A Reduced Model of Reflectometry for Wired Electric Networks

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Abstract— **Reflectometry is a technology frequently used for the diagnosis of failures in wired electric networks. For the purpose of developing advanced diagnosis methods, a reduced mathematical model of reflectometry is proposed in this paper. Based on the telegrapher's equations and on the Kirchhoff's laws, this model leads to a simple algorithm for the computation of frequency domain reflection coefficients from the characteristic parameters of the transmission lines and their connections in a star-shaped or a tree-shaped network. This algorithm implemented in a digital computer can easily simulate networks composed of different and inhomogeneous transmission lines. Comparisons between simulated reflection coefficients and real reflectometry measurements confirm the validity of the proposed model.**

I. INTRODUCTION

With the fast development of electric and electronic components in modern engineering systems, the reliability of electric transmissions lines is becoming a crucial issue. The increasing number of electric connection failures requires research on diagnosis technologies. For example, reflectometry is one of such technologies. It consists in injecting electric signals into the monitored network and in analyzing the reflection of electric waves observed at one or more network terminals [1], [2]. Nowadays, this technology provides an efficient solution for the diagnosis of *hard faults* (open or short circuits) in a single transmission line or in a network, whereas the diagnosis of *soft faults* in wired electric networks is still an open problem [3]. The purpose of this paper is to propose a mathematical model of wired electric networks intended for the diagnosis of both hard and soft faults. As soft faults may result in spatially continuous characteristic variations of transmission lines, the proposed model will cover networks composed of spatially inhomogeneous transmission lines. This new model will remain simple to be a tool convenient to fault diagnosis, as it is entirely based on the telegraphers equations for transmission lines and on the Kirchhoff's laws for network nodes. A method for numerical solution of the model equations (in the frequency domain) will be elaborated, so that the new model can serve as a numeric simulator of wired networks. While the model presented in this paper is being used as a theoretic basis in our current study for fault diagnosis in lossy and inhomogeneous wired networks, this model has already been used in [4] for studying lossless networks of simple topological structure.

In the next section, we recall the model of a single transmission line and we present a simple method to calculate the reflection coefficient. Next, in section III, we extend this theory to star-shaped and tree-shaped networks. Numerical simulation results are then presented in section IV, before the conclusion section.

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II. MODELING A SINGLE TRANSMISSION LINE

A. *Telegrapher's Equations and Zakharov-Shabat Equations.*

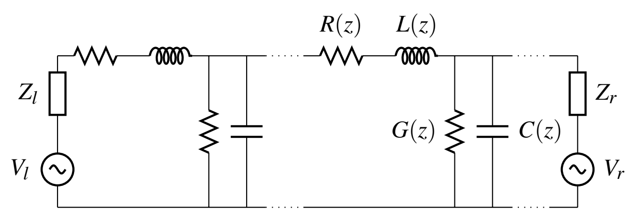

Fig. 1. A transmission line connected to a voltage source V_l with its internal impedance Z_l and to a voltage source V_r with its internal impedance Z_r.

Consider a transmission line of length l. The left end and the right end of the line correspond respectively to the coordinate values along the line $z=0$ and $z=l$. This line is connected to an alternating voltage source $V_l(k)$ of frequency k with real internal impedance $Z_l(k)$ at the left end, and a voltage source $V_r(k)$ with real internal impedance $Z_r(k)$ at the right end (see Fig 1).

The voltage and current waves along a transmission line driven by an alternating source of frequency k are of the form $V(k,z)\exp(-jkt)$ and $I(k,z)\exp(-jkt)$ (t and z being the time and the line coordinate) and are solutions of the following frequency-domain telegrapher's equations [5],

$$\begin{cases} \dfrac{\partial V(k,z)}{\partial z} - jkL(z)I(k,z) + R(z)I(k,z) &= 0 \\ \dfrac{\partial I(k,z)}{\partial z} - jkC(z)V(k,z) + G(z)V(k,z) &= 0 \end{cases} \quad (1)$$

with the boundary conditions

$$\begin{cases} V(k,0) &= -Z_l(k)I(k,0) + V_l(k) \\ V(k,l) &= Z_r(k)I(k,l) + V_r(k) \end{cases} \quad (2)$$

where $R(z), L(z), C(z)$ and $G(z)$ are respectively the distributed resistance, inductance, capacitance and conductance per unit length along the line.

For a reflectometry experiment, the boundary conditions of (1) are related to a source connected to one end of the line and a passive load connected to the other end. In order to reformulate the telegrapher's equations in terms of incident and reflected waves, some transformations are applied in what follows.

ISBN 978-963-311-370-7

The first step is to replace the space coordinate z by the wave propagation time x through the Liouville transformation [6]:

$$x(z) = \int_0^z \sqrt{L(s)C(s)}ds \tag{3}$$

After this coordinate change from z to x, by abuse of notation, $L(z(x))$ will be simply written as $L(x)$, and similarly for $C(x)$, $R(x)$, $G(x)$, $V(k,x)$ and $I(k,x)$. Then the telegrapher's equations become,

$$\begin{cases} \dfrac{\partial V(k,x)}{\partial x} & = & \left(jk - \dfrac{R(x)}{L(x)}\right) H(x) I(k,x) \\ \dfrac{\partial I(k,x)}{\partial x} & = & \left(jk - \dfrac{G(x)}{C(x)}\right) H^{-1}(x) V(k,x) \end{cases} \tag{4}$$

with:

$$H(x) = \sqrt{\frac{L(x)}{C(x)}} \tag{5}$$

which is the high frequency limit of the characteristic impedance and which would coincide with *the characteristic impedance* if the transmission line was lossless. For a line of length l, the wave traveling time over the line is $\tau = x(l)$, and for all point $z \in [0,l]$, the corresponding $x(z) \in [0,\tau]$.

In the new coordinate system, define the reflected and the incident waves:

$$\begin{cases} \nu_1(k,x) & = & \dfrac{1}{\sqrt{2}}\left(H^{-\frac{1}{2}}(x)V(k,x) - H^{\frac{1}{2}}(x)I(k,x)\right) \\ \nu_2(k,x) & = & \dfrac{1}{\sqrt{2}}\left(H^{-\frac{1}{2}}(x)V(k,x) + H^{\frac{1}{2}}(x)I(k,x)\right) \end{cases} \tag{6}$$

this change of variables leads to the Zakharov-Shabat equations,

$$\begin{cases} \dfrac{\partial \nu_1(k,x)}{\partial x} + jk\nu_1(k,x) = q_3(x)\nu_1(k,x) - q_+(x)\nu_2(k,x) \\ \dfrac{\partial \nu_2(k,x)}{\partial x} - jk\nu_2(k,x) = -q_3(x)\nu_2(k,x) - q_-(x)\nu_1(k,x) \end{cases} \tag{7}$$

with the corresponding boundary conditions obtained by rewritting (2) with (6) and where:

$$\begin{aligned} q_\pm(x) &= \frac{1}{4}\frac{\partial}{\partial x}\left[\ln \frac{L(x)}{C(x)}\right] \pm \frac{1}{2}\left(\frac{R(x)}{L(x)} - \frac{G(x)}{C(x)}\right) \\ q_3(x) &= \frac{1}{2}\left(\frac{R(x)}{L(x)} + \frac{G(x)}{C(x)}\right) \end{aligned}$$

The passage of telegraphers's equations to Zakharov-Shabat equations will allow to study the inverse problem through the inverse scattering approach [6].

B. The reflection coefficient governed by a Differential Riccati Equation.

In the inverse scattering theory, reflection coefficients are usually related to the limiting behavior of *Jost solutions* of the Zakharov-Shabat equations. For the left Jost solution($\nu_1(k,x) = f_{l_1}(k,x)$ and $\nu_2(k,x) = f_{l_2}(k,x)$) satisfying the limiting behavior,

$$\begin{cases} \lim\limits_{x\to+\infty} f_{l_1}(k,x) & = & 0 \\ \lim\limits_{x\to-\infty} f_{l_2}(k,x)e^{-jkx} & = & 1 \end{cases} \tag{8}$$

the left reflection coefficient is expressed as

$$r_l(k) = \lim_{x\to-\infty} \frac{f_{l_1}(k,x)}{f_{l_2}(k,x)} e^{2jkx} \tag{9}$$

where the factor e^{2jkx} demodulates the waves $f_{l_1}(k,x)$ and $f_{l_2}(k,x)$.

A simple method for solving the Zakharov-Shabat equations is to define a local reflection coefficient r [7] by the invariant embedding method

$$r(k,x) = \frac{\nu_1(k,x)}{\nu_2(k,x)} \tag{10}$$

For any point $x \in [0,\tau]$, this coefficient satisfies the Differential Riccati Equation (DRE) with a boundary condition at the load end of the line:

$$\begin{cases} \dfrac{\partial r(k,x)}{\partial x} & = q_-(x)r^2(k,x) + 2q_3(x)r(k,x) \\ & \quad -2jkr(k,x) - q_+(x) \\ r(k,\tau) & = \rho_r(k) \end{cases} \tag{11}$$

the coefficient $\rho_r(k)$ is defined as follows:

$$\rho_r(k) = \frac{Z_r(k) - H(\tau)}{Z_r(k) + H(\tau)} \tag{12}$$

The quantity $r(k,x)$ can be viewed as a generalization of the reflection coefficient , which is usually defined at one end of a transmission line instead of being a function of x. When $r(k,x)$ is evaluated at the source end, it coincides with the reflection coefficient $r_l(k)$ which can be measured in practice with a network analyzer.

After numerically solving the DRE (11) for $r(k,x)$, the Zakharov-Shabat equations can be easily solved for $\nu_1(k,x)$ and $\nu_2(k,x)$ with the aid of $r(k,x)$. The solution of (1) is then obtained by simply converting $\nu_1(k,x)$ and $\nu_2(k,x)$ back to $V(k,z)$ and $I(k,z)$.

$$\begin{cases} V(k,x) & = & \dfrac{H^{\frac{1}{2}}}{\sqrt{2}}(\nu_1(k,x) + \nu_2(k,x)) \\ I(k,x) & = & \dfrac{-H^{-\frac{1}{2}}}{\sqrt{2}}(\nu_1(k,x) - \nu_2(k,x)) \end{cases} \tag{13}$$

III. MODELING A STAR-SHPED OR TREE SHAPED NETWORK

A. Description of a star-shaped network.

Now, we consider a star-shaped network of lossy transmission lines, as in Fig 2, formed by $N+1$ branches b_i for $i \in \{0,\ldots,N\}$ connected together through a central node (junction) J. A source is connected at the end of branch b_0 ($x_0 = \tau_0$), and at the end of each branch b_i for $i \in \{1,\ldots,N\}$ a load is connected (including the particular cases of open or short circuits). Each branch b_i is parameterized by its per-unit-length parameters $R_i(x_i)$, $L_i(x_i)$, $G_i(x_i)$ and $C_i(x_i)$. In what follows, each function related to the branch b_i is indexed by i.

The voltage and the current in each branch of the network

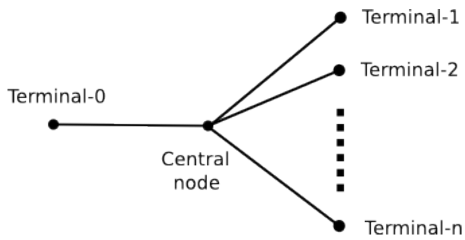

Fig. 2. Star-shaped network

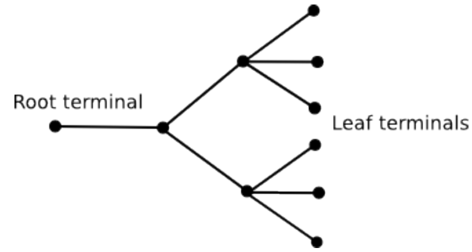

Fig. 3. Example of tree-shaped network

are modeled by (1) with boundary conditions at the terminal nodes and with the Kirchhoff's law (14) at the central node:

$$\begin{cases} \sum_{i=0}^{N} I_i(k,0) &= 0 \\ V_i(k,0) &= V_j(k,0) \quad \forall (i,j) \in \{1,\dots,N\} \end{cases} \tag{14}$$

Extending the local reflection coefficient introduced in Section 1 to each branch, we have $N+1$ DRE of the form

$$\begin{cases} \dfrac{\partial r_i(k,x_i)}{\partial x_i} &= q_{-_i}(x_i) r_i^2(k,x_i) + 2q_{3_i}(x_i) r_i(k,x_i) \\ & \quad -2jkr_i(k,x_i) - q_{+_i}(x_i) \\ r_0(k,0) &= \rho_{r_0}(k) \\ r_i(k,\tau_i) &= \rho_{r_i}(k) \qquad \forall i \in \{1,\dots,N\} \end{cases} \tag{15}$$

There is no source at the junction node J ($x_i = 0$), consequently, the voltages V_{r_0} and V_{l_i} for $i \in \{1,\dots,N\}$ are zero. After applying the Kirchhoff rules, the apparent impedances verify:

$$\frac{1}{Z_0(k,0)} = \sum_{i=1}^{N} \frac{1}{Z_i(k,0)} \tag{16}$$

where $Z_i(k,x) = \dfrac{V_i(k,x)}{I_i(k,x)}$ is the apparent impedance of each branch b_i.

B. The reflection coefficient for a star-shaped network.

The reflection coefficient for a star-shaped network viewed from terminal-0 can be computed as follows . First compute $r_i(k,x_i)$ along each branch b_i for $i \in \{1,\dots,N\}$, by solving the governing DRE (15) from terminal-i to the central node. The value of $r_0(k,x_0)$ on branch b_0 close to the central node is equal to $\rho_{r_0}(k)$. This quantity can be determined from $r_i(k,x)$ on the other branches through the Kirchhoff's law. At the junction node J, the reflection coefficient $r_i(k,0)$ can be expressed with the apparent impedance as follows:

$$r_i(k,0) = \frac{Z_i(k,0) - H_i(0)}{Z_i(k,0) + H_i(0)} \tag{17}$$

where

$$H_i(x_i) = \sqrt{\frac{L_i(x_i)}{C_i(x_i)}} \quad \forall i \in \{0,\dots,N\} \tag{18}$$

Applying the identity (16), the expression of $Z_0(k,0)$ writes

$$Z_0(k,0) = \frac{1}{\sum_{i=1}^{N} \frac{1-r_i(k,0)}{H_i(0)(1+r_i(k,0))}} \tag{19}$$

The function $\rho_{r_0}(k)$ verifies:

$$\rho_{r_0}(k) = \frac{Z_0(k,0) - H_0(0)}{Z_0(k,0) + H_0(0)} \tag{20}$$

Finally, solve the DRE for $r_0(k,x_0)$ along branch b_0 towards terminal-0 to obtain the reflection coefficient at that point.

We can solve the Zakharov-Shabat equations for $\nu_{1_i}(k,x_i)$ and $\nu_{2_i}(k,x_i)$ with the aid of $r_i(k,x_i)$ and we calculate the solutions the tension V_i and the current I_i of each branch b_i with the relation applied (13) at each branch.

Now let us consider a particular case that will be useful for verifying numerical simulation results. If the high frequency characteristic impedance as defined in (18) H_i for $i \in \{1,\dots,N\}$ is continuous at the junction node J, in the sense that $H_i(0) = H_j(0)\ \forall (i,j) \in \{0,\dots,N\}$, then the expression of $\rho_{r_0}(k)$ can be simplified as follows:

$$\rho_{r_0}(k) = \frac{1 - N + 2\sum_{i=1}^{N} \frac{r_i(k,0)}{1+r_i(k,0)}}{1 + N - 2\sum_{i=1}^{N} \frac{r_i(k,0)}{1+r_i(k,0)}} \tag{21}$$

Moreover, if

- A1 the star-shaped network is composed of lossless branches,
- A2 the terminal of each branch b_i, for $i \in \{1,\dots,N\}$, is connected to a matched load,
- A3 branch b_0 is homogeneous ($H_0(x_0)$ is constant for all $x_0 \in [0,\tau_0]$),
- A4 $H_i(x_i)$ is continuously differentiable in x_i for all $x_i \in [0,\tau_i], i \in \{1,\dots,N\}$,

then the asymptotic behavior of reflection coefficient satisfies

$$\lim_{k\to+\infty} |r_0(k)| = \frac{N-1}{N+1} \tag{22}$$

where $|r_0(k)|$ is the modulus of the reflection coefficient for a star-shaped network. See the appendix of this paper for a proof of this result.

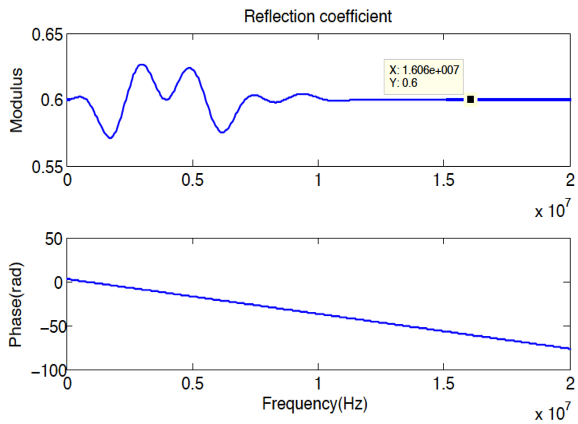

Fig. 4. Simulated reflection coefficient for a star shaped network

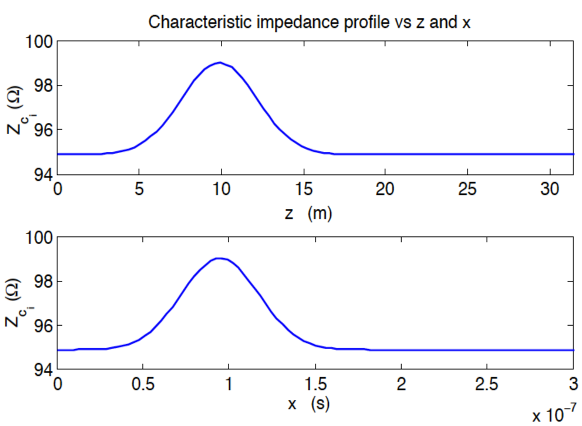

Fig. 5. Characteristic impedance profil.

C. Generalization to tree-shaped network.

The results of star-shaped network can be generalized to the tree-shaped network as in (Fig 3). The principle of the method is to treat the root level of a tree-shaped network as a star-shaped network with its branches composed of sub-networks, and each of the sub-networks (of tree-shape) is in turn treated as a star-shaped network, and so on. Conversely, a star-shaped network can be extended to a tree-shaped network by replacing each of its branches, except the branch b_0, by another star-shaped network. Each branch of the newly added star-shaped network can in turn be replaced by a star-shaped network, and so on.

To compute the reflection coefficient at the root terminal, the local reflection coefficient, $r_i(k,x_i)$ is computed along each branch directly connected to a leaf terminal, then propagated to branches of higher level, to finish by computing the reflection coefficient at the root terminal.
This algorithm can be implemented very easily in Matlab or similar softwares.

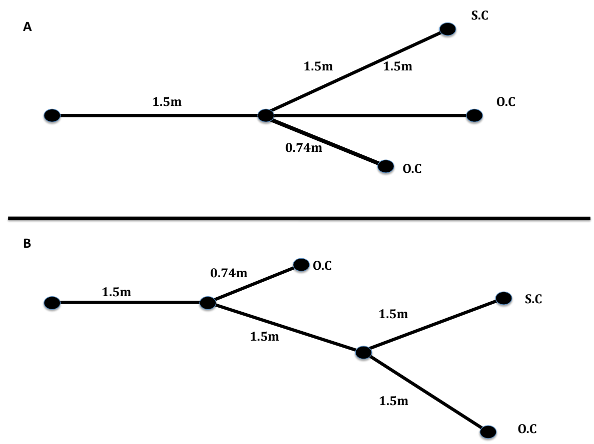

Fig. 6. Example of a star-shaped and tree-shaped networks

IV. SIMULATION STUDY

A numerical simulator of tree-shaped network has been implemented in Matlab based on the mathematical model presented in this paper. Results of simulation examples will be presented in this section. The comparison of these results with real measurements provides a validation of the model and the simulator.

A. Generation the reflection coefficient for an inhomogeneous star-shaped network.

we consider a star-shaped network of lossless transmission lines composed of 4 identical branches and a homogeneous branch b_0. The capacitance of each branch is kept to a constant value $C_i(x_i) = 0.1nF/m$ and the characteristic impedance $Z_{c_i} = \sqrt{L_i(x_i)/C_i(x_i)}$ for $i \in \{1,\ldots,4\}$ is depicted in Figure (5), in both z_i and x_i coordinates. At the end of branches 1, 2, 3 and 4 a matched load is connected. The simulated reflection coefficient $r(k)$ (modulus and phase) is shown in Figure (4). Remark that, in the high frequency regime, the modulus of reflexion coefficient converges to 0.6, this value is in agreement with the relation (22).

B. Comparison of simulations with measurements.

The purpose of this subsection is to compare simulated reflection coefficients with real reflection coefficients measured on two networks made of coaxial cables with the characteristic impedance $Z_c = 50\Omega$ and velocity $\gamma = 2.478.10^8\ m/s$.

For the first example, we consider a star-shaped network made up of 4 branches, as illustrated in Figure 6-A. The branches 1, 2, 3 are open-circuited. The reflection coefficient profile computed through the mathematical model is compared to the real reflection coefficient in Figure (6-A).

For the second example, we consider a tree-shaped network, that is shown in Figure (6-B). Two leaf branches of the tree are open-circuited, and another one is short-circuited. To obtain these results, we added the constant ohmic loss $R_i(x_i) = 1.7\Omega/m$ to each branch in the numerical simulator.

Figure (7) and Figure (8) present a good agreement between measures and simulations, both for modulus and

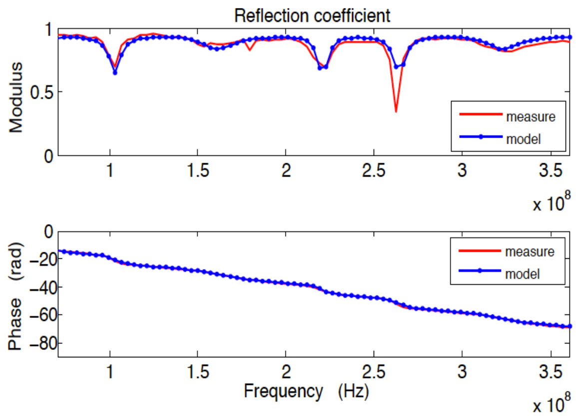

Fig. 7. Measured reflection coefficient compared to simulated reflection coefficient for a star shaped network.

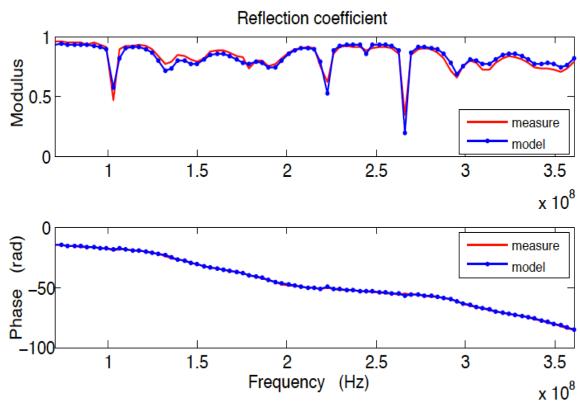

Fig. 8. Measured reflection coefficient compared to simulated reflection coefficient for a star shaped network.

phase of the reflection coefficient. The small difference in the modulus may come from measurements noises and losses in the connectors.

V. CONCLUSIONS AND FUTURE WORKS

A reduced mathematical model of reflectometry for characterizing complex wired electric networks with distributed *RLCG* parameters has been introduced in this paper. This model constitutes a first step for developing advanced methods for the diagnosis of failures in wired networks. It's also useful for numerical simulation of the studied networks, by computing reflection coefficients, as well as voltage and current waves. The comparison of simulations with real measurements provides a validation of the model and the simulator.

VI. ACKNOWLEDGMENTS

This work has been supported by the ANR 0-DEFECT project. The authors are grateful to Mostafa Smail, Lionel Pichon and Florent Loete of Laboratoire de Génie Electrique de Paris for having kindly provided the experimental data used in this paper.

APPENDIX

Now, we prove (21) and (22). In this particular case, the high frequency characteristic impedance as defined in (18), is continuous at the junction node J. Therefore,

$$H_0(0) = H_i(0)\ \forall i \in \{1,\dots,N\} \tag{23}$$

Applying this identity in (19), the apparent impedance $Z_0(k,0)$ can be expressed as follows

$$Z_0(k,0) = \frac{H_0(0)}{\sum_{i=1}^{N} \frac{1-r_i(k,0)}{1+r_i(k,0)}} \tag{24}$$

The value of $\rho_{r_0}(k)$ then verifies

$$\rho_{r_0}(k) = \frac{1-\sum_{i=1}^{N} \frac{1-r_i(k,0)}{1+r_i(k,0)}}{1+\sum_{i=1}^{N} \frac{1-r_i(k,0)}{1+r_i(k,0)}} \tag{25}$$

A simple computations imply the equation (21).

For a homogeneous branch b_0, the Riccati equation (15) satisfied by $r_0(k,x_0)$ reduces to the linear equation with a boundary condition at the load end of the branch b_0

$$\begin{cases} \dfrac{\partial r_0(k,x_0)}{\partial x_0} & = -2jkr_0(k,x_0) \\ r_0(k,0) & = \rho_{r_0}(k) \end{cases} \tag{26}$$

the solution of this equation writes

$$r_0(k,x_0) = \rho_{r_0}(k)e^{-2jkx_0} \tag{27}$$

Under the assumptions $A1$ through $A4$, we have

$$\lim_{k\to+\infty} r_i(k,0) = 0 \quad \forall i \in \{1,\dots,N\} \tag{28}$$

The two relations (21) and (28) imply

$$\lim_{k\to+\infty} r_0(k,x_0)e^{2jkx_0} \quad = \quad \frac{1-N}{1+N} \quad \forall x_0 \in [0,\tau_0] \tag{29}$$

Take the absolute value at both sides of this equation, while noticing that $r_0(k) = r_0(k,\tau_0)$ and $N \geq 1$, then the result of (22) is obtained.

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