Robust output controller design based on adaptive model reduction for parabolic PDE systems

Sivakumar Pitchaiah and Antonios Armaou†, *Senior Member, IEEE & Member AIChE*

***Abstract*— The problem of designing robust feedback controller for spatially distributed processes, described by parabolic PDE systems, is addressed by designing robust output feedback controllers using adaptive proper orthogonal decomposition methodology (APOD). Initially, an ensemble of eigenfunctions is constructed based on a relatively small data ensemble which is then recursively updated as additional process data becomes available periodically. These eigenfunctions are then utilized in deriving a reduced order model (ROM) of the PDE system by employing the Galerin's method. The obtained ROM is further utilized for the synthesis of robust feedback controllers via geometric techniques. Under the assumption that the number of measurements sensors is equal to the number of modes of the ROM, we obtain the the estimates for the states of the ROM using a static observer. Utilizing these estimated states in the robust controllers leads to robust output feedback controllers that guarantee boundedness of the state along with uncertainty attenuation in the infinite-dimensional system. As new data from the closed-loop process becomes available we update the ROM (and hence the robust controller) by employing APOD. The theoretic results are successfully applied to a representative example of dissipative PDEs with nonlinearities and uncertainty.**

I. Introduction

Many industrially relevant chemical processes involve coupling of transport phenomena (fluid flow, heat and mass transfer) and chemical reactions and are mathematically described by partial differential equation systems (PDEs). Typical examples [4] of such processes are plasma-enhanced chemical vapor deposition, Czochralski crystal growth, plug-flow and packed-bed reactors used to produce specialty chemicals and many other fluid dynamical systems. Designing feedback controllers for these distributed processes, to achieve desired product specifications is non trivial owing to the infinite dimensional mathematical descriptions.

The standard approach to design feedback controllers for these PDE systems are based upon the reduced order model (ROM) of the original infinite dimensional PDE system. One of the common approaches to compute the ROM is the method of weighted residuals, where the basis functions are found using proper-orthogonal decomposition (POD) along with method of snapshots [13], [5]. This method, however, assumes an *a priori* availability of a large ensemble of snapshots to characterize the behavior of any new trends that become available during the course of process evolution. However, generating such an ensemble is not straightforward (and may be experimentally infeasible) as it necessitates using a suitably designed input [5], [16] to excite the process sufficiently.

In our previous work [17], [9], we focused on the development of an adaptive model reduction methodology (called adaptive proper orthogonal decomposition (APOD)) to compute and recursively modify the empirical eigenfunctions, required for the formulation of ROM, in a computationally efficient manner as more information from the processes becomes available. Initially, the eigenfunctions were computed using POD employing relatively less number of snapshots that was available offline. The covariance matrix needed in the execution of POD is computed, offline, using these snapshots. The dominant eigenspace of this matrix is updated recursively (using APOD) as new snapshots from the process are added to the ensemble, simultaneously increasing or decreasing its dimensionality if required. Thus capturing any new trends that becomes available during process evolution. We maintain that as long as the dimensionality of the dominant eigenspace remains small (which is true for dissipative PDE systems), the computational burden for updating the dominant eigenspace remains small and can be easily performed online. The methodology was successfully applied in feedback control of spatially distributed processes.

The feedback controller designed in the above work, however, requires the availability of full state measurements of the process. The availability of such full state measurements is usually restricted due to limited availability of sensors. We addressed this issue [10] by requiring the availability of full state measurements only periodically (with a sufficiently large period), thus relaxing the necessity for full state measurements and extending the applicability of APOD. The designed static output feedback controller was successfully utilized to address the problem of stabilization and tracking in Kuramoto-Sivashinksy equation [2], [1] and FitzHugh-Nagumo equation [12], [3], [11].

However, the PDE models for most transport-reaction processes are not perfect; the uncertainties springing from partially known time-varying process parameters, exogenous disturbances, unmodeled dynamics. The controllers designed without considering these uncertainties might lead to unstable closed-loop system [4].

In the present work, we address the issue of robust controller design for processes modeled by uncertain dissipative *PDEs* using continuous measurements from restricted number of sensors. Combining a robust controller design

Financial support from the National Science Foundation, CAREER Award # CBET 06-44519, and American Chemical Society - Petroleum Research Fund Research Initiation Award #PRF 44300-G9, is gratefully acknowledged.

†Department of Chemical Engineering, The Pennsylvania State University, University Park, PA 16802, USA, Tel: (814) 865-5316, Email: armaou@engr.psu.edu.

ISBN 978-963-311-370-7

methodology for dissipative PDE systems with APOD and static feedback controllers that give partial information about the process behavior, we synthesize robust controllers that can account for complex process behavior, model uncertainty and irregular process domains. The proposed approach is applied to the representative example of a thin catalytic rod with uncertain process parameters. The proposed controller successfully stabilizes the closed loop system around the open-loop unstable steady state.

II. Mathematical preliminaries

We focus on the problem of feedback control of spatially distributed processes described by dissipative PDEs with the following state-space description:

$$\begin{aligned} \frac{\partial \bar{x}}{\partial t} &= \mathcal{L}\bar{x} + b(z)u + f(\bar{x}) + W(\bar{x})\theta(t), \\ y_c &= \textstyle\int_\Omega c(z)\bar{x}dz, \\ y_m &= \textstyle\int_\Omega s(z)\bar{x}dz. \end{aligned} \tag{1}$$

$$q\left(\bar{x}, \frac{d\bar{x}}{d\eta}, \ldots, \frac{d^{n_o-1}\bar{x}}{d\eta^{n_o-1}}\right) = 0 \text{ on } \Gamma \tag{2}$$

and the following initial condition

$$\bar{x}(z,0) = \bar{x_0}(z). \tag{3}$$

In the above PDE system, $\bar{x}(z,t) \in \mathbb{R}^n$ denotes the vector of state variables, $y_c \in \mathbb{R}^l$ denotes the vector of controlled outputs, t is the time, $y_m \in \mathbb{R}^p$ denotes the vector of measured outputs, $z \in \Omega \subset \mathbb{R}$ is the spatial coordinate, Ω is the domain of definition of the process and $\theta = [\theta_1, \cdots, \theta_q] \in \mathbb{R}^q$ denotes the vector of uncertain variables which may include uncertain process parameters or exogenous disturbances. $\mathcal{L}$ is a dissipative and linear (and bounded) spatial differential operator, $f(\bar{x})$ is a nonlinear vector function which is assumed to be sufficiently smooth with respect to its arguments, $c(z)$ is a known smooth vector function of z which is determined by the desired performance specifications in the domain Ω and $s(z)$ is a known smooth vector function of z which is determined by the location and type of measurement sensors (e.g., point or distributed sensing). $q(\bar{x}, \frac{d\bar{x}}{d\eta}, \ldots, \frac{d^{n_o-1}\bar{x}}{d\eta^{n_o-1}})$ is a nonlinear vector function which is assumed to be sufficiently smooth, $\left.\frac{d\bar{x}}{d\eta}\right|_\Gamma$ denotes the derivative in the direction perpendicular to the boundary, $\bar{x_0}(z)$ is the initial condition, $W(\bar{x})$ is a nonlinear vector function, $\theta(t)$ denotes uncertain process parameters or exogenous disturbances. $u = [u_1, u_2, \cdots, u_k] \in \mathbb{R}^k$ denotes the vector of manipulated inputs, $b(z) \in \mathbb{R}^{n\times k}$ is a known smooth matrix function of z of the form $[b_1(z), b_2(z), \cdots, b_l(z)]$, where $b_i(z)$ describes how the i^{th} control action $u_i(t)$ is distributed in the spatial domain Ω. We assume that for a given set of initial and boundary conditions the system of Eqs. 1-3 has a unique solution.

To simplify the presentation, we now formulate the parabolic PDE system of Eq.1 as an infinite dimensional system in the Hilbert space $\mathcal{H}(\Omega, \mathbb{R}^n)$ with $\mathcal{H}$ being the space of n-dimensional vector functions defined on Ω that satisfy the boundary conditions in Eq.2. We define the inner product and norm in $\mathcal{H}$ as follows:

$$(\phi_1, \phi_2) = \int_\Omega \phi_1(z)\phi_2(z)dz, \; ||\phi_1||_2 = (\phi_1, \phi_1)^{1/2} \tag{4}$$

where $\phi_1, \phi_2 \in \mathcal{H}[\Omega, \mathbb{R}^n]$. Defining the state function x on $\mathcal{H}[\Omega, \mathbb{R}^n]$ as $x(t) = \bar{x}(z,t), t > 0, z \in \Omega$, the operator $\mathcal{L}$ in $\mathcal{H}[\Omega, \mathbb{R}^n]$ as

$$\mathcal{A}x = \mathcal{L}\bar{x} \tag{5}$$

$$x \in D(\mathcal{A}) = \left\{ x \in \mathcal{H}[\Omega, \mathbb{R}^n]; q\left(\bar{x}, \frac{d\bar{x}}{d\eta}, \ldots, \frac{d^{n_o-1}\bar{x}}{d\eta^{n_o-1}}\right) = 0 \text{ on } \Gamma \right\}$$

and the input, controlled output and measured output operators as $\mathcal{B}u = bu$, $\mathcal{C}x = (c,x)$, $\mathcal{S}x = (s,x)$ the system of Eq.1-3 takes the form

$$\begin{aligned} \dot{x} = \mathcal{A}x + \mathcal{B}u + f(x), x(0) = x_0. \\ y_c = \mathcal{C}x, y_m = \mathcal{S}x. \end{aligned} \tag{6}$$

where $f(x) = f(\bar{x}(z,t))$ and $x_0 = \bar{x}_0(z)$. We assume that the nonlinear functions $f(x)$ and $W(x)$ are locally Lipschitz with respect to its arguments.

III. Construction of empirical eigenfunctions using adaptive proper orthogonal decomposition

In this section, we first utilize the available off-line process data and compute the empirical eigenfunctions. These eigenfunctions are updated online, using the closed loop process data, to capture the global dynamics of the system. For convenience, we slightly abuse the terminology and continue to address empirical eigenfunctions as eigenfunctions (or as basis) for the rest of the paper. Without loss of generality the presentation that follows focuses on $x(z) \in \mathbb{R}$.

A. Karhunen-Loève expansion

For comparison purposes we now present a brief overview of KLE [14], [6], also known as proper orthogonal decomposition(POD), in the context of the parabolic PDE system in Eq.1. Let $\{v_k\}_{k=1}^N$ be the collection of N data snapshots, where v_k denotes the snapshot of the system available at time t_k. Each snapshot, $\bar{x}(z,t_k)$, is the spatial profile of the system states at a particular time instant. The problem of obtaining the characteristic profile $\phi(z)$ from these snapshots v_k can be mathematically formulated as one of maximizing the following objective function.

$$\begin{aligned} &\text{Maximize } \frac{\langle (\phi, v_\kappa)^2 \rangle}{(\phi, \phi)} \\ &s.t. \; (\phi, \phi) = 1, \;\; \phi \in L^2([\Omega]) \end{aligned} \tag{7}$$

where $\langle \cdot \rangle$ is the ensemble average. The solution to the above problem is obtained as a solution of the following eigenvalue problem.

$$\int_\Omega K(z,\bar{z})\phi(\bar{z})d\bar{z} = \lambda\phi(z) \tag{8}$$

where $K(z,\bar{z})$ is the two-point correlation function given by the following expression:

$$K(z,\bar{z}) = \langle v_\kappa(z) v_\kappa(\bar{z}) \rangle \tag{9}$$

A computationally efficient method to obtain the solution of the above integral equation is provided by the method of snapshots wherein the requisite eigenfunctions are expressed as a linear combination of the available snapshots:

$$\phi(z) = \sum_k \psi^k v_k(z) \tag{10}$$

where ψ^k denotes the kth element of a vector ψ.
Substituting the above equation in Eq.8 the resultant eigenvalue problem is

$$C_N \psi = \lambda \psi \tag{11}$$

where the (κ,k)th element $C_N^{\kappa k}$ of matrix C_N is defined as

$$C_N^{\kappa k} := \frac{1}{N} \int_\Omega v_\kappa(\bar{z}) v_k(\bar{z}) d\bar{z} \tag{12}$$

and by construction vector, ψ represents the eigenvector of C_N. The solution of the above eigenvalue problem yields N eigenvectors $\psi_1, \psi_2, \cdots \psi_N$ which can be used in Eq.10 to construct N eigenfunctions $\phi_\kappa(z), \kappa = 1, \ldots, N$. By construction, matrix C_N is symmetric and positive semi-definite, and thus, its eigenvalues, λ_κ, $\kappa = 1, \ldots, N$, are real and non-negative. The relative magnitude of the eigenvalues represents a measure of the fraction of the "energy" embedded in the ensemble captured by the corresponding eigenfunctions. We order the calculated empirical eigenfunctions such that

$$\lambda_1 > \lambda_2 > \ldots > \lambda_N. \tag{13}$$

Furthermore, the computed eigenfunctions are orthogonal by construction.

B. Adaptive Proper Orthogonal Decomposition

KLE requires *apriori* availability of sufficiently large ensemble of PDE solution data in which all the possible spatial modes (including those that might appear in closed-loop evolution of the system Eq.1) are excited. This is necessary to ensure that the resulting eigenfunctions (and hence the reduced order model (in section 4) computed using these eigenfunctions) capture the global dynamics of the system during the closed-loop process evolution. This large ensemble of solution data should then be generated by computing the solutions of the PDE system for different values of $u(t)$ and different initial conditions [5]. However, it is difficult to generate such an ensemble and no well defined methodology exists for generating it.

The resulting empirical eigenfunctions, therefore, are representative of the corresponding ensemble only. As a result, the approximate ODE model of Eq.21 obtained through Galerkin's method using these empirical eigenfunctions may not be a valid approximation of the parabolic PDE model Eq.1 in a broad region of state space ($\mathcal{H}(\Omega)$). Also complete open-loop excitation of the process would be not be sufficient, as the closed-loop process will not in general stay within the same region of state space during it's evolution. As a result the optimal basis functions obtained through the open-loop system excitation may actually become irrelevant and superfluous, thus increasing the observer size unnecessarily.

One possible solution is to continue augmenting the ensemble of snapshots and subsequently recomputing the eigenfunctions as more information regarding the process becomes available. However, this would require repeated evaluations of the eigenvalue-eigenvector problem of a continuously increasing in size covariance matrix which may become computationally expensive with time and hence unsuitable for online computations.

To circumvent the latter problems, a computationally less expensive algorithm [17] that allows for construction and recursive update of eigenfunction, once new measurements from the process become available, will now be presented. This methodology called adaptive proper orthogonal decomposition (APOD) consists of two steps:

1) Initial basis Construction: We use the initially available collection of N off-line data snapshots, $\{v_k\}_{k=1}^N$, to construct the initial basis for Eq.1-3. We first construct the covariance matrix C_N then solve the following eigenvalue-eigenvector problem (Eq.11)

$$C_N \psi = \lambda \psi$$

to compute N eigenvalues. We partition the eigenspace of the covariance matrix, C_N, into two subspaces; the dominant one containing the modes which capture at least $\bar{\varepsilon}$ percent of energy in the ensemble (denoted as $\mathbb{P}$) and the orthogonal complement to $\mathbb{P}$ containing the rest of the modes (denoted as $\mathbb{Q}$). Such a partition is possible due to the fact that the dominant dynamics of dissipative PDEs are finite (typically small) dimensional [15]. Note that we define ε as the percentage energy of the ensemble captured by dominant eigenfunctions. We assume that out of N possible eigenvectors of C_N, m have the corresponding eigenvalues such that $\sum_{i=1}^{m} \lambda_i / \sum_{i=1}^{N} \lambda_i \leq \frac{\bar{\varepsilon}}{100}$; m eigenmodes of C_N capture ε percent of energy in the ensemble. These eigenvectors are then used in the following equation

$$\phi_i(z) = \sum_k \psi_i^k v_k(z),\ i = 1, \cdots, m.$$

to compute N eigenfunctions; here ϕ_i represents the i^{th} eigenfunction and ψ_i is the i^{th} eigenvector of C_N. An orthonormal basis for the subspace $\mathbb{P}$ can be obtained as:

$$Z = [\psi_1, \psi_2, \ldots, \psi_m],\ Z \in \mathbb{R}^{N \times m} \tag{14}$$

where $\psi_1, \psi_2, \ldots, \psi_m$ denote the eigenvectors of C_N that correspond to the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$. Note that the eigenfunctions computed by these eigenvectors capture the dominant dynamics of the PDE system of Eq.1-3. The orthogonal projection operators P and Q onto subspaces $\mathbb{P}$ and $\mathbb{Q}$ can be computed as

$$P = ZZ^T,\ Q = I - ZZ^T \tag{15}$$

where I denotes the identity matrix of dimension N.

2) Online basis refinement: During the course of closed loop process evolution we assume that periodically new snapshots become available. We recursively update the orthonormal basis for the subspace $\mathbb{P}$ upon the periodic arrival

of these new snapshots, possibly by increasing or decreasing the size of the basis if required and by maintaining the accuracy of basis by performing orthogonal power iteration, while the orthonormal basis for $\mathbb{Q}$ can be computed from the fact that $\mathbb{Q}$ is the orthonormal complement of $\mathbb{P}$. We maintain that the extra work required for the above process is small as long as the dimension of $\mathbb{P}$ is small (this amounts to choosing an appropriate value for $\bar{\varepsilon}$). The algorithm outlined below computes an approximation to Z without requiring the solution of the eigenvalue-eigenvector problem of the covariance matrix (Eq.11). We assume that during each step at most one eigenmode joins the subspace $\mathbb{P}$. We also assume that the process sampling is fast enough so that the appearance of new patterns is captured during process evolution. The algorithm requires the dimensionality of the covariance matrix, C_N, to remain constant, which we achieve by discarding the oldest snapshot from the ensemble as a new one is obtained. As a new snapshot from the process becomes available, the subspace $\mathbb{P}$ may change in the following three ways:

- The dimension of the dominant subspace $\mathbb{P}$ may increase i.e., one mode corresponding from $\mathbb{Q}$ becomes necessary to capture the desired percentage of energy in the ensemble.
  This is ascertained by monitoring the contribution of the dominant eigenvalue of $c_q = QC_NQ$, namely λ_{m+1} towards the total energy of the ensemble.
- Some of the eigenmodes of the subspace $\mathbb{P}$ may no longer be necessary to capture the required ε percent of the energy. In this case, the basis Z should be updated and its dimension should be simultaneously decreased. To test this the following $m \times m$ matrix $H = Z^T C_N Z$ is introduced. If only $\hat{m}$, with $\hat{m} < m$, eigenvalues of H are dominant then the basis Z is updated and its dimension is simultaneously decreased.
- The dimensionality of $\mathbb{P}$ remains unchanged. However the basis Z is updated, whenever the current basis is not accurate, to maintain the accuracy of the basis.
  The following one step power iteration $Z = \text{orth}(C_N Z)$ is executed (if necessary after analyzing the accuracy of the current basis) to maintain the accuracy of the basis after each addition of a snapshot.

Based on the new values of Z, we now compute the revised eigenfunctions $\phi_1, \phi_2, \ldots, \phi_m$ as a linear combination of the snapshots given by the following equation

$$\phi_i(z) = \sum_k \psi_i^k v_k(z), \quad i = 1, \cdots, m. \tag{16}$$

where ψ_i^k denotes the kth element of vector ψ_i.
Using these eigenfunctions, we refine our reduced order model (ROM) in Eq.21 and reconfigure the controller (Eq.24) using the updated reduced-order model. This step assures that the ROM captures new trends that appear when the process traverses through variable state space during closed-loop operation.

Remark 1: In the proposed approach for model reduction approach the snapshots used are obtained during closed-loop system evolution as opposed to all other proper orthogonal decomposition-based reduction approaches that are based on open-loop snapshots, and thus, these snapshots and the resulting ROM account for the impact of controller functional form on the process. It is important to note this intimate relation between APOD and the controller.

IV. Galerkin's method

In this section, we derive a finite-dimensional approximation (ROM) of the system of Eq.6 using the APOD computed empirical eigenfunctions. Let $\mathcal{H}_s$, $\mathcal{H}_f$ be the two subspaces of $\mathcal{H}(\Omega)$. We assume that the subspaces $\mathcal{H}_s$ and $\mathcal{H}_f$ are appropriately defined such that the $\mathcal{H}_f$ subspace includes all the fast evolving and stable process modes, while $\mathcal{H}_s$ includes the slow evolving and possibly unstable ones. By definition such a separation exists for dissipative processes and a finite number of modes belongs in $\mathcal{H}_s$, owing to the elliptic nature of the spatial differential operator.

For our model and controller construction, we assume that the eigenfunctions that capture the desired percentage of energy of the snapshot ensemble (denoted as ε in the subsection 3.2.2) also form a complete basis of $\mathcal{H}_s$. We thus define $\mathcal{H}_s = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}$ and $\mathcal{H}_f = \mathcal{H} \setminus \mathcal{H}_s$. Clearly $\mathcal{H} = \mathcal{H}_s \oplus \mathcal{H}_f$ and $\mathcal{H}_f$ is an infinite-dimensional subspace, while $\mathcal{H}_s$ is a finite-dimensional one.

Defining orthogonal projection operators $\mathcal{P}$ and Q the state $x \in \mathcal{H}(\Omega)$ can be decomposed as $x_s = \mathcal{P}x \in \mathcal{H}_s$ and $x_f = Qx \in \mathcal{H}_f$. The state x of the system of Eq.1 now can be expressed as:

$$x = x_s + x_f = \mathcal{P}x + Qx \tag{17}$$

Applying $\mathcal{P}$ and Q to the system of Eq.6 and using the above decomposition of x the system of Eq.6 can be equivalently expressed as:

$$\frac{dx_s}{dt} = \mathcal{A}_{s\alpha(t)} x_s + \mathcal{B}_{s\alpha(t)} u + f_{s\alpha(t)}(x_s, x_f) + W_{s\alpha(t)}(x_s, x_f)\theta(t)$$

$$\frac{\partial x_f}{\partial t} = \mathcal{A}_{f\alpha(t)}(x_f) + \mathcal{B}_{f\alpha(t)} u + f_{f\alpha(t)}(x_s, x_f) + W_{f\alpha(t)}(x_s, x_f)\theta(t)$$

$$y_c = \mathcal{C}_{\alpha(t)} x_s + \mathcal{C}_{\alpha(t)} x_f, y_m = \mathcal{S}_{\alpha 7(t)} x_s + \mathcal{S}_{\alpha(t)} x_f$$

$$x_s(0) = \mathcal{P}x(0) = \mathcal{P}x_0, x_f(0) = Qx(0) = Qx_0$$
$$\alpha(t) \in \mathcal{K} := \{1, \cdots, \wp\} \tag{18}$$

where $\mathcal{A}_s = \mathcal{P}\mathcal{A}(x_s + x_f)$, $\mathcal{B}_s = \mathcal{P}\mathcal{B}$, $f_s = \mathcal{P}f$, $\mathcal{A}_f = Q\mathcal{A}(x_s + x_f)$, $\mathcal{B}_f = Q\mathcal{B}$ and $f_f = Qf$ and the notation $\partial x_f / \partial t$ is used to denote that the state x_f belongs in an infinite dimensional subspace ($\mathcal{H}_f$). $\wp$ is the total number of times APOD was invoked during the closed-loop process operation, $\alpha : [0, \infty) \to \mathcal{K}$ is the switching signal which is assumed to be a piecewise continuous (from the right) function of time. Note that the updates made by APOD in the empirical eigenfunctions will also result in an update of $\mathcal{A}_s$, f_s, u, W_s, $\mathcal{A}_f$, f_f and W_f in the above equation. As a result $\alpha(t)$, which takes on different values in the finite index set $\mathcal{K}$, is used to to index $\mathcal{A}_s$, f_s, W_s, u, $\mathcal{A}_f$, f_f, W_f.

Using that $\varepsilon = \frac{|\lambda_{m+1}|}{|\lambda_1|}$, the system of Eq.18 can be written in the following form:

$$\frac{dx_s}{dt} = \mathcal{A}_{s\alpha(t)}x_s + \mathcal{B}_{s\alpha(t)}u + f_{s\alpha(t)}(x_s,x_f) + W_{s\alpha(t)}(x_s,x_f)\theta(t)$$

$$\varepsilon\frac{\partial x_f}{\partial t} = \mathcal{A}_f\varepsilon_{\alpha(t)}(x_f) + \varepsilon\mathcal{B}_{f\,\alpha(t)}u + \varepsilon f_{f\,\alpha(t)}(x_s,x_f) + \varepsilon W_{f\,\alpha(t)}(x_s,x_f)\theta(t)$$

$$y_c = \mathcal{C}_{\alpha(t)}x_s + \mathcal{C}_{\alpha(t)}x_f, y_m = \mathcal{S}_{\alpha(t)}x_s + \mathcal{S}_{\alpha(t)}x_f \tag{19}$$

where $\mathcal{A}_{f\varepsilon_{\alpha(t)}}$ is an unbounded differential operator defined as $\mathcal{A}_{f\varepsilon_{\alpha(t)}} = \varepsilon\mathcal{A}_{f\,\alpha(t)}$. As the operators $\mathcal{A}_{s\alpha(t)}$, $\mathcal{A}_{f\varepsilon_{\alpha(t)}}$ are of same order of magnitude the above system in Eq. 19 is in the standard singularly perturbed form [7] with x_s being the slow states and x_f being the fast states. Introducing the fast timescale $\tau = \frac{t}{\varepsilon}$ and setting $\varepsilon = 0$ we obtain the following infinite dimensional fast subsystem of Eq.19

$$\partial x_f/\partial t = \mathcal{A}_{f\varepsilon_{\alpha(t)}}x_f \tag{20}$$

Assumption 4.1: The infinite dimensional fast subsystem in Eq. 20 is assumed to be exponentially stable, as long as $\varepsilon << 1$.

Assumption 4.2: There exists known positive constants, θ_{bk} such that $|\theta_k(t)| \le \theta_{bk}$, $k = 1,\cdots,q$

Lemma 4.1: Consider the infinite dimensional system in Eq.1, the residual signal, r(t), given by the norm of the state reconstruction error $r(t) = \|x(t) - x_r(t)\|$ remains bounded and the bound $r_0(t)$ is given by:

$$r_0(t) = \sqrt{2e^{2(k_1+k_2+k_3)t}r(0)}$$

where k_1,k_2 and k_3 are positive numbers, x_r is the reconstructed state vector of x given by $x_r = \sum_{i=1}^{m} x_{si}\phi_i$. The proof of the above lemma will be presented in [8].

At the beginning of each time interval, wherein APOD is invoked, APOD updates the slow subspace ($\mathcal{H}_s$) and ensures that the value of ε to be small (i.e., $\varepsilon << 1$). However, during the closed-loop process operation (till APOD is invoked for the next time) the value of ε may increase thus rendering the fast subsystem in Eq. 20 to be no longer exponentially stable. To avoid such scenarios, we increase the dimensionality of the slow subspace $\mathcal{H}_s$, if required, by shifting the eigenfunctions from the fast subspace $\mathcal{H}_f$ till the bound on the residual signal, r(t), given by the norm of the state reconstruction error $r(t) = \|x(t) - x_r(t)\|$ in lemma 4.1 is satisfied thus eliminating the possibility of value of ε to increase. As a result the exponential stability of infinite- dimensional subsystem in Eq. 20 is guaranteed.

Setting $\varepsilon = 0$ in the system of Eq.18, we have that $x_f = 0$ and thus the following m-dimensional x_s subsystem is obtained:

$$\frac{dx_s}{dt} = \mathcal{A}_s(x_s,0) + \mathcal{B}_s u + f_s(x_s,0)$$

$$y_{cs} = \mathcal{C}x_s, y_{ms} = \mathcal{S}x_s \tag{21}$$

$$x_f \equiv 0$$

where the subscript s in y_{cs} and y_{ms} indicates that these outputs are associated with the x_s subsystem.

V. Robust nonlinear static Output feedback control

In this section, we utilize the m dimensional subsystem in Eq.21 in designing robust nonlinear static output feedback controllers for the parabolic PDE system of the form given in Eq. 1-3. As the availability of full state measurements is usually restricted in practice due to limited availability of sensors, we assume the availability of snapshots (for updating the ROM of Eq.21 using APOD) periodically. The controller, however, is designed based on continuous point measurements available from restricted number of sensors.

We consider the synthesis of static output feedback controller of the following form:

$$u = \mathcal{F}(y_m) \tag{22}$$

where $\mathcal{F}(\cdot)$ is a smooth vector function.

In order to estimate of the states x_s (namely $\tilde{x}_s$) of the finite dimensional system of Eq.21, from limited number of continuous point measurements, y_m we impose the following requirement on the number of available measurements.

Assumption 5.1: The number of available point measurements, p, is equal to or greater than the number of modes of the x_s subsystem, m, i.e. $p \ge m$, and the inverse of the operator $\mathcal{S}$, $\mathcal{S}^{\perp}$, exists, so that $\tilde{x}_s = \mathcal{S}^{\perp}y_m$.

When the finite-dimensional space, $\mathcal{H}_s$, is expressed using the empirical eigenfunctions as basis functions the operator $\mathcal{S}^{\perp}$ becomes a matrix and is given by

$$\mathcal{S}^{\perp} = (\mathcal{S}^T\mathcal{S})^{-1}\mathcal{S}^T \tag{23}$$

Note that the existence of inverse of matrix $\mathcal{S}$ depends on the location and shape of the measurement sensors; this amounts to properly choosing $s(z)$ in Eq.1.

Assumption 5.2: $l = m$, i.e., the number of control actuators is equal to the number of slow modes and the inverse of matrix $\mathcal{B}_s$ exists.

Theorem 5.1: Consider the system of Eqs.1-3 for which assumptions holds and the finite-dimensional system of Eq. for which assumptions 4.2 & 5.2 holds, under the robust output feedback controller.

$$\begin{aligned} u &= \mathcal{B}_{s_{\alpha(t)}}^{-1}((\Lambda_s - \mathcal{A}_{s\alpha(t)})\mathcal{S}^{\perp}y_m - f_{s\alpha(t)}(\mathcal{S}^{\perp}y_m,0)\dots \\ &- \chi(\sum_{k=1}^{q} W_{s\alpha(t),k}(\mathcal{S}^{\perp}y_m,0)\theta_{bk})\frac{\mathcal{S}^{\perp}y_m}{|\mathcal{S}^{\perp}y_m| + \phi}) \end{aligned} \tag{24}$$

where χ, ϕ are adjustable parameters with $\chi > 1$ and $\phi > 0$. Then there exists positive real numbers (δ,ϕ^*) such that for each $\phi \le \phi^*$, there exists $\varepsilon^*(\phi)$ such that if $\phi \le \phi^*$, $\varepsilon \le \varepsilon^*(\phi)$ and $max\{|x_s(0), \|x_f(0)\|_2, \|\theta\|, \|\dot{\theta}\|\} \le \delta$. Then the above controller guarantees the boundedness of the state of the closed-loop system

The proof of the above theorem will be presented in [8].

VI. Application to diffusion reaction process

We use the above methodology to stabilize an unstable steady state of a typical diffusion-reaction process with a time varying uncertainty. Specifically, we consider a zero-th order exothermic reaction $A \rightarrow B$ taking place on a thin catalytic rod. The temperature of the rod is adjusted by means of an actuator (by cooling the rod) located along the length of the rod. Assuming that the reactant A is present in excess, the spatial profile of the dimensionless temperature of the rod is described by the following parabolic PDE

$$\begin{aligned} \frac{\partial x}{\partial t} &= \frac{\partial^2 x}{\partial z^2} + \beta_{T,n}(e^{-\gamma/(1+x)} - e^{-\gamma}) \\ &+ \beta_U (b(z)u(t) - x) + e^{-\gamma/(1+x)}\theta(t), \end{aligned} \tag{25}$$

subject to the following boundary condition and initial conditions

$$x(0,t) = 0,\ x(\pi,t) = 0,\ x(z,0) = 0.05. \tag{26}$$

The dimensionless rod temperature is given as $x = \frac{T-T_0}{T_0}$, where T is the temperature of the reactor in ${}^0K$ and T_0 is the reference temperature used. The domain of this process is $\Omega = [0,\pi]$; z is the spatial coordinate along the axis of the rod, $\beta_{T,n}$ denotes the nominal dimensionless heat of reaction, γ denotes the dimensionless activation energy, β_U denotes the dimensionless heat transfer coefficient, $u(t)$ denotes the magnitude of actuation, $\theta(t)$ denotes a time varying uncertainty in the dimensionless heat of reaction and $b(z)$ accounts for the spatial profile of the actuator. A spatially distributed actuation with $b(z) = H(z-0.3\pi) - H(z-0.7\pi)$, where $H(\cdot)$ denotes the standard Heaviside function was considered. Full state measurements (snapshots) of the system are assumed to be available periodically at a period of $t = 0.25$. On the other hand, continuous point measurements were assumed to be available from 6 point measurement sensors placed uniformly across the domain of the process $(-\pi,\pi)$. The sensor shape distribution function, $s(z)$ at these respective positions would be $s_i(z) = \delta(z-P_i); i = 1,\ldots,6$, here P_i is the location of i^{th} sensor. The nominal values of the parameters used were $\beta_{T,n} = 50$, $\gamma = 4$, and $\beta_U = 2$. In this numerical study, the slowly varying uncertainty in the process model (Eqs.25-26) is assigned to be $\theta(t) = \beta_{T,n} sin(0.524t)$.

For the above system the spatial differential operator $\mathcal{L}$ in Eq.1 takes the form

$$\mathcal{L}x = \frac{\partial^2 x}{\partial z^2}$$

the nonlinear function $f(x)$ can be represented as

$$f(x) = \beta_T(e^{-\gamma/(1+x)} - e^{-\gamma}) - \beta_U x$$

and the process uncertainty $\mathcal{W}(x)\theta = e^{\frac{-\gamma}{1+x}}\theta(t)$. Note that the process uncertainty is a non vanishing term in the PDE.

Numerical results

Figure 1 presents the evolution of the PDE for $u(t) = 0$ from an initial condition of $x(z,0) = 0.05$. It is observed that the system evolves away from the above steady-state leading to a very poor open-loop behavior for $x(z,t)$. Hence, we conclude that the steady-state $x(z,t) = 0$ is an unstable one. The control problem can be formulated as designing a state feedback controller that stabilizes the rod temperature close to the spatially open-loop unstable steady-state.

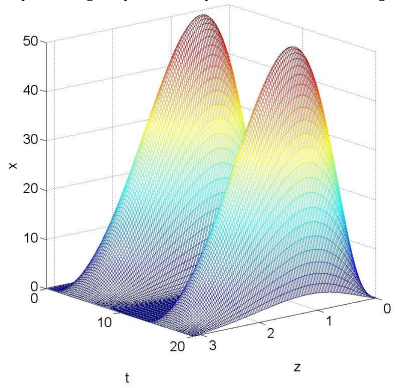

Fig. 1. (a) Open-loop profile of Eq.25 for extended time interval.

In order to obtain a finite dimensional approximation of the infinite dimensional system of Eq.25, initially an ensemble of 100 snapshots was generated for $u(t) = 0$. Each snapshot is a spatial profile obtained at a fixed time instant from numerical simulations of Eq.25-Eq.26. Note that an exhaustive sampling of the state-space of the PDE for a number of different initial conditions and magnitudes of actuation was *not* required during the ensemble generation phase. Application of POD at this initial stage resulted in a two dominant eigenfunction which captured more than 99.9% of the energy embedded in the ensemble. To stabilize the system at the spatially uniform steady-state $x(z,t) = 0$ the controlled output, $y_{c,i}$, is chosen to be the first dominant eigenmode.

Based on this eigenfunction a ROM was subsequently derived by applying Galerkin's method to the PDE system of Eq.25-26. Subsequently, the ROM was utilized to design a output-feedback robust controller in Eq.24. The adjustable parameters χ, ϕ were assigned to be 1.2 and 0.005 respectively. During closed-loop process operation it was assumed that snapshots of the process evolution are available every $t_s = 0.25$ seconds.

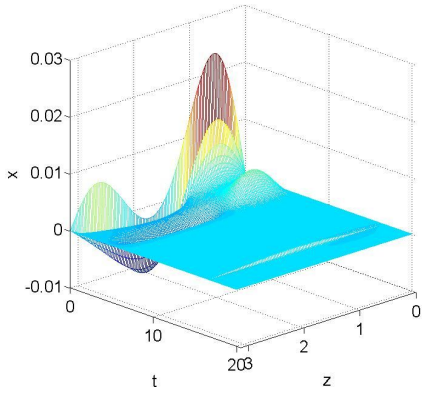

Fig. 2. Closed loop temperature profile of Eq.25 using distributed actuation $b(z) = H(z-0.3\pi) - H(z-0.7\pi)$.

Figure 2 presents the spatiotemporal profile of the rod temperature under closed-loop operation. Clearly the con-

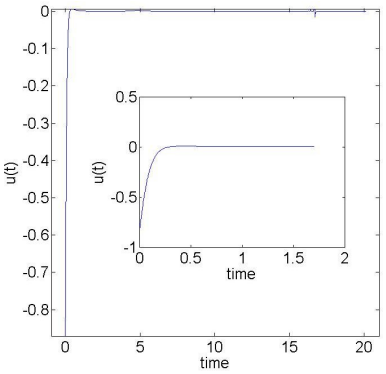

Fig. 3. Control action needed to stabilze Eq.25 calculated using robust controller; no measurement noise.

troller regulates the process in a neighborhood close to the desired setpoint $x(z,t) = 0$, attenuating the effect of the uncertain variable, β_T, on the process. Figure 3 presents the corresponding control action, $u(t)$ that was utilized to achieve the above control objective. We note that the control action is a smooth function of time, without any discontinuities and chattering. As the new process measurements from the closed-loop operation were included in the ensemble while simultaneously old snapshots were removed, the slow subspace ($\mathcal{H}_s$) was updated thus ensuring the validity of the ROM throughout the closed-loop process operation.

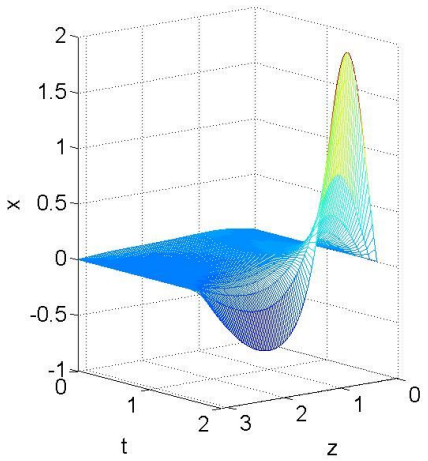

Fig. 4. Closed loop temperature profile of Eq.25 using distributed actuation $b(z) = H(z-0.3\pi) - H(z-0.7\pi)$ (no uncertainty compensation).

Finally to present the importance of the uncertainty compensation in the designed controller (Eq.24), we present the evolution of the close-loop rod temperature profile (Eq.25) without uncertainty compensation in figure 4. Clearly, process becomes unstable as the controller does not account for the uncertainty.

REFERENCES

[1] A. ADROVER AND M. GIONA, *Modal reduction of PDE models by means of snapshot archetypes*, Physica D, 182 (2003), pp. 23–45.

[2] A. ARMAOU AND P. D. CHRISTOFIDES, *Dynamic optimization of dissipative PDE systems using nonlinear order reduction*, Chem. Eng. Sci., 57 (2002), pp. 5083–5114.

[3] A. ARMAOU, C. I. SIETTOS, AND I. G. KEVREKIDIS, *Time-steppers and 'coarse' control of distributed microscopic processes*, Int. J. Robust & Nonlin. Contr., 14 (2004), pp. 89–111.

[4] P. D. CHRISTOFIDES, *Nonlinear and Robust Control of PDE Systems*, Birkhäuser, New York, 2001.

[5] M. D. GRAHAM AND I. G. KEVREKIDIS, *Alternative approaches to the Karhunen-Loève decomposition for model reduction and data analysis*, Comp. & Chem. Eng., 20 (1996), pp. 495–506.

[6] P. HOLMES, J. L. LUMLEY, AND G. BERKOOZ, *Turbulence, Coherent Structures, Dynamical Systems and Symmetry*, Cambridge University Press, New York, 1996.

[7] P. V. KOKOTOVIC, H. K. KHALIL, AND O. J., *Singular perturbations in Control: Analysis and Design*, Academic Press, London, 1986.

[8] S. PITCHAIAH AND A. ARMAOU, *Robust output feedback control of distributed parameter systems using adaptive proper orthogonal decomposition*, Manuscript in preparation.

[9] S. PITCHAIAH AND A. ARMAOU, *Adaptive model reduction for output feedback control of dissipative PDE systems*, In the proceedings of Mathematical Theory of Networks and Systems, (2008).

[10] S. PITCHAIAH AND A. ARMAOU, *Output feedback control of distributed parameter systems using adaptive proper orthogonal decomposition*, Ind. & Eng. Chem. Res., accepted (2010).

[11] S. Y. SHVARTSMAN AND I. G. KEVREKIDIS, *Nonlinear model reduction for control of distributed parameter systems: a computer assisted study*, AICHE J., 44 (1998), pp. 1579–1595.

[12] C. I. SIETTOS, A. ARMAOU, A. G. MAKEEV, AND I. G. KEVREKIDIS, *Microscopic/stochastic timesteppers and coarse control: a kinetic Monte Carlo example*, AIChE J., 49 (2003), pp. 1922–1926.

[13] L. SIROVICH, *Turbulence and the dynamics of coherent structures: part I, II and III*, Quarterly of Applied Mathematics, XLV (1987), pp. 561–590.

[14] ———, *Turbulence and the dynamics of coherent structures: part I, II and III*, Quarterly of Applied Mathematics, XLV (1987), pp. 561–590.

[15] R. TEMAM, *Infinite-Dimensional Dynamical Systems in Mechanics and Physics*, New york: Springer-Verlag, 1988.

[16] A. VARSHNEY AND A. ARMAOU, *Optimal operation of GaN thin film epitaxy employing control vector parameterization*, AIChE J., 52 (2006), pp. 1378–1391.

[17] A. VARSHNEY, S. PITCHAIAH, AND A. ARMAOU, *Feedback control of dissipative distributed parameter systems using adaptive model reduction*, AICHE J., 55 (2008), pp. 906–918.