Anti-Palindromic Pencil Formulation for Open-Loop Stackelberg Strategy in Discrete-Time

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Abstract— **The Stackelberg strategy offers, in gametheoretic framework an adapted concept to obtain equilibrium for hierarchical games. For linear-quadratic games, Stackelberg strategy with open-loop information structure leads to solve non-symmetric Riccati equations, by assumming the invertibility of some weighting matrices. This paper provides an anti-palindromic pencil approach to formulate it. Thus it allows to relaxe invertibility assumptions and furthermore to take advantage of the recent literature of numerical methods for palindromic pencils.**

Index Terms— **Palindromic pencil; Game theory; Nonsymmetric Riccati equations; Stackelberg strategy.**

I. Introduction

The Stackelberg strategy from game theory [1–3], called in honor of the economist H. Stackelberg who introduced this concept [4] offers a framework for control design for systems where there exists a hierarchy between control inputs (the players). That is, in the case of two-player nonzero-sum game, there is a *leader* and a *follower*. The leader is able to enforce his control on the follower.

A strategy could be obtained only when the information structure is precised. By considering an open-loop information structure, the current state is not available and the players are commited to follow a predetermine control law. The Stackelberg strategy with an open-loop information structure has been widely studied [5, 6], particularly for the linear-quadratic case, where the necessary conditions are solved by nonsymmetric Riccati equations [7], obtained by the invariant subspaces of a characteristic matrix, which is symplectic in discrete-time [8] and Hamiltonian in continuous-time [9]. Such a characteristic matrix implies a symmetry for its eigenspectrum [10] and thus some helpful theoretic and numerical properties. A symplectic pencil approach was proposed in [11] for the open-loop information structure, under invertibility assumptions of some weighting matrices.

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Such a symmetry is also encountered for some matrix polynomials, called palindromic or anti-palindromic [12, 13]. The properties associated with this structure allow to obtain numerical methods which reduce the ill-conditioning of eigenvalue problems [14, 15]. Several applications could benefit of this palindromic pencil approach, like mechanical vibrations [16, 17], or Linear-Quadratic problem [18].

The aim of this paper is to provide an anti-palindromic pencil approach which improves the symplectic pencil one for solving open-loop Stackelberg strategy applied on linear-quadratic games. This paper is organized as follows. In Section II, the problem is stated. Section IV is dedicated to the main result, before a conclusion in Section V.

II. Problem formulation

Consider a discrete-time, two-player non-zero game associating the linear dynamic

$$x(k+1) = Ax(k) + B_1u_1(k) + B_2u_2(k), \quad x(0) = x_0, \qquad (1)$$

where $x(k) \in \mathbb{R}^n$, $u_1(k) \in \mathbb{R}^{r_1}$ and $u_2(k) \in \mathbb{R}^{r_2}$ are respectively the state, the control of the follower and of the leader. A criterion, or cost function is associated with each player $i \in \{1,2\}$ and defined on a finite-time horizon by

$$J_i(u_1,u_2) = \frac{1}{2}x'(N)K_{iN}x(N) + \sum_{k=0}^{N-1} L_i(x(k),u_1(k),u_2(k)), \qquad (2)$$

where

$$L_i(x,u_1,u_2) = \frac{1}{2}\left(x'Q_ix + u_1'R_{i1}u_1 + u_2'R_{i2}u_2\right). \qquad (3)$$

All weighting matrices in the criterion J_i are constant and symmetric with $Q_i \geq 0$, $K_{if} \geq 0$, $R_{12} \geq 0$, $R_{21} > 0$ and $R_{ii} > 0$. $R_{ii} > 0$ denotes the convexity of J_i with respect to the control u_i and $R_{21} > 0$ allows the leader to obtain information about the follower's control. For a sake of clarity, we denote $S_i = B_iR_{ii}^{-1}B_i^T$, $S_{ij} = B_jR_{jj}^{-1}R_{ij}R_{jj}^{-1}B_j^T$, $\forall (i,j) \in \{1,2\}^2$.

Let define the rational reaction set of the follower

$$\mathscr{R}_1(u) = \left\{\tilde{u}_1 \in \mathscr{U}_{ad,1} \;\middle|\; J_1(\tilde{u}_1,u) \leq J_1(u_1,u), \forall u_1 \in \mathscr{U}_{ad,1}\right\}. \qquad (4)$$

$\mathscr{U}_{ad,i}$ denotes the set of admissible controls $u_i(k)$ for the player i.

The Stackelberg strategy (u_1^*, u_2^*) is thus defined by

$$\begin{cases} u_1^* \in \mathscr{R}_1(u_2^*) & \text{and} \\ \forall u_2 \in \mathscr{U}_{ad,2} & \max\limits_{u_1 \in \mathscr{R}_1(u_2^*)} J_2(u_1,u_2^*) \leq \max\limits_{u_1 \in \mathscr{R}_1(u_2)} J_2(u_1,u_2). \end{cases} \qquad (5)$$

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Note that the definition (5) emphasizes the hierarchical and not obvious structure of the game. Actually the control of the leader is a solution of an optimization problem constrained by an other optimization problem related to the follower.

In this paper, we are interested in the framework of open-loop information structure, that is no measurement of the state of the system is available and the players are committed to follow a predetermined strategy based on their knowledge of the initial state, the system's model and the criteria.

The necessary condition for obtaining open-loop Stackelberg strategies are well known in the literature [5, 6, 19] and are based on an Hamiltonian approach for the leader and the follower. The open-loop Stackelberg strategy is associated with the following Popov-system:

$$\begin{cases} \sigma x &= Ax + B_1 u_1 + B_2 u_2, \\ \sigma\gamma &= A\gamma + B_1 u_3, \\ \lambda_1 &= Q_1 x + A'\sigma\lambda_1, \\ \lambda_2 &= Q_2 x + Q_1\gamma + A'\sigma\lambda_2, \\ 0 &= (B_1)'\sigma\lambda_1 + R_{11}u_1, \\ 0 &= (B_2)'\sigma\lambda_2 + R_{22}u_2, \\ 0 &= (B_1)'\sigma\lambda_2 + R_{21}u_1 + R_{11}u_3, \end{cases} \tag{6}$$

where σ is the time-shift operator defined by acting on a vector valued sequence $w = (w)_{k\in\mathbb{Z}}$, defined as:

$$(\sigma w)(k) = w(k+1); \tag{7}$$

λ_1 and λ_2 are the costate related to the dynamical constraint and γ is the costate related to the hierarchical position of the leader. The control u_3 is a fictitious player. In the finite-time horizon case, the transversality conditions are given as follows:

$$x(0) = x_0; \quad \gamma(0) = 0 \tag{8}$$

and

$$\lambda_1(N) = K_{1N}x(N); \quad \lambda_2(N) = K_{2N}x(N) - K_{1N}\gamma(N). \tag{9}$$

The transversality conditions consisting of conditions at the initial time (8) and at the final time (9), such a problem is called *two-point boundary value problem*, which is generally solved via a numerical method: the shooting method [20]. In the sequel of the paper, we investigate the framework of games with infinite-time horizon, that is we take the limit $N \to +\infty$. Thus the system is still associated with the Popov-system (6), but not anymore with the transversality conditions (8) and (9).

The following section will present several definitions of matrix pencils in order to allow the main result in Section IV.

III. Definitions related to matrix pencils

The matrix pencil [21, 22] approach is theoretically and numerically suitable in automatic control and is particularly well adapted to solve standard algebraic Riccati equations [23, 24].

Definition 1 (see [21]): For $(n_1, n_2) \in \mathbb{N}^* \times \mathbb{N}^*$, let M, $N \in \mathbb{R}^{n_1\times n_2}$. The first order polynomial matrix in the indeterminate z belonging to $\mathbb{C}$

$$zM - N \tag{10}$$

is called *matrix pencil* associated with the matrices M and N. The pencil $zM - N$ is said *regular* if and only if the matrices M and N are square ($n_1 = n_2$) and $\det(zM - N)$ is not trivial. In all other cases ($n_1 \neq n_2$ or $\det(zM - N) \equiv 0$), the pencil is said to be *singular*.

Definition 2: Let consider the matrix pencil $zM - N$, with $M, N \in \mathbb{R}^{n_1\times n_2}$. The *normal rank* of the matrix pencil $zM - N$, $r_{(M,N)}$ is the value of $\text{rank}(zM - N)$ for almost all $z \in \mathbb{C}$. In the case of regular matrix pencil, by definition $r_{(M,N)} = n_1 = n_2$. In the case of singular matrix pencil, at least one of the inequalities $r_{(M,N)} < n_1$ or $r_{(M,N)} < n_2$ holds.

Definition 3: The set

$$\Lambda_f(M,N) = \left\{ z \in \mathbb{C} \mid \text{rank}(zM - N) < r_{(M,N)} \right\} \tag{11}$$

is called the *finite spectrum* of the matrix pencil $zM - N$. When the matrix pencil $zM - N$ is regular, the finite spectrum is given by

$$\Lambda_f(M,N) = \{ z \in \mathbb{C} \mid \det(zM - N) = 0 \}. \tag{12}$$

The finite spectrum with multiplicity counted will be noted $\Lambda_f^*(M,N)$. In addition of these finite eigenvalues, the matrix pencil $zM - N$ could have infinite ones gathered into $\Lambda_\infty^*(M,N)$ (multiplicity counted), corresponding to null eigenvalues of the reciprocal matrix pencil $M - zN$. The spectrum of the matrix pencil is the union $\Lambda(M,N) = \Lambda_f^*(M,N) \cup \Lambda_\infty^*(M,N)$. Each (finite or infinite) element of $\Lambda(M,N)$ is called a *generalized eigenvalue* of the matrix pencil $zM - N$. Note that the spectrum of a matrix pencil is a set which could be empty, finite or infinite.

Definition 4: Consider a square pencil with an even dimension, that is $n_1 = n_2 = 2q$, then the pencil $zM - N$ is called *symplectic*, if the relation holds

$$M\mathscr{J}_q M' = N\mathscr{J}_q N', \tag{13}$$

where $\mathscr{J}_q = \begin{bmatrix} 0_q & I_q \\ -I_q & 0_q \end{bmatrix}$.

Definition 5: The pencil $zM - N$ is called *palindromic*, if the relation holds $M = -N'$, that is the pencil has the following structure

$$zM - N = zM + M'. \tag{14}$$

The pencil $zM - N$ is called *anti-palindromic*, if $M = N'$, that is it has the structure

$$zM - N = zM - M'. \tag{15}$$

The eigenspectrum of a symplectic pencil exhibits a symmetry [10]: if λ is an eigenvalue of a symplectic pencil, this is also the case for $\frac{1}{\lambda}$. For real valued symplectic pencil, the eigenspectrum can be decomposed into quadruplets $(\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}})$. The palindromic and anti-palindromic pencils are characterized also by a symmetry of their eigenspectrum: if λ is an eigenvalue of such a pencil, $\frac{1}{\lambda}$ is also an eigenvalue [13]. For real valued (anti-) palindromic pencils, one gets the same decomposition as for a symplectic pencil with quadruplets $(\lambda, \bar{\lambda}, \frac{1}{\lambda}, \frac{1}{\bar{\lambda}})$. Such a similar eigenspectrum property induces a strong link between these both pencil structures.

The Popov-system can be rewritten into:

$$\left(\sigma \begin{bmatrix} 0 & I & 0 \\ \tilde{A}' & 0 & 0 \\ \tilde{B}' & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{A} & \tilde{B} \\ I & -\tilde{Q} & 0 \\ 0 & 0 & -\tilde{R} \end{bmatrix}\right) \begin{bmatrix} \lambda_2 \\ \lambda_1 \\ x \\ \gamma \\ u_3 \\ u_2 \\ u_1 \end{bmatrix} (k), \tag{16}$$

with

$$\tilde{A} = \text{diag}(A;A); \; \tilde{B} = \begin{bmatrix} 0 & B_2 & B_1 \\ B_1 & 0 & 0 \end{bmatrix}; \tag{17}$$

$$\tilde{Q} = \begin{bmatrix} Q_2 & Q_1 \\ Q_1 & 0 \end{bmatrix}; \; \tilde{R} = \begin{bmatrix} 0 & 0 & R_{11} \\ 0 & R_{22} & 0 \\ R_{11} & 0 & R_{21} \end{bmatrix}. \tag{18}$$

Remark 1: It is noteworthy that the matrix $\tilde{R}$ is symmetric, by considering the extended input vector $\begin{bmatrix} u_3 & u_2 & u_1 \end{bmatrix}$ instead of $\begin{bmatrix} u_1 & u_2 & u_3 \end{bmatrix}$, as in the literature [25]. Furthermore $\tilde{R}$ is invertible if and only if R_{11} and R_{22} are invertible.

The relation (16) induces that the solution of open-loop Stackelberg game, with infinite-time horizon, can be obtained via the computation of disconjugate deflating subspaces [26] of the pencil

$$z \begin{bmatrix} 0 & I & 0 \\ \tilde{A}' & 0 & 0 \\ \tilde{B}' & 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & \tilde{A} & \tilde{B} \\ I & -\tilde{Q} & 0 \\ 0 & 0 & -\tilde{R} \end{bmatrix}. \tag{19}$$

The pencil (19) does not exhibit an intuitive eigenspectrum symmetry. Nevertheless, as presented in [11], when $\tilde{R}$ is invertible, solving Stackelberg strategy is strongly related to the symplectic pencil:

$$z \begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{Q} & I \end{bmatrix} - \begin{bmatrix} I & \tilde{B}(\tilde{R})^{-1}\tilde{B}' \\ 0 & \tilde{A}' \end{bmatrix}. \tag{20}$$

Furthermore, when $\tilde{R}$ and $\tilde{A}$ are invertible, the disconjugate deflating subspace of the pencil (20) can be reformulated as the disconjugate invariant subspace [9] of the symplectic matrix

$$\begin{bmatrix} I & \tilde{B}(\tilde{R})^{-1}\tilde{B}' \\ 0 & \tilde{A}' \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{Q} & I \end{bmatrix} \tag{21}$$

and is related to Coupled Algebraic Riccati Equations (CARE) [7].

The main contribution of this paper is to provide a anti-palindromic pencil formulation for solving open-loop Stackelberg strategy, without assumption about the invertibility of the matrix $\tilde{R}$. The advantage of such a formulation is to preserve the obvious symplectic symmetry and to allow dedicated numerical tools.

IV. MAIN RESULT

Theorem 1: The solutions of Stackelberg strategy with open-loop information structure are given by the deflating subspaces of the anti-palindromic pencil

$$z\tilde{M} - \tilde{M}', \tag{22}$$

with

$$\tilde{M} = \begin{bmatrix} 0 & \tilde{A} & \tilde{B} \\ I & -\tilde{Q} & 0 \\ 0 & 0 & -\tilde{R} \end{bmatrix}. \tag{23}$$

Proof: From the Popov-system (6), let us consider the change of variables

$$\mu_1(k) = \lambda_1(k) - \lambda_1(k+1), \tag{24}$$

$$\mu_2(k) = \lambda_2(k) - \lambda_2(k+1). \tag{25}$$

The two first equations of (6) are not modified. In order to make appear the new variables μ_1 and μ_2, we substract the third equation at time $k+1$ to the same equation at time k. One gets

$$-Q_1 x + \mu_1 = -Q_1 \sigma x + A' \sigma \mu_1. \tag{26}$$

By the same way on the fourth equation, we obtain

$$-Q_2 x - Q_1 \gamma + \mu_2 = -Q_2 \sigma x - Q_1 \sigma \gamma + A' \sigma \mu_2. \tag{27}$$

The same substraction on the three last equations of (6) leads to

$$(B_1)' \sigma \mu_1 - R_{11} \sigma u_1 = -R_{11} u_1, \tag{28}$$

$$(B_2)' \sigma \mu_2 - R_{22} \sigma u_2 = -R_{22} u_2, \tag{29}$$

$$(B_1)' \sigma \mu_2 - R_{21} \sigma u_1 - R_{11} \sigma u_3 = -R_{21} u_1 - R_{11} u_3. \tag{30}$$

These equalities can be reformulated into the following system

$$\left(\sigma \begin{bmatrix} 0 & I & 0 \\ \tilde{A}' & -\tilde{Q} & 0 \\ \tilde{B}' & 0 & -\tilde{R} \end{bmatrix} - \begin{bmatrix} 0 & \tilde{A} & \tilde{B} \\ I & -\tilde{Q} & 0 \\ 0 & 0 & -\tilde{R} \end{bmatrix}\right) \begin{bmatrix} \mu_2 \\ \mu_1 \\ x \\ \gamma \\ u_3 \\ u_2 \\ u_1 \end{bmatrix} (k), \tag{31}$$

which is associated to the anti-palindromic pencil (22). ∎

The eigenvalue problem with a palindromic structure arises in different active fields, as for examples: vibrations of fast trains [16, 17] or time-delay systems [27]. A rich literature offers several numerical tools these last years, which take into account the palindromic structure to solve such an eigenproblem. We can cite for instance, Jacobi-like algorithm [15], the structure Schur form for pencils [27], QZ form and structure-preserving doubling algorithms [16, 17, 28] or QR form [14, 29] and references therein.

V. Conclusion

An anti-palindromic pencil formulation was provided to give, via the recent literature associated with, a numerical tool for solving Stackelberg strategy with an open-loop information structure. This approach allows in addition to relaxe the inverbility assumption of weighting matrices which is widespread in the literature.

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