

Fast Controls and Their Calculation

Alexander Daryin and Yulia Minaeva

Abstract—New technologies, such as control in quantum systems, may require that the control would act on a very small time horizon. Another requirement is that the control should be designed in a closed-loop form. A possible response to this demand is the use of *fast controls* [1]. They are introduced as bounded approximations of generalized impulse controls (belonging to the class of higher-order distributions).

I. IDEAL ZERO-TIME CONTROLS

Consider a linear differential equation in distributions:

$$\dot{x}(t) = A(t)x(t) + B(t)u + f^{(\alpha)} - f^{(\beta)},$$

$$x \in \mathbb{R}^n, \quad t \in [\alpha, \beta],$$

with k times continuously differentiable matrices $A(t) \in \mathbb{R}^{n \times n}$, $B(t) \in \mathbb{R}^{n \times m}$. The terms $f^{(\alpha)}$ and $f^{(\beta)}$ are two distributions [2], [3] from $D_{k,n}^*[\alpha, \beta]$ concentrated in points t_α and t_β respectively.

Recall that the space $D_{k,m}^*$ consists of k times differentiable functions $\varphi(t) : [\alpha, \beta] \rightarrow \mathbb{R}^m$ with support set contained in $[\alpha, \beta]$, endowed with a norm

$$\mathcal{G}[\varphi] = \max_{t \in [\alpha, \beta]} \gamma[\gamma_0(\varphi(t)), \dots, \gamma_k(\varphi^{(k)}(t))].$$

Here γ_k, γ are some finite-dimensional norms in vector spaces \mathbb{R}^m and \mathbb{R}^{k+1} respectively. The norm $\mathcal{G}[\varphi]$ defines a conjugate norm $\mathcal{G}^*[u]$ in the space $D_{k,m}^*[\alpha, \beta]$.

The generalized control $u(t)$ is chosen from the space $D_{k,m}^*[\alpha, \beta]$ and must ensure the existence of distribution $x(t) \in D_{k-1,n}^*[\alpha, \beta]$ with support on $[t_\alpha, t_\beta]$. This control may be represented in the form [4]

$$u(t) = \sum_{j=0}^k \frac{d^{j+1}U_j(t)}{dt^{j+1}}, \quad U_j \in BV[t_\alpha, t_\beta].$$

Here $BV[t_\alpha, t_\beta]$ is a space of functions $U(t)$ of bounded variation.

In [5], [6] it was shown that a completely controllable linear system may be steered from one state to another in fixed time by an ordinary impulse control ($k = 0$)

$$u(t) = \sum_{i=1}^N u_i \delta(t - \tau_i),$$

where the number of impulses $N \leq n$. Here the points τ_i are time moments when the impulses are applied to the system.

For $k > 0$ the controls include, apart from delta-functions, higher-order derivatives of delta-functions, which extends capabilities of the control. In particular, for $m \geq n - 1$ a

Moscow State (Lomonosov) University, Department of Computational Mathematics and Cybernetics, 1, Vorobiev gory, 119991 Moscow Russia. daryin@cs.msu.su, yminaeva@gmail.com.

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completely controllable linear system may be steered from one state to another by the control of type

$$u(t) = \sum_{j=0}^m u_j \delta^{(j)}(t - \tau). \quad (1)$$

in zero time.

The problem of feedback control in the class of distributions is solved using Hamilton–Jacobi–Bellman type variational inequalities [1].

II. FAST CONTROLS

Control (1) is an “ideal” one. Bounded functions approximating (1) are known as *fast controls*, since they are physically realizable and may steer a system to a given state in arbitrary small time [1]. Such controls may be found, for example, in the following form:

$$u_\Delta(t) = \sum_{j=0}^m u_j \Delta_{h_j}^{(j)}(t - \tau), \quad (2)$$

where $\Delta_h^{(j)}(t)$ approximate the derivatives of delta-function:

$$\Delta_h^{(0)}(t) = h^{-1} \mathbf{1}_{[0,h]}(t),$$

$$\Delta_h^{(j)}(t) = h^{-1} \left(\Delta_h^{(j-1)}(t) - \Delta_h^{(j-1)}(t-h) \right).$$

Here arises the problem of how to choose the parameters of control (2) — the coefficients h_j and vectors u_j . These parameters should be chosen due to physical requirements on the realizations of control.

III. DISCONTINUOUS APPROXIMATIONS

First, we consider fast controls with various restrictions:

1) bounded time of control:

$$\max_j \{(j+1)h_j\} \leq H;$$

2) hard bounds on control:

$$\|u_\Delta(t)\| \leq \mu;$$

3) separate hard bounds on approximations of generalized functions of all orders included in the control:

$$\|u_{\Delta,j}(t)\| \leq \mu_j,$$

$$u_{\Delta,j}(t) = u_j \Delta_{h_j}^{(j)}(t - \tau).$$

The indicated restrictions lead to the problems of moments of similar type. For example, for an approximation of the derivative $\delta^{(n)}(t)$ such a moment problem has the following solution:

$$\Delta_h^n(t) = \frac{1}{4} (-1)^n n! \left(\frac{2}{h}\right)^{(n+1)} \text{sign } U_n(t/h), \quad (3)$$

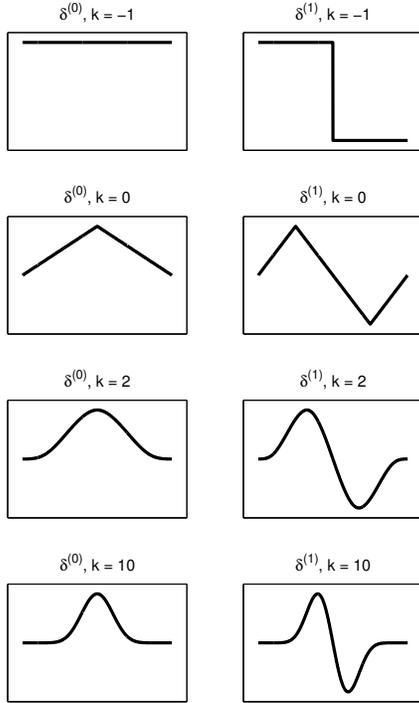


Fig. 1. Approximations of $\delta^{(n)}(t)$ (Part I)

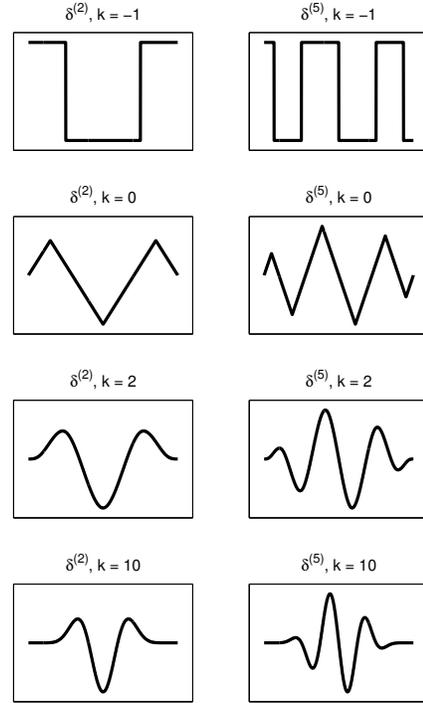


Fig. 2. Approximations of $\delta^{(n)}(t)$ (Part II)

where $U_n(t)$ is the Chebyshev polynomial of the second kind:

$$U_n(t) = \cos(n \arccost).$$

The approximation (3) is piecewise constant (and hence discontinuous), equal to $\pm \frac{1}{4}n! (\frac{2}{h})^{(n+1)}$ between Chebyshev points $t_k = h \cos \frac{\pi j}{n+1}$, $j = 0, \dots, n + 1$.

A. Haar Series

Another promising approach is to represent fast controls in terms of Haar series [7], since the basic element (wavelet) of this orthogonal system of functions is essentially an approximation of $\delta'(t)$:

$$\chi_n^{(k)}(t) = \begin{cases} \sqrt{2^n}, & t \in (\frac{2k-2}{2^{n+1}}, \frac{2k-1}{2^{n+1}}); \\ -\sqrt{2^n}, & t \in (\frac{2k-1}{2^{n+1}}, \frac{2k}{2^{n+1}}); \\ 0, & \text{otherwise}; \end{cases}$$

$$n = 0, 1, \dots, \quad k = 1, \dots, 2^n.$$

IV. SMOOTH APPROXIMATIONS

After that, we consider continuous or smooth approximations. To do this, we impose the bound on k -th derivative of the approximation $\|u_{\Delta}^{(k)}(t)\| \leq \mu$. It turns out that a k times smooth approximation of $\delta^{(n)}(t)$, $\Delta_{h,k}^n(t)$, is a

normalized $(k + 1)$ -fold integral of $\Delta_{h,k}^{n+k+1}(t)$. Here $k = -1$ corresponds to discontinuous approximations $\Delta_h^n(t)$, and $k = 0$ leads to continuous (but not smooth) approximations.

Approximations $\Delta_{h,k}^n(t)$ are piecewise polynomials of order k , with $k-1$ derivatives continuous at the junction points. The coefficients of these polynomials may be calculated recursively by explicit formulae.

V. EXAMPLES

In Figs. 1, 2 we present our approximations of $\delta(t)$ and its derivatives, with various degrees of smoothness.

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