Multievolution scattering systems and the multivariable Schur class

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***Abstract*— We show using the multievolution scattering systems formalism how to obtain the decompositions of multivariable Schur-class functions which are analogous, to a certain extent, to Agler's decompositions of Schur–Agler-class functions. In particular, this gives a new class of d-tuples of commuting strict contractions on a Hilbert space which satisfy the multivariable von Neumann inequality.**

It is well known that in the classical one variable case, there are very strong relations between function theory, operator theory, system theory, and scattering theory. In particular, the Schur class $\mathcal{S}(\mathcal{F},\mathcal{F}_*)$, consisting of analytic functions on the unit disk $\mathbb{D}=\{z\in\mathbb{C}\colon |z|<1\}$ whose values are contractive linear operators from a Hilbert space $\mathcal{F}$ to a Hilbert space $\mathcal{F}_*$, can be identified with each of the following:

- (i$_1$) the class of analytic $\mathcal{L}(\mathcal{F},\mathcal{F}_*)$-valued functions f on $\mathbb{D}$ such that $\|f(T)\|\le 1$ for any strict contraction T on a Hilbert space $\mathcal{H}$, i.e., von Neumann's inequality holds [19];
- (ii$_1$) the class of analytic $\mathcal{L}(\mathcal{F},\mathcal{F}_*)$-valued functions f on $\mathbb{D}$ such that
$$I_{\mathcal{F}_*}-f(z)f(w)^*=(1-z\overline{w})K(z,w)$$
for all $(z,w)\in\mathbb{D}\times\mathbb{D}$, where $K(z,w)$ is a positive $\mathcal{L}(\mathcal{F}_*)$-valued kernel on $\mathbb{D}\times\mathbb{D}$ (the Pick kernel associated with f) [16], [15], [11];
- (iii$_1$) the class of transfer functions of a discrete-time conservative 1D systems with the input and output spaces $\mathcal{F}$ and $\mathcal{F}_*$, respectively [9], [8] (see also [4]);
- (iv$_1$) the class of scattering functions of (single-evolution) scattering systems with the outgoing and incoming wandering subspaces $\mathcal{F}$ and $\mathcal{F}_*$, respectively [14] (see also [1]);
- (v$_1$) the class of characteristic functions of completely non-unitary contractive operators from $\mathcal{F}$ to $\mathcal{F}_*$ [13], [17].

Which of these characterizations are still valid (in an appropriate form) for the d-variable Schur class $\mathcal{S}_d(\mathcal{F},\mathcal{F}_*)$ consisting of contractive $\mathcal{L}(\mathcal{F},\mathcal{F}_*)$-valued analytic functions on the unit polydisk $\mathbb{D}^d$? It turns out that the general picture is different for $d=2$ and for $d>2$. First of all, to the best of our knowledge, there is no appropriate notion of the characteristic function of a d-tuple of commuting contractions, let alone the generalization of the characterization (v$_1$) of the class $\mathcal{S}_d(\mathcal{F},\mathcal{F}_*)$ to the case of $d\ge 2$. However, for $d=2$ there are analogues of (i$_1$)–(iv$_1$) which all coincide with $\mathcal{S}_2(\mathcal{F},\mathcal{F}_*)$. For (i$_2$), a single strict contraction is replaced by a pair $T=(T_1,T_2)$ of commuting strict contractions on a Hilbert space $\mathcal{H}$ [3]; (ii$_1$) is replaced by

- (ii$_2$) there exist positive $\mathcal{L}(\mathcal{F}_*)$-valued kernels $K_i(z,w)$, $i=1,2$, on $\mathbb{D}\times\mathbb{D}$ such that
$$I_{\mathcal{F}_*}-f(z)f(w)^* = (1-z_1\overline{w}_1)K_1(z,w)+(1-z_2\overline{w}_2)K_2(z,w)$$
for all $(z,w)\in\mathbb{D}\times\mathbb{D}$ [2];

in (iii$_2$), transfer functions of conservative 2D Roesser systems are involved [2], [7]; in (iv$_2$), two-evolution scattering systems are considered (we will discuss multievolution scattering systems later). In the case of $d>2$, the natural generalizations, (i$_d$)–(iii$_d$), of the classes (i$_1$)–(iii$_1$) are all equal [2] (see also [7]). However, due to the existence of counterexamples to von Neumann's inequality in more than 2 variables [18], [10], each of them forms a proper subclass, $\mathcal{SA}_d(\mathcal{F},\mathcal{F}_*)$ (which is called the Schur–Agler class), of the d-variable Schur class $\mathcal{S}_d(\mathcal{F},\mathcal{F}_*)$. Still, the latter can be characterized via (iv$_d$), i.e., via multievolution scattering systems [6], [5].

A d-evolution scattering system is a collection $\mathfrak{S}=(\mathcal{K},\mathcal{U},\mathcal{F},\mathcal{F}_*)$, where $\mathcal{K}$ is a Hilbert space (the ambient space), $\mathcal{U}=(\mathcal{U}_1,\ldots,\mathcal{U}_d)$ is a d-tuple of commuting unitary operators on $\mathcal{K}$ (the evolutions of the system), and $\mathcal{F}$ and $\mathcal{F}_*$ are wandering subspaces for $\mathcal{U}$, i.e. $\mathcal{U}^n\mathcal{F}\perp\mathcal{F}$, $\mathcal{U}^n\mathcal{F}_*\perp\mathcal{F}_*$, $n\in\mathbb{Z}^d\setminus\{0\}$. A d-evolution scattering system $\mathfrak{S}$ is called minimal if the smallest closed subspace of $\mathcal{K}$ containing $\mathcal{F}$ and $\mathcal{F}_*$ and invariant for $\mathcal{U}$ and $\mathcal{U}^*$ is the whole space $\mathcal{K}$. The subspaces $\mathcal{W}:=\bigoplus_{n\in\mathbb{Z}^d_+}\mathcal{U}^n\mathcal{F}$ and $\mathcal{W}_*=\bigoplus_{n\in\mathbb{Z}^d\setminus\mathbb{Z}^d_+}\mathcal{U}^n\mathcal{F}_*$ are called the outgoing subspace and the incoming subspace. The system $\mathfrak{S}$ is called causal if $\mathcal{W}\perp\mathcal{W}_*$. In this case, the ambient space $\mathcal{K}$ has the orthogonal decomposition $\mathcal{K}=\mathcal{W}_*\oplus\mathcal{V}\oplus\mathcal{W}$, where $\mathcal{V}$ is called the scattering subspace. Next, let $\mathfrak{S}$ be a causal d-evolution scattering system. Define the Fourier representation operators Φ and Φ_* from $\mathcal{K}$ to $L^2(\mathbb{T}^d,\mathcal{F})$ and to $L^2(\mathbb{T}^d,\mathcal{F}_*)$, respectively, by $\Phi: h\mapsto\sum_{n\in\mathbb{Z}^d}(P_{\mathcal{F}}\mathcal{U}^{-n}h)z^n$, $\Phi_*: h\mapsto\sum_{n\in\mathbb{Z}^d}(P_{\mathcal{F}_*}\mathcal{U}^{-n}h)z^n$, where $\mathbb{T}^d=\{z\in\mathbb{C}^d\colon |z_k|=1,\ k=1,\ldots,d\}$ is the unit d-torus, and $L^2(\mathbb{T}^d,\mathcal{X})$ denotes the Lebesgue space of measurable norm-square integrable $\mathcal{X}$-valued functions on $\mathbb{T}^d$, for a Hilbert space $\mathcal{X}$. Then Φ is a coisometry with the initial space equal $\mathcal{W}$ and with $\Phi\mathcal{W}=H^2(\mathbb{T}^d,\mathcal{F})$, while Φ_* is a coisometry with the initial space equal $\mathcal{W}_*$ and with $\Phi_*\mathcal{W}_*=H^2(\mathbb{T}^d,\mathcal{F}_*)^\perp$. Here $H^2(\mathbb{T}^d,\mathcal{X})$ denotes the Hardy

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space, i.e., the subspace in $L^2(\mathbb{T}^d, \mathcal{X})$ consisting of functions determined by boundary values of analytic functions on $\mathbb{D}^d$. It turns out that the operator $\Phi_*\Phi^*$ maps $H^2(\mathbb{T}^d, \mathcal{F})$ into $H^2(\mathbb{T}^d, \mathcal{F}^*)$, moreover it is a multiplication operator: $\Phi_*\Phi^* = M_S$. The corresponding multiplier $S = S_{\mathfrak{S}}$, which is called the scattering function of $\mathfrak{S}$, belongs to $\mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$. Conversely, given any $f \in \mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$, there is a minimal causal scattering system $\mathfrak{S}^f$ with scattering function $S_{\mathfrak{S}^f} = f$. Such a scattering system is determined by f uniquely up to a unitary equivalence.

Analyzing the geometry of the scattering subspace $\mathcal{V}$ of a certain model scattering system for $f \in \mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$, namely, invariant subspaces of $\mathcal{V}$ under chosen evolutions $\mathcal{U}_p$ or $\mathcal{U}_q$), one obtains

Theorem 1 ([12]): $f \in \mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$ if and only if for any $p, q \in \{1, \dots, d\}$, $p < q$, there exist positive $\mathcal{L}(\mathcal{F}_*)$-valued sesquianalytic kernels $K^I_{pq}(z, w)$ and $K^{II}_{pq}(z, w)$ on $\mathbb{D}^d \times \mathbb{D}^d$ such that

$$I_{\mathcal{F}_*} - f(z)f(w)^* = \prod_{k\neq p}(1 - z_k\overline{w_k})K^I_{pq}(z, w) + \prod_{k\neq q}(1 - z_k\overline{w_k})K^{II}_{pq}(z, w)$$

for all $(z, w) \in \mathbb{D}^d \times \mathbb{D}^d$.
Theorem 1 gives a new version of (ii$_d$), which characterizes now the whole d-variable Schur class $\mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$ and coincides with (ii$_2$) above when $d = 2$.

It is also of interest to replace the set of d-tuples of commuting strict contractions by its maximal subset so that the von Neumman inequality holds for all Hilbert spaces $\mathcal{F}, \mathcal{F}_*$ and for all $f \in \mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$, i.e., to find the right version of (i$_d$) corresponding to the whole d-variable Schur class $\mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$. Though not maximal, a subset with this property was found in [12] using the decomposition of multivariable Schur-class functions from Theorem 1. For a d-tuple $T = (T_1, \dots, T_d)$ of commuting bounded linear operators on a Hilbert space and a vector $\beta \in \{0, 1\}^d$, set $\Delta^\beta_T := \sum_{0\le\alpha\le\beta}(-1)^{|\alpha|}T^\alpha T^{*\alpha}$, where the inequality $\alpha \le \beta$ for $\alpha, \beta \in \{0, 1\}^d$ means that $\alpha_j \le \beta_j$, $j = 1, \dots, d$, $|\alpha| = \alpha_1 + \cdots + \alpha_d$, $T^\alpha = T_1^{\alpha_1} \cdots T_d^{\alpha_d}$, and $T^* = (T_1^*, \dots, T_d^*)$. We will say that a d-tuple T of commuting strict contractions on a Hilbert space belongs to the class $\mathcal{P}^d_{p,q}$, with $p < q$, if the operators $\Delta^{e-e_p}_T$ and $\Delta^{e-e_q}_T$ are positive semidefinite, where $e = (1, \dots, 1)$ and e_k is the k-th standard basis vector for the module $\mathbb{Z}^d$.

Theorem 2 ([12]): Let $T \in \mathcal{P}^d_{p,q}$. Then $\|f(T)\| \le 1$ for any Hilbert spaces $\mathcal{F}, \mathcal{F}_*$ and for all $f \in \mathcal{S}_d(\mathcal{F}, \mathcal{F}_*)$.

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