Emergence of Lévy flights in Distributed Consensus Systems

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***Abstract*— In this paper, we consider a multi-agent model which generates a collective super-diffusion behavior. Although such complex behaviors are ubiquitous in many natural and human-made systems, discovering mechanisms of their emergence is mostly an open research area. Our model is motivated to solve consensus problem under limitations on information exchange including link failures and additive noise. We use orthogonal decomposition approach to analyze the system and establish several equivalent necessary and sufficient conditions for Mean Square (MS) stability of part of this system. We show that the emergence of the super-diffusion behavior is introduced by the loss of MS stability and prove it to be Lévy flights for a special system. This work is the first, to the best of our knowledge, to establish the intimate relationship between propagation of uncertainties in networks, the MS stability robustness and the emergence of Lévy flights, which may have far reaching consequence on the understanding and engineering of complex systems.**

Keywords: Distributed consensus, Lévy flights, Link failures, Additive noise, Distributed averaging.

I. Introduction

Lévy flights are a particular generalized class of random walks in which the increments during the walk obey the power law distribution with scaling factor in certain range. Lévy flights correspond to large fluctuation (called super-diffusion) behaviors and have been observed in many natural sciences as well as economics and many other fields. Examples include biological searching patterns [1], [2], the distribution of human travel [3], finantial series of stock markets [4], [5] and more recently, photons in hot atomic vapors [6]. In the context of engineering, the light performs Lévy flights random walk has been generated for potential applications in medical imaging, random lasing and image reconstruction [7]. Although Lévy flights have been reported in various fields, the universal accepted explanation for such complex behaviors has not been found.

In this paper, we consider a multi-agent model which generates a collective super-diffusion behavior[1] in the sense that each agent behaves as a Lévy flight. Our model is inspired by natural behaviors — swarming and flocking, and motivated from the study of distributed coordination in networked control systems. This is the first time that the origin of the collective Lévy flights behavior is identified in a simple but natural multi-agent linear system in the presence of fading coupling and imprecise information exchange. As many natural and human-made systems exhibit fluctuation behaviors collectively, we hope the analysis of our model could bring some insights into the mechanisms of their emergence.

Our work is originated from the study of distributed consensus (agreement) problems. Generally speaking, distributed consensus means that all the nodes (agents) of a networked system agree to a common value, referred as their collective decision, based only on local information exchange. Distributed consensus has attracted much interest in computer science, control and signal processing communities as it has many application areas, such as load balancing, clock synchronization, distributed sensor fusion, distributed optimization, social networks and multi-agents coordination, etc.

One common feature of networked system is inter-agent communication. It is well known that the uncertainties on information exchange, such as fading connections and additive noise, will bring detrimental effect to the system or even the loss of stability. Therefore, it would be of practical value to understand the limitations and robustness of distributed consensus model under these communication constraints. Distributed consensus in the presence of stochastic fading connections has been investigated in [8]. In that paper, the authors consider symmetric switchings and provide analysis of the convergence speed of the system for small probability of link failure. Distributed averaging in the presence of additive noise has been studied in [9], where in that paper, the authors showed that consensus algorithm is sensitive to additive noise and focused on designing the algorithm to minimize the steady state total mean square deviation of the individual variables from their average. [10] considers formation problems in the presence of fading connections and additive noise, and observe the power law distribution in terms of the difference between the states of two agents when noise is added in the loop. Our work is also related with recent studies on distributed consensus over networks of time varying interconnections or random networks, see *e.g.*, [11], [12], [13], [14] to name a few, where different notions of consensus in terms of different assumptions on network connectivity are established.

In contrast to these works, we consider how fading connections and additive noise *together* affect distributed consensus and observe the emergence of Lévy flights. We assume that each agent is a simple adder who accumulates the weighted sum of the values which are the differences between the agent and her neighbors. The natural and simple nature

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[1]In this paper, the super-diffusion behavior (or complex behavior) refers to the particular discrete-time stochastic process whose increments converge to a random variable that has unbounded variance and thus a heavy tailed distribution.

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of the model ensures that it can be easily implemented without memory for each agent and more importantly, when only *relative* measurements are available (*e.g.* the case in formation problems [16]).

To understand the emergence of the collective super-diffusion behaviors, we provide analysis of the convergence property of our model. We construct the state space representation of the system and decompose the system into two subsystems called the deviated system and the conserved system. The deviated system allows us to capture evolution of the system in the directions that are orthogonal to the direction of all ones, while the conserved system remains unchanged in the mean sense. We establish the equivalence relationship between MS stability of the deviated system and MS consensus of the original system. We then derive several equivalent necessary and sufficient conditions for MS stability. Using special coordinate transformations, we show that the emergence of complex behaviors is introduced by the loss of MS stability and verify them as Levy flights for a special system.

The main contribution of the paper is that we identify a collective super-diffusion behavior, ubiquitous in many natural and human-made systems, as the system behavior of a simple linear coupled multi-agent system, elucidate the mechanism for its emergence and provide necessary and sufficient conditions to check it. On the other hand, the paper also contributes to solve distributed consensus problems under constraints on information exchange.

The paper is organized as follows. In section II, we introduce the model and clarify the statistical assumptions about its paramters. In section III, we analyze the model without considering additive noise and establish conditions for MS stability. We prove that the collective super-diffusion behavior emerges if and only if the networked system is mean square unstable and prove it to be Levy flights for a special system. In section V, we provide an example to illustrate the collective Levy flights behavior and verify our theoretical results. Finally, we provide some concluding remarks and point out some future research directions.

II. The Model

We consider a set of n identical Linear Time Invariant (LTI) discrete time systems called agents (or nodes), which are connected over a network. The index set of agents is thus given by $\mathcal{V} = \{1, \ldots, n\}$. We use $x_i \in \mathbb{R}$ to denote the state of agent i, where x_i may represent physical quantities such as speed, position, temperature, *etc*. Our multi-agent model is motivated to solve distributed agreement problems, *i.e.*, the agents' *objective* is to cooperatively reach to a common value or stay relatively close to each other via a sequence of local information exchange (with neighboring nodes only). Moreover, we consider limitations on information exchange including link failures and additive noise. Specifically, at each integer valued time index k, we assume each agent has the unreliable information from its neighbors and updates its own state according to the following iteration:

$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} \xi_{ij}(k)[x_j(k) - x_i(k)] + v_i(k), \quad (1)$$

where N_i denotes the set of all possible neighbors of node i, $\beta > 0$ is a constant called the update gain which denotes the coupling between two agents. Here, $\xi_{ij}(k)$ is a Bernoulli random variable characterizing the fading property of the channel e_{ij} at time k. We assume $\xi_{ij}(k)$ are independent across both the time index k and spatial index i, j, and identically distributed for each k. The distribution of ξ_{ij} is given by:

$$\xi_{ij} = \begin{cases} 1 & \text{with Prob. } \mu_{ij} \\ 0 & \text{with Prob. } 1 - \mu_{ij} \end{cases}$$

Note that μ_{ij} denotes the probability that link (i, j) is connected. $v_i(k)$ are Gaussian random variables independently identically distributed across both k and i with zero mean and variance $\sigma^2_{v_i}$. We assume $v_i(k)$ are independent of $\xi_{ij}(k)$ for all i, j, k. Let $x(k) \in \mathbb{R}^n$ be the concatenation of $x_i(k)$, We assume that $x(0)$ is a random number independent of all $v(k)$ and $\xi_{ij}(k)$. In this paper, we adopt the following assumption which ensures that all links are not always connected or disconnected.

Assumption 2.1: We assume that $0 < \mu_{ij} < 1$ for all $i \in \mathcal{V}$ and $j \in N_i$.

In the model, we assume each agent is naive, *i.e.*, it does not have the ability to protect the message from the channel unreliable behavior using coding. The noise $v(k)$ shows that there is imperfect information exchange between agents, i.e., the message received by the node from her neighbor may be corrupted by some small value. Model (1) is inspired by nature – flocking and swarming behaviors, which represents many natural unsophisticated biological and social networked systems.

In (1), we assume all the possible neighbors of agent i are in the set N_i. From N_i, we can induce a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, called *conectivity graph* of the network, where each element of the link set $\mathcal{E}$) is given by (i, j) implying that there is a link from node j to node i. In this setup, the network has a fixed topology with random failing connections. On the other hand, our setup captures the randomness of the connectivity of the network. We can associate a graph $\mathcal{G}(k) = (\mathcal{V}, \mathcal{E}(k))$ with $\xi_{ij}(k)$ and $\mathcal{E}(k) = \{(i, j) | \xi_{ij}(k) > 0\}$. We also define the mean connectivity graph as $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ where $\tilde{\mathcal{E}} = \{(i, j) | \mathbf{E}(\xi_{ij}(k)) > 0\}$. The graph G and $\tilde{G}$ have the same topology as we avoid the trivial case $\mu_{ij} = 0$. We put the following mild assumption on the connectivity graph of the network.

Assumption 2.2: The connectivity graph $\mathcal{G}$ is strongly connected.

This assumption ensures in expectation, the information of each node can be obtained by every other node through a directed path. We will use this assumption in this paper without special claim.

We next provide the following definition about Lévy flights.

Definition 2.3: Let $X_1, X_2, \dots,$ be a sequence of random variables, the random process $\{X_i\}_{i=1}^{\infty}$ is called a Lévy flight if $X_k - X_{k-1}$ converges to a random variable R as $k \to \infty$ and $\lim_{t\to\infty} t^{\alpha}\Pr(|R| > t)$ is finite, where $1 < \alpha < 2$ is called the scaling factor.

Note that the slight difference between our definition of Lévy flights and the traditional definition of Lévy flights [18] is that we allow the increments of the random variables converge to a power law distribution instead of restricting that every increment should be a power law. Our definition takes care of the dynamic evolution of systems and still captures the super-diffusion behavior property, which we think may be more suitable to describe this scaling invariant behavior in natural and social dynamical systems.

III. Convergence Analysis of the Model

In this section, we provide convergence analysis of model (1) without additive noise. This not only lays the foundation to understand the mechanism for the emergence of complex behaviors, but also contributes to solving distributed consensus problem in the presence of link failures.

A. State Space Construction

Before we proceed to analyze the model, we introduce the following decomposition of graph Laplacian that will be used later. For any graph Laplacian L, it can be decomposed as

$$L = B \times C$$

where B likes a permutation matrix and each row of C corresponds to the components of L associated with each link. Let the total number of links be denoted by p, then B is of dimension $n \times p$ and C is of dimension $p \times n$. To illustrate this decomposition, consider the Laplacian

$$L = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

then $L = BC$, where B and C are given by

$$B = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}.$$

Now, we consider model (1) without additive noise, which is given by

$$x_i(k+1) = x_i(k) + \beta \sum_{j\in N_i} \xi_{ij}(k)[x_j(k) - x_i(k)]. \quad (2)$$

Let $\Delta_{ij}(k) = \xi_{ij}(k) - \mu_{ij}$, then $\Delta_{ij}(k)$ are independent across k and i, j with mean $\mathbf{E}(\Delta_{ij}(k)) = 0$ and variance $\sigma_{ij}^2 = \mu_{ij}(1-\mu_{ij})$. The above equations then can be written compactly as

$$x(k+1) = Ax(k) + B\Delta Cx(k) \quad (3)$$

with appropriate matrices A, B and C, where $x(k)$ is a concatenation of $x_i(k)$. Let p denote the total number of links of the connectivity graph, then Δ is a $p \times p$ diagonal matrix with Δ_{ij} on its diagonal. We would like to point out that the objective of this kind of decomposition is that it separates the zero mean uncertainties from the system, which allows us to use standard techniques to analyze the convergence property of the system.

The following Lemma provides structural properties of the matrices A, B and C, which are crucial in our further analysis of the model.

Lemma 3.1: For state equation (3), all row sums of A are equal to 1, all row sums of C are equal to 0 and B has full row rank.

We next provide a definition characterizing the convergence property of system (3).

Definition 3.2: (**MS consensus**) Let $x(k)$ be generated by (3). The networked system is said to achieve *MS consensus* if $\lim_{k\to\infty} \mathbf{E}(x_i(k) - x_j(k))^2 = 0$ for all $i \in \mathcal{V}$.

Remark 3.3: In the presence of random connectivity, MS consensus is a strong notion of consensus and implies almost sure consensus considered in [12], *i.e.*, $||x_i(k)-x_j(k)|| \to 0$ almost surely. To see this, suppose that $||x_i(k)-x_j(k)||$ does not converge to zero on a set of positive measure, then it's clear that $\mathbf{E}(x_i(k) - x_j(k))^2$ does not converge to zero.

B. System Decomposition

An explicit fact from the construction of the state space is that A always has an eigenvalue of 1 as all the row sums of A are 1, *i.e.*,$A\mathbf{1} = \mathbf{1}$. Let γ be the left eigenvector of A associated with the eigenvalue 1, *i.e.*, $\gamma' A = \gamma'$, and assume by normalization $\gamma'\mathbf{1} = 1$, we define $P := (I - \mathbf{1}\gamma')$. The matrix P has the following properties.

Lemma 3.4: Consider $P = I - \mathbf{1}\gamma'$, we have that

1) $P^2 = P$,
2) $PA = AP$,
3) $C_iP = C_i$,
4) $N(P) = \text{span}\{\mathbf{1}\}$.

where $N(P)$ denote the null space of P.

We next decompose the system state $x(k)$ into two states which induces the orthogonal decomposition of the system (3). The objective for this decomposition is that it splits the original system into two subsystems where the dynamics of one keeps unchanged in the mean sense, and the stability property of the other is intimately associated with the emergence of complex behaviors.

Definition 3.5: The deviated state is defined as $x_d(k) := Px(k)$ and the conserved state is defined as $x_c(k) := Fx(k)$, where $F = I - P = \mathbf{1}\gamma'$ and the subscripts stand for obvious meaning.

From the above definition, $x_d(k)$ measures how far each component of $x(k)$ is from the weighted average of all of them and $x_c(k)$ is the projection of $x(k)$ into the space span$\{\mathbf{1}\}$ with each component being the weighted average of $x_i(k)$. Note that $x = x_d + x_c$. Once we define the two set of states, the deviated system and the conserved system, with inherited meaning from their states, can respectively be obtained from (2) as

$$x_d(k+1) = PAPx_d(k) + PB\Delta Cx_d(k), \quad (4)$$

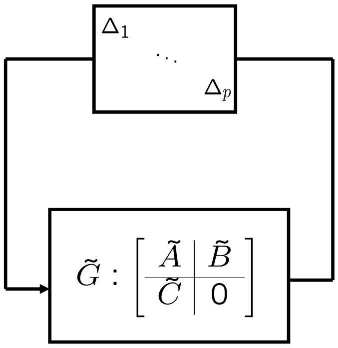


Fig. 1. The deviated system is represented as the interconnection between the structured uncertainty Δ and the deterministic plant $\tilde{G}$. This separation allows us to test the MS stability of the system using the input-output maps of the mean system $\tilde{G}$.

$$x_c(k+1) = x_c(k) + FB\Delta C x_d(k), \tag{5}$$

where in the above derivations, we use the results of Lemma 3.4. To explain the name for the conserved system, we take the expectation of (5), as Δ is independent of $x_d(k)$, we have $\mathbf{E}(x_c(k+1)) = \mathbf{E}(x_c(k))$, which explicitly shows that $\mathbf{E}(x_c(k))$ is a conserved quantity. For the deviated system, we leave it to be detailed explored in the next section.

C. Conditions for MS Stability

In this section, we shall focus on the second moment stability of the deviated system. The main reason for this analysis, as seen later, is that it lays the foundation for our further investigation of mechanisms of complex behaviors. Besides, the results in this section provide conditions to solve the distributed agreement problem in the presence of constraints on information exchange.

We start by providing definitions that characterize the convergence property of the deviated system. To this end, we first define

$$\begin{aligned} M(k) &:= \mathbf{E}\{x(k)x'(k)\}, \\ M_d(k) &:= \mathbf{E}\{x_d(k)x_d'(k)\}. \end{aligned}$$

Definition 3.6: System (4) is mean stable if $\lim_{k\to\infty} \mathbf{E}(x_d(k)) = 0$.

Definition 3.7: System (4) is MS stable if $M_d(k)$ is well defined for all k and $\lim_{k\to\infty} M_d(k) = 0$.

At this point, we are particularly interested in the relationship between MS consensus of the networked system and MS stability of the deviated system. Indeed, we show these two notions are equivalent to each other. This result, as we see in the next section, establishes the intrinsic connection between MS instability and the emergence of complex behaviors.

Theorem 3.8: The deviated system is MS stable if and only if the networked system achieves MS consensus (cf. Definition 3.2).

In the sequel, with a little abuse of terminology, we make no distinction between that the networked system being MS stable and the deviated system being MS stable.

We next seek the conditions under which system (4) is MS stable. Note that the deviated system (4) can also be written as

$$H: \quad \begin{aligned} x_d(k+1) &= \tilde{A}x_d(k) + \tilde{B}w(k), \\ z(k) &= \tilde{C}x_d(k), \\ w(k) &= \Delta z(k), \end{aligned} \tag{6}$$

where $\tilde{A} = PAP$, $\tilde{B} = PB$ and $\tilde{C} = CP$. Here H can be seen as the interconnection of a deterministic system $\tilde{G}$ defined as

$$\tilde{G}: \quad \left[\begin{array}{c|c} \tilde{A} & \tilde{B} \\ \hline \tilde{C} & 0 \end{array}\right],$$

and a diagonal structured uncertainty Δ, which is shown in Figure 1. This representation, as seen subsequently, allows us to use robust control theory to assess the stability property of the system. More importantly, it provides a physical interpretation for the loss of MS stability from the small gain theorem: the MS instability is introduced when the variance of the switches can not be tolerated by the mean system $\tilde{G}$.

We next define the operator on $\mathbb{S}^n$ (space of symmetric matrices of dimension n) as

$$\tilde{\mathcal{A}}(X) = \tilde{A}X\tilde{A}' + \sum_{i=1}^{p} \sigma_i^2 \tilde{B}(i)\tilde{C}_i X \tilde{C}_i' \tilde{B}(i)',$$

and the system

$$G: \quad \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right],$$

where $\tilde{B}(i)$ denote the i-th column of $\tilde{B}$ and $\tilde{C}_i$ denote the i-th row of $\tilde{C}$. The following conditions are all necessary and sufficient for MS stability of the networked system.

Theorem 3.9: System (4) is MS stable if and only if either of the following conditions holds.

1) there exists an $X > 0$ such that $\tilde{\mathcal{A}}(X) < X$.
2) $\rho(\tilde{\mathcal{A}}) < 1$
3) $\rho(\hat{G} \cdot \Sigma) < 1$

where $\hat{G} = \begin{pmatrix} \|G_{11}\|_2^2 & \dots & \|G_{1p}\|_2^2 \\ \vdots & \dots & \vdots \\ \|G_{p1}\|_2^2 & \dots & \|G_{pp}\|_2^2 \end{pmatrix}$ and Σ is a diagonal matrix with σ_i as its diagonal elements.

Proof. The proof for the first condition is similar to the proof in [21], Chapter 9. The sufficient part of the second condition is obvious while the necessary part can be found in [15], Proposition 2.6.

We next show that $\tilde{G}$ and G have the same input-output transfer matrix. We expand the transfer matrix of $\tilde{G}$ for $|z| > 1$ as

$$\begin{aligned} & \tilde{C}(zI - \tilde{A})^{-1}\tilde{B} \\ =\ & CP\left(\frac{I}{z} + \frac{PAP}{z^2} + \frac{PA^2P}{z^3} + \cdots\right)PB, \\ =\ & C\left(\frac{I}{z} + \frac{A}{z^2} + \frac{A^2}{z^3} + \cdots\right)B, \\ =\ & C(zI - A)^{-1}B, \end{aligned}$$

where we use the properties of P in Lemma 3.4. The equivalence between the first condition and the third condition then follows from Theorem 6.4 of [20] with proper scaling matrix Σ. ■

Remark 3.10: The system G and $\tilde{G}$ are the same from the input-output point of view, and condition 3 implies that the MS stability of the deviated system can be assessed from the original system (3) without decomposition. Indeed, (A, B, C) and $(\tilde{A}, \tilde{B}, \tilde{C})$ are two non-minimal realizations of the same input-output map. It's not hard to show that the only unobservable mode of (A, B, C) is at 1 with corresponding direction of $\mathbf{1}$ (see [19] for the proof of a more general setup). This mode, after the projection of P, turns to be the unique uncontrollable and unobservable mode of $(\tilde{A}, \tilde{B}, \tilde{C})$. We also would like to point out that mean stability of the deviated system (cf. Definition 3.6) is equivalent to the stability of the transfer matrix G (or $\tilde{G}$). Therefore, with mean stability, we can compute $\rho(\hat{G})$ and determine MS stability. However, mean stability is not necessary for the third condition as when the system is mean unstable, $\rho(\hat{G})$ is $+\infty$ and the third condition is violated (assuming at least one $\mu_{ij} > 0$ to avoid the trivial case).

Remark 3.11: If all the links have the same probability of link connection, the third condition turns to be $\sigma^2 < 1/\rho(\hat{G})$, and $1/\rho(\hat{G})$ can be interpreted as the stability margin of the system, or a measure on the quality of communication service that the networked system can tolerate, *i.e.*, the largest variance of ξ_{ij}. We refer the reader to [20] for complete discussion and references of related works.

Although MS consensus ensures the second moment of the difference between any two states converges to zero, it does not automatically guarantee that all states converge to the same *fixed* value, which is a desirable property in many consensus applications. Our next result will establish this convergence property of our model.

Theorem 3.12: Consider (3), as $k \to \infty$, $x(k)$ converges in the mean square sense to a random vector in the space span$\{\mathbf{1}\}$ with mean $\mathbf{1}\gamma' x(0)$ and bounded anvariance iff the networked system is MS stable.

Remark 3.13: Note that our system under the current assumption and in absence of additive noise is guaranteed to converge to a common value although such value is not the average consensus value in general. This is often referred as agreement in the literature. However, if we restrict our setup, to a symmetric graph with symmetric switchings then the system converges to the average consensus and we recover the result of [8] as a special case. s

IV. Mechanisms of the Collective Super-Diffusion Behavior

In this section, we show the emergence of super-diffusion behaviors is introduced by the MS instability of the networked system. For a special two node system, we verify that each state is a Lévy flight. We first provide the definition of MS instability.

Definition 4.1: The networked system is called MS unstable if $\rho(\tilde{A}) > 1$ or equivalently $\rho(\hat{G}\Sigma) > 1$.

Note that according to this definition, the necessary and sufficient conditions for MS instability can be easily deduced from Theorem 3.9, which we omit for brevity.

We recall that the model (1) is given by

$$x_i(k+1) = x_i(k) + \beta \sum_{j \in N_i} \xi_{ij}(k)[x_j(k) - x_i(k)] + v_i(k).$$

The above set of equations for all i can be written compactly as

$$x(k+1) = W(k)x(k) + v(k), \tag{7}$$

with appropriate matrix $W(k)$. We next construct a new set of states as $z_1 = x_1$, $z_2 = x_1 - x_2, \ldots, z_n = x_{n-1} - x_n$. This states transformation can be written compactly as $z = Ux$, where

$$U = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 1 & -1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 1 & -1 \end{bmatrix}$$

It's easy to verify that U is nonsingular and thus define a coordinate transformation. System (7) under the new coordinate is given by

$$z(k+1) = \tilde{W}(k)z(k) + Uv(k),$$

where $\tilde{W}(k) = UW(k)U^{-1}$. We next provide a lemma that characterizes the structural properties of $\tilde{W}(k)$.

Lemma 4.2: For each k, the matrix $\tilde{W}(k)$ has the following structure

$$\tilde{W}(k) = \begin{bmatrix} 1 & \tilde{W}_1(k) \\ 0_{n-1\times 1} & \tilde{W}_2(k) \end{bmatrix}$$

where $0_{n-1\times 1}$ denotes a column vector of zeros with dimension of $n-1$ and $\tilde{W}_1(k)$ and $\tilde{W}_2(k)$ are matrices with appropriate size.

We are now ready to state our main result in this section which shows that under certain conditions, the increments of each state converge to a heavy tailed distribution iff the networked system is MS unstable.

Theorem 4.3: Consider system (7), if the Lyapunov exponent

$$\eta = \lim_{k\to\infty} \frac{1}{k} \log ||\tilde{W}_2(1) \ldots \tilde{W}_2(k)|| < 0, \tag{8}$$

then as $k \to \infty$, $x_i(k+1) - x_i(k)$ converges with probability 1 (w.p.1)[2] to a random variable R. Furthermore, R has unbounded variance for all $i \in \mathcal{V}$ if and only if the networked system is MS unstable.

Remark 4.4: Condition (8) is difficult to check as it involves the computation of the Lyapunov exponent. However, when the system loses MS stability, the error states z_e may still converge to R w.p.1. (at least converges to R in

[2]In the paper, we use $W(k)$ to indicate the time evolution of the system. An alternative notation of $W(k)$ is $W(\omega)$ as each $W(k)$ is i.i.d. over k. We can also induce the measure for $W(k)$ from the measure of ξ_{ij} and thus a measure for $\tilde{W}_2(k)$, say dp_k. Since $\tilde{W}_2(k)$ are i.i.d., $dp_k = dp$ for all k. When we say converge w.p.1, we refer the convergence with respect to the probability measure $dP = \prod_{j\geq 0} dp_j$.

distribution is possible) In this situation, the increments of each state converge to a stationary heavy tailed distribution.

Having showed that all states could have heavy tailed distribution, at this point, we are interested in whether the limiting distribution has a regular tail, *e.g.*, power law. For generic cases, the famous result in [22] does not apply to our system. Our conjecture is that the additive Gaussian noise assumed in our model could produce Lévy flights as its support is dense in $\mathbb{R}$. However, for rigorous treatment of the problem, the renewal theory for our system need to be developed which is nontrivial and we leave it for future investigation. In this paper, instead, we consider a special system and prove that its states are Levy flights under some mild conditions.

A. Lévy Flights in a Special System

We consider the following two node system:

$$\begin{aligned} x_1(k+1) &= x_1(k) + \beta\xi_1(k)(x_2(k) - x_1(k)) + v_1(k) \\ x_2(k+1) &= x_2(k) + \beta\xi_2(k)(x_1(k) - x_2(k)) + v_2(k), \end{aligned} \tag{9}$$

where we assume the probability of link connection are identical for both links and denoted by μ. Using states transformation $z_1 = x_1$ and $z_2 = x_1 - x_2$. The system then can be rewritten as

$$\begin{aligned} z_1(k+1) &= z_1(k) + \beta\xi_1 z_2(k) + v_1(k) \\ z_2(k+1) &= (1 - \beta(\xi_1 + \xi_2))z_2(k) + v_2(k) - v_1(k). \end{aligned} \tag{10}$$

Using Theorem 5 in [22], we show in the following that each states in (9) is a Lévy flight.

Theorem 4.5: Consider (9) and let $M = 1 - \beta(\xi_1 + \xi_2)$. Assume that $\mathbf{E}\log|M| < 0$, and $\log|1-\beta|$ and $\log|1-2\beta|$ generates a group dense in $\mathbb{R}^3$. Then $|x_i(k+1) - x_i(k)|$ converges to a stationary solution R_i w.p.1 for $i = 1, 2$ as $k \to \infty$ and

$$\lim_{t\to\infty} t^\alpha \Pr(|R_i| > t)$$

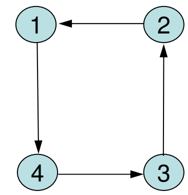
exists and is finite. Moreover, the slope α satisfies $1 < \alpha < 2$ if $\mathbf{E}|M| < 1$ and the system is MS unstable.

Remark 4.6: The condition $E|M| < 1$ ensures the stability of the first moment of z_2, *i.e.*, $\lim_{k\to\infty} E|z_2(k)|$ is bounded. We also point out that the condition for mean stability (cf. Definition 3.6) is $|E(M)| < 1$, which only guarantees the stability of the mean system and is a weaker notion of stability than the first moment stability. However, MS stability ensures the second moment stability of z_2. One can interpret our result as that the system shows Lévy flights if it loses second moment stability but maintains first moment stability.

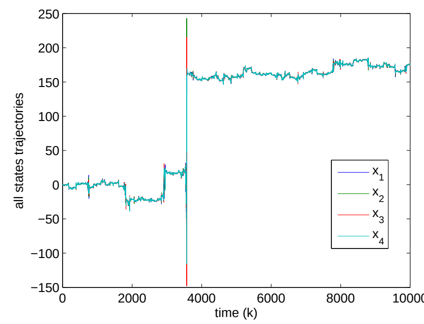
V. An Illustrative Example

In this section, we use an example to illustrate the collective super diffusion behavior generated form our model and

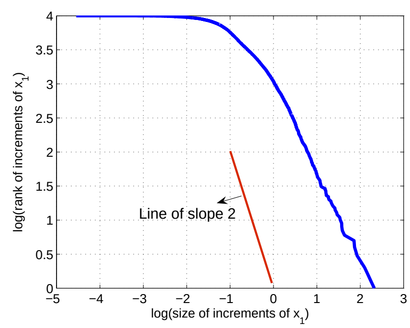
[3]The condition that $\log|1-\beta|$ and $\log|1-2\beta|$ generates a group dense in $\mathbb{R}$ is indeed not restrictive. For two real numbers, if one is rational and the other is irrational, they automatically generate a dense group in $\mathbb{R}$. If all of them are irrational, we need them to be independent. If both of them are rational, they can not generate a group dense in $\mathbb{R}$. We leave the reader to verify these simple facts.



(a) The connectivity graph.



(b) States' trajectories.



(c) log-log plot of rank vs. size of the increments of x_1.

Fig. 2. An example to illustrate the emergence of the complex behavior: (a) shows the network topology; (b) provides state's trajectories of all agents. Although the states does not converge to any value, they stays relatively close to each other with intermittent large jumps; (c) shows the log-log plot of rank of $x_1(k+1) - x_1(k)$ versus size of $x_1(k+1) - x_1(k)$ for $k = 0 \to 1 \times 10^4$, while this type of plots for other states is very similar to x_1. This empirical plot suggests that the increments of x_1 obeys the power law distribution and has unbounded variance as the slop of the tail is slightly less than 2. The non-smooth tail in (c) is most likely introduced by the lack of enough empirical data.

numerically verify our theoretical results. We consider the connectivity graph showing in 2(a). The model parameters are given by $\beta = 1.18$, $\mu = 0.5$, where for simplicity we assume all the channels have the same probability of link failure. The initial condition of each state is a fixed number randomly chosen from the normal distribution with mean 0 and variance 1. Figure 2(b) shows a remarkable new behavior of the collective system. As can be noticed, there are abrupt jumps in the agents states, and between jumps periods where the agents' states are reasonably close to each others. However, shorter portions of the data reveal similar patterns and certain scale invariance. Finally, Figure 2(c) shows the empirical distribution of the rank vs the size of the increments of the state of one agent in log-log plot.

This way for detecting power-laws in empirical data-sets is more reliable than the frequency binning used to plot empirical density distributions [17]. Note that the slope of the tail is less than two. This indicates that the tail of the density has slope less than 3 which corresponds to a process of unbounded variance. According to Definition 2.3, this example suggests that the states generated by model (1) are Lévy flights.

From the topology and model parameters, we derive the state space of the mean system G and obtain that $\sigma^2\rho = 1.0793 > 1$, which shows that the system is MS unstable. This numerical result agrees with our main result that the complex behavior is introduced by the MS instability.

VI. Conclusions

In this paper, we have analyzed the model which generates a collective super-diffusion behavior and elucidated the mechanisms for such complex behaviors. Our model reveals that the propagation of uncertainties in the network could produce complex behaviors, which we hope to be helpful in understanding widely existing Levy flights phenomena. In this paper, we only consider linear coupling between agents, it would be interesting to see if any other coupling between agents could produce Lévy flights and if it is possible that Lévy flights can be localized, which we plan for future investigation.

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