# Optimal Output Consensus Control and Outlier Detection

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***Abstract*— In this paper we study the output consensus problem for systems of agents with linear continuous time invariant dynamics, and derive control laws that minimize a conical combination of the energies of the agents control signals, while only using local information. We show that the optimal control requires the connectivity graph to be complete and in general requires measurements of the state errors. We identify the cases where the optimal control is only based on output errors, and show that in the infinite time horizon case, the optimal control can always be expressed as a dynamic control that is only based on the output errors. We also give a Lemma for the position of the equilibrium point for a large class of agent dynamics. As a second part of this paper we consider the problem of outlier detection, in which an agent wants to deduce if an other agent is using the consensus controller, or if it is an outlier that uses a different controller. We introduce the outlier detection equation.**

## I. INTRODUCTION

There has been a very extensive study on the consensus problem in the past few years [3], [4], [5], [8], [10]. Apparently among all the collective behaviors of multiple agents, consensus is one of the simplest, while still important behaviors. So far the consensus problem that has been studied widely concerns agents with first or second order dynamics. For example, a pioneer work is the famous Vicsek model [9], in which a consensus scheme was proposed based on a simple discrete-time model for the headings of $n$ autonomous agents moving in a plane. Then some theoretical explanations for the consensus behavior of the Vicsek model were given in [3], [5], [11], etc. [4] solved the average-consensus problem of first order multi-agent system with strongly connected and balanced digraph. In [8], [10], [12], [13], to name a few, the consensus of second order multi-agent system is discussed. Various connectivity conditions are assumed to assure the consensus. A survey on consensus problem was given in [14], [15].

Recently, more general linear models have been used in for example [6], [16], [17], [18], [19]. However, in [16], [17], [18], it is assumed that the relative state errors are available for the local control design. In [6] a necessary and sufficient condition is given for the solvability of the consensus problem based on local dynamic output feedback control with fixed connection topology. But how to design the local dynamic output feedback remains a very difficult problem and it seems that at present no general answer is available. In [19] the consensusability of linear time-invariant multi-agent systems is studied, where the admissible consensus protocol is based on static output feedback. In [1] a more general linear model is considered, where the dynamics of each agent can be of any order. Different from [16], [17], [18], the case where only the output error with the neighbors can be measured is studied there. In this paper we study the output consensus problem for general linear systems. We formulate the consensus problem as an optimal control problem and show that the optimal control can be expressed as a consensus protocol where the connectivity graph is completely connected. Not surprisingly, the optimal control requires the measurement of state errors in general. In this paper we identify the cases where the optimal control is only based on the output errors. We also show that in the infinite time horizon case, the optimal control can always be expressed as a dynamic control that is only based on the output errors. Based on the optimal consensus protocol we have designed we also discuss the problem of outlier detection in the paper.

The rest of the paper is organized as follows. Section II is our problem formulation. Section III considers the optimal consensus problem in finite time. In Section IV the infinite time case is studied. In Section V, an outlier detection algorithm is presented.

## II. PROBLEM FORMULATION

In this paper we consider a system of $N$ agents, where each agent $i$ has the dynamics

$$\dot{x}_i = Ax_i(t) + Bu_i(t), \quad (1)$$
$$y_i = Cx_i, \quad (2)$$

where $x_i(t_0) = x_0$, $x_i(t) : \mathbb{R} \to \mathbb{R}^n$, $u_i(t) : \mathbb{R} \to \mathbb{R}^m$ and $y_i(t) : \mathbb{R} \to \mathbb{R}^p$, $A \in \mathbb{R}^{nxn}$, $B \in \mathbb{R}^{nxm}$ and $C \in \mathbb{R}^{pxn}$. It is assumed that $B$ and $C$ are full rank matrices. Let us define $X(t) = [x_1(t)^T, x_2(t)^T, ...., x_N(t)^T]^T \in \mathbb{R}^{nN}$, $U(t) = [u_1(t)^T, u_2(t)^T, ..., u_N(t)^T]^T \in \mathbb{R}^{mN}$ and $Y(t) = [y_1(t)^T, y_2(t)^T, ..., y_N(t)^T]^T \in \mathbb{R}^{pN}$. In this setting we will consider the following problem.

*Problem 2.1:* For any $T > t_0$, construct a control $U(t)$ for the system of agents such that the agents reach consensus in $y_i$ at time $T$, while minimizing the following cost functional

$$\int_{t_0}^{T} \sum_{i=1}^{N} a_i u_i^T u_i dt \quad (3)$$

where $a_i \in \mathbb{R}^+$, $i = 1, 2, ..., N$.

In this problem the agents, *e.g.* mobile robots, reach consensus in $y_i$ while minimizing a weighted sum of the

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energies of the input control signals, where the weights are positive constants.

Before we proceed let us define $\boldsymbol{a} = [a_1, a_2, ..., a_N]$ and define the matrix

$$R(\boldsymbol{a}) = (\sum_{i=1}^{N} a_i)^{-1}(\boldsymbol{a}\mathbf{1}_N^T - \mathrm{diag}(\sum_{i=1}^{N} a_i)), \quad (4)$$

where $\mathbf{1}_N$ is a vector of dimension $N$ with all entries equal to zero.

## III. Consensus in finite time

Let us define

$$W(t) = \int_t^T Ce^{A(T-s)}BB^Te^{A^T(T-s)}C^T ds \quad (5)$$

and

$$G(t) = \int_0^{T-t} Ce^{-Ar}BB^Te^{-A^Tr}C^T dr. \quad (6)$$

*Theorem 3.1:* For $T < \infty$ the solution to Problem 2.1 is

$$U(X, t) = R(\boldsymbol{a}) \otimes (B^Te^{A^T(T-t)}C^TW(t)^{-1}Ce^{A(T-t)})X(t). \quad (7)$$

The optimal control in general requires knowledge of state errors, unless some special condition is imposed:

*Theorem 3.2:* Suppose $ker(C)$ is A-invariant. For $T < \infty$ the solution to Problem 2.1 is

$$U(Y, t) = R(\boldsymbol{a}) \otimes B^TC^TG(t)^{-1}Y(t). \quad (8)$$

*Remark 3.1:* Even though the feedback controllers in (7) and (8) are bounded for $t \in [t_0, T)$, computational difficulties arise as $t \to T$, since $W(T)$, and $G(T)$ are not invertible.

*Remark 3.2:* An open loop version of (7) is presented in the text.

*Proposition 3.1:* The controller defined in (8) uses only local information, *i.e.* differences of the outputs of the agents.

*Proof:* From (8) we get that

$$u_i(t, Y) = B^TC^TG(t)^{-1}(\sum_{i=1}^{n} a_i)^{-1}\sum_{j=1}^{n} a_j(y_j - y_i) \quad (9)$$

■

Let us define $y_c = \frac{1}{\sum_{i=1}^N a_i}\sum_{i=1}^N a_iy_i$,and $Y_c$ which is a $pN$ dimensional vector of $N$ stacked $y_c$.

*Lemma 3.1:* Suppose $A$ has not full rank, and that $x_i(0) \in ker(A), i = 1, ..., N$, then the consensus point or equilibrium for the system of agents using the controller (7) or (8) is $y_c(0)$.

## IV. Consensus in infinite time

*Theorem 4.1:* When the time horizon $T = \infty$, the optimal control

$$U = R(\boldsymbol{a}) \otimes (B^TP_0)X$$

where $P_0$ is the positive semi-definite solution to

$$A^TP_0 + P_0A = -P_0BB^TP_0.$$

When only the output $y = Cx$ is available for control action, an observer has to be designed. If we require such an observer should minimize in some average sense the estimation error $\|y - C\hat{x}\|$, then under the assumption that $(A, C)$ is detectable and $A$ does not have any eigenvalue on the imaginary axis we have

$$\dot{\hat{x}} = A\hat{x} - BB^TP_0\hat{x} - Q(y - C\hat{x}), \quad (10)$$

where

$$AQ + QA^T = -QC^TCQ.$$

Let us now get back to the optimal consensus problem. As the above discussion shows in the case of infinite time the optimal solution is

$$U = R(\boldsymbol{a}) \otimes (B^TP_0)X. \quad (11)$$

On the other hand, it is easy to see that the control (11) can be written as

$$u_i = -B^TP_0(x_i - x_c)$$

where $x_c = \frac{1}{\sum_{i=1}^N a_i}\sum_{i=1}^N a_ix_i$ satisfies the free drift equation

$$\dot{x}_c = Ax_c.$$

Let $\delta_i = x_i - x_c$, then

$$\dot{\delta}_i = A\delta_i + Bu_i.$$

Similar to the case discussed in [1], we can design the following optimal observer for $\delta_i$:

$$\dot{\hat{\delta}}_i = (A - BB^TP_0)\hat{\delta}_i - Q(\frac{1}{\sum_{i=1}^N a_i}\sum_{j=1}^N a_j(y_i - y_j) - C\hat{\delta}_i)$$

*Proposition 4.1:* Suppose $(A, B)$ is stabilizable and $(A, C)$ is detectable, and $A$ has no eigenvalue on the imaginary axis. Then the following dynamic output control

$$\begin{aligned} u_i &= -B^TP_0\hat{\delta}_i && (12) \\ \dot{\hat{\delta}}_i &= (A - BB^TP_0)\hat{\delta}_i - && (13) \\ & Q(\frac{1}{\sum_{i=1}^N a_i}\sum_{j=1}^N a_j(y_i - y_j) - C\hat{\delta}_i), \end{aligned}$$

solves the optimal consensus problem in infinite time.

## V. Outlier Detection

Consider a system of $N$ agents each with the dynamics (1), and each agent $i$ is supposed to use the controller $u_i$ in (9). In this setting we define an outlier at $t \in [0, T]$ and $Y \in \mathbb{R}^{pN}$ as an agent $i$ that use a controller $\hat{u}_i$ such that

$$B(\hat{u}_i(t, Y) - u_i(t, Y)) \neq 0. \quad (14)$$

Now suppose agent $i$ is using (9) and wants to find out if agent $j$ is an outlier ($i \neq j$). To use the controller (9) agent $i$ has to be able to measure the differences between its output and all other agents outputs. Now let $u_i, u_j$ denote the controller for agents $i$ and $j$ respectively, defined according to (9). Agent $i$ can construct both $u_i$ and $u_j$. Let $\hat{u}_i$ and $\hat{u}_j$

be the actual controllers that the agents use. Now we have that

$$B(\hat{u}_i - \hat{u}_j) = \dot{x}_i - \dot{x}_j + A(x_i - x_j) \qquad (15)$$

We assume $B$ and $C$ have full rank and $CB$ is invertible, *i.e.* a square system with relative degree $[1, ..., 1]$. We also assume that $ker(C)$ is $A$-invariant. We get that

$$CB(\hat{u}_i - \hat{u}_j) = \dot{y}_i - \dot{y}_j + P(y_i - y_j), \qquad (16)$$

where $P \in \mathbb{R}^{pxp}$.

Now we have that

$$\begin{aligned} CB(u_i - u_j) - CB(\hat{u}_i - \hat{u}_j) &= && (17)\\ CB(u_i - u_j) - \dot{y}_i + \dot{y}_j - P(y_i - y_j) &= && (18)\\ CB(\hat{u}_j - u_j), & && (19) \end{aligned}$$

since we know that $\hat{u}_i = u_i$. From this we get that agent $i$ is not an outlier if and only if

$$CB(u_i - u_j) - \dot{y}_i + \dot{y}_j - P(y_i - y_j) = 0, \qquad (20)$$

or equivalently

$$\dot{y}_i - \dot{y}_j = CB(u_i - u_j) - P(y_i - y_j). \qquad (21)$$

Now let us define the *outlier detection equation*

$$\dot{z}_{ij} = CB(u_i - u_j) - P(y_i - y_j) \qquad (22)$$

where $z_{ij}(t_0) = y_i(t_0) - y_j(t_0)$. Now if there is a $t \in [t_0, T]$ such that $z_{ij}(t) \neq y_i(t) - y_j(t)$, then $j$ is an outlier. However due to measurement noise a more suitable way to measure if $j$ is an outlier is to check whether

$$||z_{ij}(t) - (y_i(t) - y_j(t))|| > \alpha, \qquad (23)$$

and if so $j$ is an outlier, where $\alpha$ is a positive constant and $||.||$ is a suitable norm.

*Remark 5.1:* For the general case where $ker(C)$ is not necessarily $A$-invariant, a dynamic detection scheme that requires the estimation of relative state errors would be needed.

## References


[1] J. Wang, Z. Liu and X. Hu, "Consensus of High Order Linear Multi-agent Systems Using Output Error Feedback", *Proc. 2009 CDC*, 2009, Shanghai.

[2] Z. Lian and A. Deshmukh, "Performance prediction of an unmanned airborne vehicle multi-agent system", *Euro. J. Operational Research*, vol. 172, no. 2, 2006, pp 680-695.

[3] A. Jadbabaie, J. Lin, and A. S. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules", *IEEE Trans. Auto. Contr.*, vol. 48, no. 6, 2003, pp 998-1001.

[4] R. Olfati-Saber and R. Murray, "Consensus problems in networks of agents with switching topology and time-delays", *IEEE Trans. Auto. Contr.*, vol. 49, no. 9, 2004, pp 1520-1533.

[5] W. Ren and R. W. Beard, "Consensus seeking in multi-agent systems using dynamically changing interaction topologies", *IEEE Trans. Auto. Contr.*, Vol. 50, No. 5, 2005, pp 655-661.

[6] A. Fax and R. Murray, "Information flow and cooperative control of vehicle formations", *IEEE Trans. Auto. Contr.*, Vol. 49, No. 9, 2004, pp 1453-1464.

[7] Z. Lin, B. Francis and M. Maggiore, "Necessary and sufficient graphical conditions for formation control of unicycles", *IEEE Trans. Auto. Contr.*, Vol. 50, No. 1, 2005, pp 121-127.

[8] R. Olfati-Saber, "Flocking for multi-agent dynamic systems: algorithms and theory", *IEEE Trans. Auto. Contr.*, Vol. 51, No. 3, 2006, pp 401-420.

[9] T. Vicsek, A. Cziroók and E. Ben-Jacob, et al, "Novel type of phase transition in a system of self-deriven particles", *Phys. Rev. Letters*, Vol. 75, No. 6, 1995, pp 1226-1229.

[10] Y. Hong, L. Gao, D. Cheng, and J. Hu, "Coordination of multi-agent systems with varying interconnection topology using common Lyapunov function", *IEEE Trans. Auto. Contr.*, vol. 52, no. 5, 2007, pp 943-948.

[11] Z. Liu and L.Guo, "Connectivity and synchronization of Vicsek's model", *Science in China*, vol. 37, no.8, 2007, pp 979-988.

[12] G. Xie and L. Wang, "Consensus control for a class of networks of dynamic agents: switching topology", *Proc. the 2006 American Control Conference*, 2007, pp 1382-1387.

[13] D. Cheng, J. Wang and X. Hu, "An extension of LaSalle's invariance principle and its application to multi-Agent consensus", *IEEE Trans. Auto. Contr.*, vol. 53, no. 7, 2008, pp 1765-1770.

[14] R. Olfati-Saber, J. Fax and R. Murray, "Consensus and cooperation in networked multi-agent systems", *Proc. of IEEE*, Vol. 95, No. 1, 2007, pp 1-17.

[15] W. Ren, R. Beard, and E. Atkins, Information consensus in multivehicle cooperative control, *IEEE Control systems magazine*, April, 2007, pp 71-82.

[16] W. Ren, K. Moore, Y. Chen, "High-Order and Model Reference Consensus Algorithms in Cooperative Control of Multi-Vehicle Systems", *ASME Journal of Dynamic Systems, Measurement, and Control*, vol. 129, no. 5, 2007, pp 678-688.

[17] F. Xiao, L. Wang, "Consensus problems for high-dimensional multi-agent systems", *IET Control Theory Appl.*, vol. 1, no. 3, 2007, pp 830-837.

[18] J. Wang, D. Cheng, and X. Hu, "Consensus of Muti-Agent Linear Dynamic Systems", *Asian J. Contr.*, vol. 10, no. 2, 2008, pp. 144-155.

[19] C. Ma, J. Zhang, "Necessary and sufficient conditions for consensusability of linear multi-agent systems", *submitted to IEEE Trans. on Automatic Control*, revised, 2008.

[20] R. Horn, C. Johnson, *Matrix Analysis*, New York: Cambbridge Univ. Press, 1985.

[21] A. Lindquist, J. Sand , "An introduction to mathematical systems theory".