the paper. The dual form of the problem is presented in section III. In section IV we discuss the special case, when the uncertainty vector is real-valued. The numerical example is considered in section V.

II. Notations

By $\mathbb{R}$ (or $\mathbb{C}$) we denote the field of real (or complex) numbers. The unit circle and the open unit disc in $\mathbb{C}$ are denoted by $\mathbb{T}$ respectively $\mathbb{D}$

$$\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\},\ \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}.$$

Let $Y \subset \mathbb{C}^n$ be any measurable set and $1 \leq p \leq \infty$. Denote by $L^p(Y)$ the standard Lebesgue space of functions $f : \mathbb{T} \to Y$ equipped with the norm

$$\|f\|_p = \left\{ \begin{array}{ll} (\int_{\mathbb{T}} |f(z)|^p dm(z))^{\frac{1}{p}}, & 1 \leq p < +\infty, \\ \operatorname{ess\,sup}_{z\in\mathbb{T}} |f(z)|, & p = +\infty \end{array} \right.$$

where $|\cdot|$ denotes the usual 2-norm in $\mathbb{C}^n$

$$|f| = \sqrt{|f_1|^2 + |f_2|^2 + \ldots + |f_n|^2}.$$

The Hardy class $\mathbf{H}^p(Y)$ consists of functions analytic in $\mathbb{D}$ and such that

$$\|f\|_p = \sup_{0<r<1} (\int_{\mathbb{T}} |f(rz)|^p dm(z))^{\frac{1}{p}} < \infty.$$

The Hardy class $\mathbf{H}^\infty$ is the space of bounded analytic functions in $\mathbb{D}$ with norm

$$\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|.$$

A function θ analytic in $\mathbb{D}$ is called an *inner function* if $\theta \in \mathbf{H}^\infty$ and

$$|\theta(z)| = 1 \text{ for almost all } z \in \mathbb{T}.$$

A function h analytic in $\mathbb{D}$ is called an *outer function* if there exists a real function $g \in \mathbf{L}^1$ and a complex number c of modulus 1 such that

$$h(\lambda) = c\exp\left(\int_{\mathbb{T}} \frac{z+\lambda}{z-\lambda} g(z)dm(z)\right), \ \ \lambda \in \mathbb{D}$$

If $f \in \mathbf{H}^p$, then f admits the representation $f = \theta h$, where θ is an inner function and h is an outer function in $\mathbf{H}^p$.

We define the *Blaschke product* as follows. For $\lambda \in \mathbb{D}$ we put

$$b_\lambda(z) = \frac{|\lambda|}{\lambda}\,\frac{\lambda - z}{1-\bar{\lambda}z}, \ \ \lambda \neq 0, \ \ and \ \ b_0(z) = z.$$

Let $\{\lambda_i\}_{i\geq 0}$ be a sequence in $\mathbb{D}$ satisfying the *Blaschke condition*

$$\sum_{i\geq 0}(1-|\lambda_i|) < \infty.$$

Let c be a complex number of modulus 1. Then the product

$$B(z) = c\prod_{i\geq 0} b_{\lambda_i}(z)$$

converges for all $z \in \mathbb{D}$ and is not identically equal to 0. Function B is called a *Blaschke product*.

Let $\mathbf{H}_0^p(Y)$ denote

$$\mathbf{H}_0^p(Y) = z\mathbf{H}^p(Y) = \{f \in \mathbf{H}^p(Y) | f(0) = 0\}.$$

The disk algebra $\mathbf{A}(Y)$ is by definition the subspace of $\mathbf{H}^\infty$ that consists of analytic functions in $\mathbb{D} \subset Y$ that can be extended continuously to the closed unit disk.

The set $\mathbf{RH}^\infty$ is the set of all functions from $\mathbf{H}^\infty$ that are rational with reals coefficient.

The brief notations $\mathbf{A}$, $\mathbf{H}^\infty$ etc. will be used if $Y = \mathbb{C}^n$ and the dimension of the space is clear from context.

The superscript $^\top$ stands for transposition and † stands for pseudoinverse. The bar denotes the complex conjugate and * denotes conjugate transpose. The prefix $\mathcal{B}$ denotes the unit ball in the corresponding space, and $\mathcal{S}$ is the unit sphere.

III. Preliminaries

The problem of finding $h \in \mathbf{H}^\infty$ such that condition (1) holds is considered as a *primal problem*. We would like to solve the problem for ν as large as possible. Denote this largest ν as ν_{opt}.

The problem was stated in [2] as follows

$$\gamma_{opt}(\nu) = \sup_{h\in\mathcal{B}\mathbf{H}^\infty} ess\inf_{z\in\mathbb{T}}\inf_{\delta\in\Delta} \mathrm{Re}\,(F+\nu\delta^\top G(z))h(z).$$

It is immediate that $(1) \Leftrightarrow \gamma_{opt}(\nu) > 0$.

We choose the unit ball in $\mathbf{H}^\infty$ as an optimization set since the function

$$\mathrm{Re}\,(F+\nu\delta^\top G(z))h(z)$$

depends linearly on h and then any bounded set containing the origin as an interior point can be chosen. However, the unit ball has an easier interpretation from the classical results and is more appropriate for our task.

The duality result was obtained in [2].

Theorem 2. *We have*

$$\gamma_{opt}(\nu) = \inf_{\substack{w\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)\\ \delta\in\mathbf{L}^\infty(\Delta)}} \ \inf_{p\in\mathbf{H}_0^1} \|(F+\nu\delta^\top G)w - p\|_1.$$

Using Theorem 2 we state the dual problem as follows.

Dual problem. Given $F = (\ 1 \ \ 0\ )$, $G = (\ G_1 \ \ G_2\ )$, $\nu > 0$ and a convex compact set Δ, find, if possible, a sequence of functions $\{(w_i,\,\delta_i,\,p_i)\}$ such that $w_i \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)$, $\delta_i \in \mathbf{L}^\infty(\Delta)$, $p_i \in \mathbf{H}_0^1$, and

$$\|(F+\nu\delta_i^\top G)w_i - p_i\| \to 0, \text{ as } i \to \infty.$$

The dual problem can be split into two parts: regular and singular ones.

Theorem 3. *Let $F \in \mathbf{A}^{1\times n}$, $G \in \mathbf{A}^{m\times n}$ and denote $\Phi_\delta = F + \nu\delta^\top G$. Then the optimal value ν_{opt} takes the following form*

$$\nu_{opt} = \min\{\nu_{opt|c}, \nu_{opt|s}\}$$

with the regular part

$$\nu_{opt|c} = \inf\{\nu | \exists w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)\backslash 0, \delta \in \mathbf{L}^\infty(\Delta)\colon \Phi_\delta w \in \mathbf{H}_0^1\} \quad (2)$$

and the singular part

$$\nu_{opt|s} = \inf\{\nu | \exists z \in \mathbb{T}, \ \delta \in \Delta : \Phi_\delta(z) = 0\}. \quad (3)$$

IV. Optimization of the uncertainty bound via dual problem for a real-valued uncertainty.

The primal problem is infinite dimensional. In [3] it was proposed a numerical method to solve the problem by successive finite-dimensional approximation. A feasible solution of the primal problem gives a lower bound ν to the optimal value ν_{opt}. However, the algorithm is unable to indicate if there is no solution for a given ν since at each step we solve a finite-dimensional approximation. In order to obtain the upper bound to the optimal value ν_{opt} we use the duality result (Theorem 3).

Calculation of an upper bound for $\nu_{opt|s}$ can be organized as a finite-dimensional convex programming at each z. But for $\nu_{opt|c}$ it becomes infinite-dimensional. In [3] it was proposed the dual algorithm to estimate $\nu_{opt|c}$. In this section we discuss the solution of the dual problem for the real-valued uncertainty. It turns out that in this case the dual problem becomes a finite-dimensional in the space of variables (semi-infinite convex programming).

Let us start with a slight relaxation of the considered optimization set. Recall that

$$\gamma_{opt}(\nu) = \sup_{h\in\mathcal{B}\mathbf{H}^\infty} \inf_{z\in\mathbb{T}} \inf_{\delta\in\Delta} \operatorname{Re}\left(\alpha + \nu\delta^\top(G_1\alpha + G_2\beta)\right).$$

Note that
$\inf_{\delta\in\Delta} \operatorname{Re}\left(F + \nu\delta^\top G(z)\right)h(z) =$
$\operatorname{Re} Fh(z) - \nu \sup_{\delta\in-\Delta} \operatorname{Re}\delta^\top G(z)h(z)$.
For $\Delta \ni 0$ we get $\sup_{\delta\in\Delta} \operatorname{Re}\delta^\top G(z)h(z) \geq 0$ for all z. It means that instead of the condition $|h| \leq 1$ we can demand that $|Fh| \leq 1$ for $\gamma_{opt} < +\infty$. Furthermore it turns out that it is sufficient to demand $\alpha(0) = 1$.

We denote by $\mathbf{H}^\infty_1$ the subspace of $\mathbf{H}^\infty$ which contains the analytical functions h such that $h(0) = 1$, i.e.

$$\mathbf{H}^\infty_1 = \{h :\ h \in \mathbf{H}^\infty,\ h(0) = 1\}.$$

Now we use the class $\mathbf{H}^\infty_1$ as an optimization set.

Denote by

$$\tilde{\gamma}_{opt} := \sup_{\substack{\alpha\in\mathbf{H}^\infty_1\\ \beta\in\mathbf{H}^\infty}} ess\inf_{z\in\mathbb{T}} \inf_{\delta\in\Delta} \operatorname{Re}\left(\alpha + \nu\delta^\top(G_1\alpha + G_2\beta)\right).$$

The new set will not lead to infinite values of $\tilde{\gamma}_{opt}$. Indeed, α is an analytic function and according to the mean value theorem

$$\inf_{z\in\mathbb{T}} \operatorname{Re}\alpha(z) \leq \operatorname{Re}\alpha(0).$$

Then

$$\tilde{\gamma}_{opt} \leq \sup_{\substack{\alpha\in\mathbf{H}^\infty_1\\ \beta\in\mathbf{H}^\infty}} ess\inf_{z\in\mathbb{T}} \operatorname{Re}\alpha \leq 1.$$

It is clear that $\gamma_{opt}(\nu_{opt}) = \tilde{\gamma}_{opt}(\nu_{opt}) = 0$. However the hyperplane $\mathbf{H}^\infty_1$ allows us to obtain a stronger dual condition on the pair (δ, w). In particular, it gives a finite-dimensional dual variable in case of real-valued uncertainty δ. Furthermore, if $\gamma_{opt}(\nu) \geq 0$ for all ν, the new $\tilde{\gamma}_{opt}(\nu)$ takes both positive and negative values, which makes the numerical search for the optimal ν_{opt} (as the solution to $\tilde{\gamma}_{opt} = 0$) smoother.

Now we present the duality result

Theorem 4. *We have*

$$\tilde{\gamma}_{opt}(\nu) = \inf_{\delta,w} \mathit{Re}\, p(0),$$

where $p = (1 + \nu\delta^\top G_1)w \in \mathbf{H}^1$, $w \in \mathcal{S}\mathbf{L}^1(\mathbb{R}_+)$, $\delta \in \mathbf{L}^\infty(\Delta)$ *and* $\delta^\top G_2 w \in \mathbf{H}^1_0$.

Proof: See Appendix.

According to the Theorem 4, $\delta^\top G_2 w \in \mathbf{H}^1_0$ and $(1 + \nu\delta^\top G_1)w \in \mathbf{H}^1$. Therefore we have the following conditions on the functions δ and w

$$\begin{array}{l}(1 + \nu\delta^\top G_1)w = p_1\\ \delta^\top G_2 w = zp_2,\end{array} \tag{4}$$

where $p_1, p_2 \in \mathbf{H}^1$.

Now we consider the case when δ is a real-valued vector. We show that in this case the dual problem becomes finite dimensional in space of variables (semi-definite convex programming). The main idea is the following. According to conditions (4) a function $zp_2G_2^\dagger$ has to be real since δ, w are real-valued. Therefore every entry of p_2 has to be finite dimensional. Since w is real-valued, p_1 has a finite dimension. Furthermore $\delta^\top G_2 w$ is an analytic function. Therefore all unstable poles of $\delta_i^\top w$ have to be canceled with zeros of G_2. It turns out that dimensions of p_1 and p_2 are closely related to the number of unstable zeros in G_2. Let us show that.

As $G_2 \in \mathbf{A}$, hence we can factorize G_2 as $G_2 = G_{2_i}G_{2_o}$, where G_{2_i} is an inner factor and G_{2_o} is an outer factor. Then

$$\delta^\top wG_{2_i}G_{2_o} = zp_2.$$

We get

$$\begin{aligned}\delta^\top wG_{2_i} &= zp_2G_{2_o}^{-1},\\ \delta^\top wG_{2_i} &= zp_2' := zp_2,\\ \delta^\top wG_{2_i}G_{2_i}^* &= zp_2G_{2_i}^*.\end{aligned}$$

As function G_2 is from the disk algebra $\mathbf{A}$ and without zeros on $\mathbb{T}$ (in the regular case), then the inner function G_{2_i} is a Blashke-Potapov product. It means that G_{2_i} can be written in the following way

$$G_{2_i} = U\prod_{j=1}^k \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & \frac{\lambda_j - z}{1-\overline{\lambda_j}z}\mathbb{I}_{q_j} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix},$$

where $r_j + q_j + s_j = n$ and U is a unitary matrix. Then

$$\begin{aligned} zp_2G_{2_i}^* &= zp_2\prod_{j=1}^k \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & -\frac{\overline{\lambda_j} - \overline{z}}{1-\lambda_j\overline{z}}\mathbb{I}_{q_j} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*\\ &= zp_2\prod_{j=1}^k \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & -\frac{z^{-1}(1-\overline{\lambda_j}z)^2}{|z-\lambda_j|^2}\mathbb{I}_{q_j} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*. \end{aligned}$$

Let $p_2^j = \frac{\hat{p}_2^j}{(1-\overline{\lambda_j}z)^2}$, where $\hat{p}_2^j$ is a polynomial. Then we have

$$\begin{aligned} \delta^\top wG_{2_i}G_{2_i}^* =& zp_2\prod_{j=1}^k \begin{pmatrix} \mathbb{I}_{r_j} & \mathbb{O} & \mathbb{O}\\ \mathbb{O} & -\frac{z^{-1}(1-\overline{\lambda_j}z)^2}{|z-\lambda_j|^2}\mathbb{I}_{q_j} & \mathbb{O}\\ \mathbb{O} & \mathbb{O} & \mathbb{I}_{s_j}, \end{pmatrix} U^*\\ =& \frac{zz^{-k}}{\prod_{j=1}^k |z-\lambda_j|^2}\begin{pmatrix} q_1 & \ldots & q_n \end{pmatrix} U^*, \end{aligned}$$

where

$$q_i = z^{k_i}\prod_{j=1}^k (|z-\lambda_j|^2)^{s_i}\hat{p}_2^i.$$

As δ and w are real-valued, then every $\hat{p}_2^j$ has to be such that the following equality is satisfied

$$\begin{aligned}&\frac{z^{1-k}}{\prod_{j=1}^k |z-\lambda_j|^2}(z^{k_1}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_1}\hat{p}_2^1\dots\\&\qquad z^{k_n}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_n}\hat{p}_2^n)U^* =\\&\frac{z^{k-1}}{\prod_{j=1}^k |z-\lambda_j|^2}(z^{-k_1}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_1}\overline{\hat{p}_2^1}\dots\\&\qquad z^{-k_n}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_n}\overline{\hat{p}_2}^n)U^\top.\end{aligned} \tag{5}$$

It means that $\hat{p}_2^j$ are polynomials of degree $\leq 2(k-1)$. Now put the expression of the $\delta^\top w$ into the first equation in (4)

$$\begin{aligned}w &= p_1 - \nu\delta^\top w G_1\\&= p_1 - \nu\frac{z^{1-k}}{\prod_{j=1}^k|z-\lambda_j|^2}(q_1 \quad \dots \quad q_n)U^*(G_{2_i}G_{2_i}^*)^{-1}G_1\\&= \frac{z^{-k}\hat{p}_1\prod(z-\lambda_j) - \nu z^{1-k}(q_1\dots q_n)U^*(G_{2_i}G_{2_i}^*)^{-1}G_1}{\prod_{j=1}^k|z-\lambda_j|^2},\end{aligned}$$

where $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k(1-\overline{\lambda_j}z)}$ and $\hat{p}_1$ is a polynomial. As w is real-valued, then $\hat{p}_1$ has to be such that the following equality is satisfied

$$\begin{aligned}&z^{-k}\hat{p}_1\prod(z-\lambda_j) - \nu z^{1-k}(\ q_1 \quad \dots \quad q_n\ )U^*G_1 =\\&z^k\overline{\hat{p}_1}\prod(\overline{z-\lambda_j}) - \nu z^{k-1}(\ q_1 \quad \dots \quad q_n\ )U^\top\overline{G_1}.\end{aligned} \tag{6}$$

Therefore $\hat{p}_1$ is a polynomial of degree $\leq \max(k, \max(\deg G_{1_j}) + k - 1)$.

As $\int_{\mathbb{T}} w dm = 1$, then we have one more condition on p_1 and p_2:

$$\int_{\mathbb{T}} p_1 - \nu z p_2 U^{-1}(G_{2_i}G_{2_i}^*)^{-1}T_1 dm = 1 \tag{7}$$

According to the discussion above we formulate the main result as follows.

Main result: Let the uncertainties vector δ be real-valued. Given the uncertainty bound ν, solve the following problem

$$\min \operatorname{Re} p_1(0). \tag{8}$$

Here $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k(1-\overline{\lambda_j}z)}$ is such that $\hat{p}_1$ is a polynomial, $\deg \hat{p}_1 \leq \max(k, \max(\deg G_{1_j}) + k - 1)$, and satisfies equation (6),

k is a number of zeros λ_j of function $\det B_2$, where B_2 is such that $G_{2_i} = UB_2$

$\hat{p}_2$ is a polynomial, $\deg \hat{p}_2 \leq 2(k-1)$, satisfying the equation (5),

the functions $p_1 = \frac{\hat{p}_1}{\prod_{j=1}^k(1-\overline{\lambda_j}z)}$ and $p_2 = \frac{z^{-k}}{\prod_{j=1}^k|z-\lambda_j|^2}(z^{k_1}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_1}\hat{p}_2^1\dots z^{k_n}\prod_{j=1}^k(|z-\lambda_j|^2)^{s_n}\hat{p}_2^n)$ satisfy the equation (7).

We can solve the problem (8) numerically by the semi-infinite convex programming. Then we can construct the primal optimal solution via the alignment principle formulated in [1]: *the optimal controller is equal to the inverted plant with the optimal uncertainty strategy*. Since we have a zeros/poles cancelation in case of optimal δ, the optimal controller will be of low order. We will show how we solve the problem (8) using a numerical example.

V. Numerical example: Robust stabilization, real-valued uncertainty.

In this section we solve the robust stabilization problem for the system with real-valued uncertainty by the dual method. The problem is formulated as follows.

Given uncertain plant P

$$\begin{pmatrix} y \\ z \end{pmatrix} = P\begin{pmatrix} w \\ u \end{pmatrix},$$

$$w = \nu\delta^\top z,$$

where $\delta \in \Delta$ for some convex compact $\Delta \ni 0$, the problem is to find a controller $u = Ky$ that robustly stabilizes the plant for ν as a large as possible. In case when w is a scalar input, the Youla parametrization leads to the following equivalent problem: find a function $Q \in \mathbf{RH}^\infty$ that maximizes ν such that

$$1 + \nu\delta^\top(T_1(z) + T_2(z)Q(z)) \neq 0\, \forall z \in \mathbb{T},\, \forall\delta \in \Delta.$$

By Theorem 1 the problem can be reduced to the form (1). In our case we take $\Delta = [-1, 1]$, i.e. the uncertainty vector δ is real-valued.

Let $T_1 = z^2 + 1.5z + 0.7$ and $T_2 = z^2 + \frac{1}{2}$. As T_1/T_2 is not in $\mathbf{H}^\infty$, then the solution to the optimization problem is not trivial.

Denote $R = \prod_{j=1}^m(z-\lambda_j)$, where $\{\lambda_j\}$ is the set of all zeros of T_2. In our case $R = z^2 + \frac{1}{2}$. Using the main result (see section IV) we have

$$\begin{aligned}&\delta w R = z p_2,\\&\delta w R R^* = z p_2 R^*,\\&\varphi(z) := \delta w|R|^2 = z p_2 R^* = z p_2 z^{-2}(1 + \frac{1}{2}z^2) = z^{-1}\hat{p}_2,\end{aligned}$$

where $p_2 = \frac{\hat{p}_2}{1+\frac{1}{2}z^2}$.

The function $\varphi(z) = z^{-1}\hat{p}_2$ has to be real, i.e. $z^{-1}\hat{p}_2 = z\overline{\hat{p}_2}$. It means that $\varphi(z)$ is a quasi-polynomial of degree 1 with symmetric coefficients, i.e.

$$\varphi(z) = c_0 z^{-1} + c_1 + c_0 z$$

and

$$\delta w = \frac{c_0 z^{-1} + c_1 + c_0 z}{|R|^2}.$$

Now

$$w = \frac{p_1|R|^2 - \nu\varphi T_1}{|R|^2} = \frac{\hat{p}_1 z^{-2}R - \nu\varphi T_1}{|R|^2} = \frac{\psi}{|R|^2},$$

where $p_1 = \frac{\hat{p}_1}{1+\frac{1}{2}z^2}$. The function $\psi = \hat{p}_1 z^{-2}R - \nu\varphi T_1$ is real. Therefore $\psi(z)$ is a quasi-polynomial of degree 2 with symmetric coefficients, i.e.

$$\psi(z) = d_0 z^{-2} + d_1 z^{-1} + d_2 + d_1 z + d_0 z^2.$$

Now our task is to minimize $\mathrm{Re}\, p_1(0)$. Using the calculations above we obtain that

$$p_1(z) = \frac{\hat{p}_1(z)}{1+\frac{1}{2}z^2} = \frac{(\psi + \nu\varphi T_1)z^2}{R(1+\frac{1}{2}z^2)}$$

and

$$p_1(0) = 2d_0.$$

The quasi-polynomials φ and ψ have to satisfy the following conditions

$$\begin{aligned} &1)\quad |\varphi| \le \psi \\ &2)\quad \int_{\mathbb{T}} \frac{\psi}{|R|^2} dm = 1 \\ &3)\quad \psi(\frac{-i}{\sqrt{2}}) + \nu\varphi T_1((\frac{-i}{\sqrt{2}})) = 0 \\ &4)\quad \psi(\frac{i}{\sqrt{2}}) + \nu\varphi T_1((\frac{i}{\sqrt{2}})) = 0 \end{aligned} \tag{9}$$

The first two conditions in (9) are due to the conditions on δ and w, i.e. $|\delta w| \le w$ and $\int_{\mathbb{T}} w dm = 1$. The last two conditions came from the fact that p_1 has to be an analytic function, i.e. all unstable poles need to be canceled.

As the problem is still infinite-dimensional on z, then we must consider a finite grid of points on $\mathbb{T}$. We get the finite-dimensional linear program

$$\min_X \; fX \text{ subject to}$$

$$A_{12}X \le 0$$

$$A_{22}X = 1$$

$$A_{32}X = 0,$$

where vector $X = (\, c_0 \;\; c_1 \;\; d_0 \;\; d_1 \;\; d_2 \,)$ absorbs the coefficients of the functions φ and ψ and $f = (\, 0 \;\; 0 \;\; 1 \;\; 0 \;\; 0 \,)$.

We run the algorithm in the linear programming form for different values of ν. We stop the optimization at $\nu = 2.5812$ when $\tilde{\gamma}(\nu) = 6.907e - 006$. We calculate the optimal uncertainty

$$\delta_{opt} = \frac{46341.8z(z^2+0.1181z+1)}{(z+2702)(z+34.27)(z+0.02918)(z+0.0003701)}.$$

Using the alignment principle, we get the following optimal controller

$$Q_{opt} = \frac{z^2+1.618z+1.354}{z^2+0.1181z+1}.$$

Note that the controller is of lower order.

The plot of the closed-loop pole map is shown in Figure (1).

Now we will consider a more complicated T_1, i.e. $T_1 = \frac{z^5+3z^4+2z^3+4z^2+5z+3}{z^3-z^2-4z+12}$ and solve the same robust stabilization problem. First we try to find the solution of the primal problem by using the algorithm in [3]. We stop at $\nu = 10$ when $\gamma = 4.0828e-005$. The order of optimization reached is 19×88 and the optimal controller is of order 19. However, the dual algorithm gives

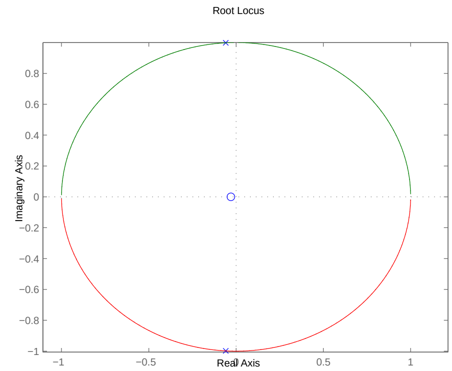

Fig. 1. The closed-loop pole map

us the optimal bound $\nu = 10.1784$ when $\tilde{\gamma} = 1.7776e-006$ and the optimal controller has a lower order

$$Q_{opt} = \frac{-(z+2.60)(z^2+1.55z+1.23)(z^2-0.98z+1.81)}{(z+2.51)(z^2+0.18z+1)(z^2-3.50z+4.79)}$$

The plot of the closed-loop pole map is shown in Figure (2).

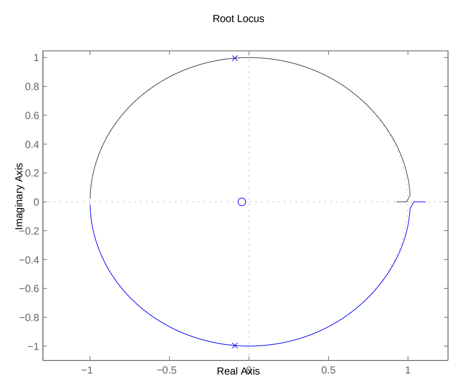

Fig. 2. The closed-loop pole map

VI. Conclusions and Future Works.

In this paper we presented the Linear Programming algorithm which solves the problem of robustly stabilizing controller design when the uncertainty parameter appears linearly in the closed-loop characteristic polynomial (rank-one problem). The initial problem is infinite-dimensional. The algorithm was derived as finite-dimensional approximation of the initial problem. We discussed the special case when an uncertainty vector is real-valued. In this case we showed that estimating of the uncertainty bound ν became the optimization problem on a finite-dimensional parameter. It should be interesting to develop the algorithm for a matrix uncertainty.

Appendix

A. Proof of Theorem 4

Proof: In [2] it was shown that

$$ess\inf_{z\in\mathbb{T}}\inf_{\delta\in\Delta}\varphi_\delta(z)=\inf_{w\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)}\inf_{\delta\in\mathbf{L}^\infty(\Delta)}\int_{\mathbb{T}}\varphi_\delta(z)wdm,$$

where $\varphi_\delta(z)=\mathrm{Re}\,(\alpha(z)+\nu\delta^\top(G_1(z)\alpha(z)+G_2(z)\beta(z)))$. Thus

$$\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}} ess\inf_{z\in\mathbb{T}}\inf_{\delta\in\Delta}\mathrm{Re}\,(\alpha+\nu\delta^\top(G_1\alpha+G_2\beta))=$$

$$\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}}\inf_{w,\delta}\mathrm{Re}\int_{\mathbb{T}}(\alpha+\nu\delta^\top(G_1\alpha+G_2\beta))wdm,$$

where $w\in S\mathbf{L}^1(\mathbb{R}_+)$ and $\delta\in\mathbf{L}^\infty(\Delta)$.

Now we introduce two measures $d\mu=w\,dm$ and $d\xi=\delta^\top d\mu$ and define a set of measures M as follows:

$$M=\left\{\begin{pmatrix}\mu\\ \xi\end{pmatrix}:\ \begin{array}{l}\mu \text{ is a Borel probability measure on } \mathbb{T},\\ \xi\ll\mu,\ \frac{d\xi^T}{d\mu}\in\Delta.\end{array}\right\}.$$

Then

$$\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}}\inf_{w,\delta}\mathrm{Re}\int_{\mathbb{T}}(\alpha+\nu\delta^\top(G_1\alpha+G_2\beta))wdm=$$

$$\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}}\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M}\mathrm{Re}\int_{\mathbb{T}}(\alpha d\mu+\nu d\xi(G_1\alpha+G_2\beta)),$$

.

The set $(\mathbf{H}_1^\infty,\mathbf{H}^\infty)$ is convex.

The set M is convex and *weakly compact since Δ is a convex compact set.

The function $\mathrm{Re}\int_{\mathbb{T}}(\alpha d\mu+\nu d\xi(G_1\alpha+G_2\beta))$ is concave on $h=\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$, convex and continuous on M.

By the Ky Fan's min-max theorem (see [5]), the order of $\sup$ and $\inf$ can be interchanged

$$\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}}\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M}\mathrm{Re}\int_{\mathbb{T}}(\alpha d\mu+\nu d\xi(G_1\alpha+G_2\beta))=$$

$$\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M}\sup_{\substack{\alpha\in\mathbf{H}_1^\infty\\ \beta\in\mathbf{H}^\infty}}\mathrm{Re}\int_{\mathbb{T}}(\alpha d\mu+\nu d\xi(G_1\alpha+G_2\beta)).$$

We denote $df_1(\mu,\xi,\nu)=d\mu+\nu d\xi G_1$ and $df_2(\xi)=d\xi G_2$. Then

$$\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M}\sup_{\alpha\in\mathbf{H}_1^\infty,\beta\in\mathbf{H}^\infty}\mathrm{Re}\int_{\mathbb{T}}(\alpha d\mu+\nu d\xi(G_1\alpha+G_2\beta))=$$

$$\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M}(\sup_{\alpha\in\mathbf{H}_1^\infty}\mathrm{Re}\int_{\mathbb{T}}df_1(\mu,\xi,\nu)\alpha+$$

$$\nu\sup_{\beta\in\mathbf{H}^\infty}\mathrm{Re}\int_{\mathbb{T}}df_2(\xi)\beta)=\tilde{\gamma}_{opt}(\nu).$$

It follows that $df_2\in\mathbf{H}_0^1\,dm$ as otherwise, by choosing an appropriate $\beta\in\mathbf{H}^\infty$ we can get infinity in the second term. By the same reason, we obtain that $df_1=p\,dm$, $p\in\mathbf{H}^1$. Therefore $\int_{\mathbb{T}}df_2(\xi)\beta=0$, $\forall\beta\in\mathbf{H}^\infty$ and $\int_{\mathbb{T}}df_1(\mu,\xi,\nu)\alpha=p(0)$, $\forall\alpha\in\mathbf{H}_1^\infty$. We get finally that

$$\tilde{\gamma}_{opt}(\nu)=\inf_{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M_1}\mathrm{Re}\,p(0)$$

where by M_1 we denote the set of measures defined as follows:

$$M_1=\left\{\begin{pmatrix}\mu\\ \xi\end{pmatrix}\in M:\ \begin{array}{l}d\mu+\nu d\xi G_1=p\,dm,\ p\in\mathbf{H}^1,\\ d\xi G_2\in\mathbf{H}_0^1\,dm\end{array}\right\}.$$

Approximating the set M_1 by absolutely continuous measures with respect to dm we get

$$\tilde{\gamma}_{opt}(\nu)=\inf_{w,\delta}\mathrm{Re}\,p(0)$$

where $w\in\mathcal{S}\mathbf{L}^1(\mathbb{R}_+)$, $\delta\in\mathbf{L}^\infty(\Delta)$, $\delta^\top G_2w\in\mathbf{H}_0^1$ and $(1+\nu\delta^\top G_1)w=p\in\mathbf{H}^1$.

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