Controllability of Networked Systems

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***Abstract*— In this paper we investigate the controllability properties associated with networked control systems whose information exchange takes place over a static communication network. The control signal is assumed to be injected into the network at a given input node and its influence is propagated through the network through a nearest-neighbor interaction rule employed to ensure network cohesion. In particular, the problem of driving a collection of mobile robots to a given target destination is studied, and conditions are given for this to be possible, based on tools from algebraic graph theory. The main result is a necessary and sufficient condition for an interaction topology to be controllable, given in terms of the network's external, equitable partitions.**

I. INTRODUCTION

A number of decentralized control laws have been designed for networked multi-agent systems to achieve a vast array of objectives such as swarming, flocking, alignment, cohesion, rendezvous, formation maintenance, and coverage, just to name a few. (See for example, [1], [2], [5], [7], [10].) But, what has not received that much attention is the question if such networks can be effectively controlled in the first place.

In this paper we follow the *leader-follower* (or *anchor-floater*) idea where the input signal is injected at a particular input node (the leader/anchor), while the remaining nodes (followers/floaters) execute a particular interaction law. Under this construction, as discussed in [6], [9], [11], certain network topologies are better than others when it comes to being able to effectively control the system. For instance, if the network topology is given by a complete graph and the interaction dynamics is the nearest-neighbor averaging rule, what can effectively be controlled is just the centroid of the network, i.e., even though there are lots of agents in the network, the dimension of the controllable subspace is one. (Or d if the state of each node takes on values in $\mathbb{R}^d$.)

We will recall some of the main results for controllability of single-leader leader-follower networks and then extend these results by giving a network characterization of the controllable subspace in terms of the quotient graphs and the associated dynamics over these reduced order graphs, induced by a particular choice of node partitions.

II. SINGLE-LEADER NETWORKS

Consider a network comprising of dynamical nodes $i = 1, \ldots, N$, whose dynamics evolve according to the nearest neighbor-averaging rule known as the *consensus equation* (see e.g., [4], [7], [8], [12])

$$\dot{x}_i = -\sum_{j \in \mathcal{N}_i} (x_i - x_j). \tag{1}$$

Here $x_i \in \mathbb{R},\ i = 1, \ldots, N$ is the state of the individual nodes (assumed, without loss of generality, to be scalar) and $\mathcal{N}_i$ is the set of neighboring nodes to node i. We will assume that the neighborhood information in the network is encoded through a static and undirected graph structure $G = (V, E)$, with the node set $V = \{1, \ldots, N\}$ being the set of nodes, the edge set is a set of unordered pairs $E \subseteq V \times V$, and $\mathcal{N}_i = \{j \in V \mid (i, j) \in E\}$. The consensus equation is known to provide cohesion to the network in the sense that each state asymptotically approaches the stationary average of all the agents in the network if and only if the underlying graph is connected. (For an introduction to such and other *agreement protocols*, see [7], [10].)

Now, assume that we would like to control this network by injecting a control signal at one of the nodes, say node N, as

$$\dot{x}_N = v, \tag{2}$$

while all other nodes satisfy the dynamics given in Equation 1. This amounts to actively controlling a single so-called leader-node, and then using the network to propagate the control signals to the rest of the nodes. And, what we are interested in is to try to characterize the controllability properties of this network from a purely graph theoretic vantage point, i.e., to relate controllability to the underlying graph topology. For this we need some basic tools from algebraic graph theory. (For a good introduction to this topic, see [3].)

Let Δ be the $N \times N$ *degree matrix* associated with the graph, with entries given by

$$[\Delta]_{i,j} = \begin{cases} \deg(i) & \text{if } i = j \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

where the degree $\deg(i) = |\mathcal{N}_i|$ is the numbers of neighbors to node i (and $|\cdot|$ denotes cardinality). Similarly the *adjaceny matrix* $\mathcal{A}$ is given by

$$[\mathcal{A}]_{i,j} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 \text{ otherwise.} \end{cases} \tag{4}$$

The final matrix, the *graph Laplacian*, needed for our discussion is given by $L = \Delta - \mathcal{A}$, and we choose to decompose L as

$$L = \left[\begin{array}{c|c} -A & -B \\ \hline B^T & \lambda \end{array} \right], \tag{5}$$

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where $A = A^T$ is $(N-1)\times(N-1)$, B is $(N-1)\times 1$, and $\lambda \in \mathbb{R}$. The point behind this decomposition is that if we gather the states from all non-leader nodes as $x = [x_1, \ldots, x_{N-1}]^T$ and assume that we control node N directly (instead through its derivative), i.e., $x_N = u$, the dynamics of the controlled network can be rewritten on a very familiar form as

$$\dot{x} = Ax + Bu, \qquad (6)$$

as shown in [9].

Now, what we would like to know is what the controllability properties are associated with this system. In particular, we would like to avoid rank tests and instead obtain characterizations of what the network topology should look like in order to render the system completely controllable. To this end, we need to introduce a few additional graph theoretic constructs. By a *partition* of the graph $G = (V, E)$ we understand a map $\pi : V \to \{C_1, \ldots, C_K\}$, where we say that $\pi(i)$ denotes the *cell* that node i gets mapped to, and we use $\text{dom}(\pi)$ to denote the *domain* to which π maps, i.e., $\text{dom}(\pi) = \{C_1, \ldots, C_K\}$. Similarly, the operation $\pi^{-1}(C_i) = \{j \in V \mid \pi(j) = C_i\}$ returns the set of nodes that are mapped to cell C_i. An example of such a node partition is given in Figure 1.

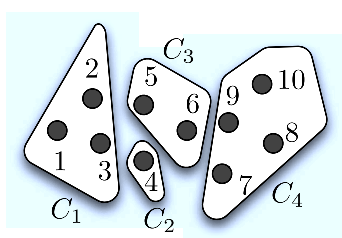

Fig. 1. A partition of the node set into four cells, with $\pi^{-1}(C_1) = \{1,2,3\}, \pi^{-1}(C_2) = \{4\}, \pi^{-1}(C_3) = \{5,6\}, \pi^{-1}(C_4) = \{7,8,9,10\}$.

By the *node-to-cell degree* $\text{deg}_\pi(i, C_j)$ we understand the number of neighbors that node i has in cell C_j under the partition π, i.e.,

$$\text{deg}_\pi(i, C_j) = |\{k \in V \mid \pi(k) = C_j \text{ and } (i,k) \in E\}|. \qquad (7)$$

Definition 2.1: A partition π is equitable *if all nodes in a cell have the same node-to-cell degree to all other cells, i.e., if, for all $C_i, C_j \in \text{dom}(\pi)$,*

$$\text{deg}_\pi(k, C_j) = \text{deg}_\pi(\ell, C_j),\ \forall k, \ell \in \pi^{-1}(C_i). \qquad (8)$$

As each node in a given cell has the same number of neighbors in adjacent cells (as long as the partition is equitable), we can in this case talk about the *cell-to-cell degree*. in other words, given an equitable partition π, $\text{deg}_\pi(C_i, C_j) = \text{deg}_\pi(k, C_j),\ \forall k \in \pi^{-1}(C_i)$.

This is almost the construction one needs to completely characterize the controllability properties of the network. However, what we need to do is produce partitions that are equitable between cells (i.e., all agents in a given cell have the same number of neighbors in adjacent cells), but where we do not care about the structure inside a cell. This leads us to the notion of an *external equitable partition* (EEP):

Definition 2.2: A partition π is an EEP *if, for all $C_i, C_j \in \text{dom}(\pi)$, where $C_i \neq C_j$,*

$$\text{deg}_\pi(k, C_j) = \text{deg}_\pi(\ell, C_j),\ \forall k, \ell \in \pi^{-1}(C_i). \qquad (9)$$

We are particularly interested in EEPs that place the leader node in a singleton cell, i.e., partitions where $\pi^{-1}(\pi(N)) = \{N\}$, and we refer to such EEPs as *leader-invariant*. Moreover, we say that a leader-invariant EEP is *maximal* if its domain has the smallest cardinality, i.e., if it contains the fewest possible cells, and we let $\pi^\star$ denote this maximal, leader-invariant EEP. (We note that given a graph G, $\pi^\star$ always exists uniquely.) Examples of these constructions are shown in Figure 2.

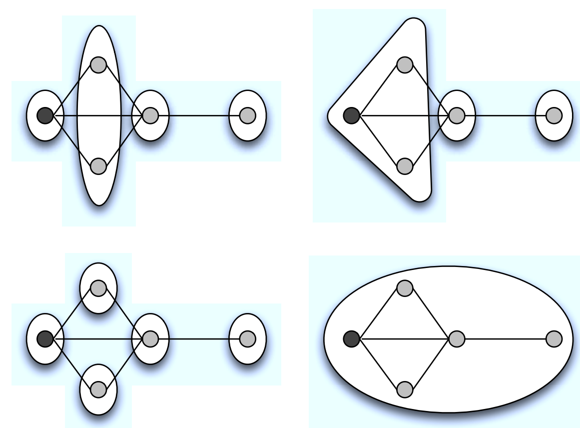

Fig. 2. A graph is given together with its four EEPs. The leader-node (black node) is in a singleton cell in the two left-most figures and, as such, they correspond to leader-invariant EEPs. Of these two leader-invariant EEPs, the top-left partition has the fewest number of cells and that partition is thus maximal. (We note that this maximal partition is not trivial since one cell contains two nodes.)

Using the construction, we can state the following key theorem from [6].

Theorem 2.1: The networked system in Equation 6 is completely controllable if and only if $\pi^\star$ is trivial and G is connected, i.e., $\pi^{\star -1}(\pi^\star(i)) = \{i\},\ \forall i \in V$.

We omit the proof of this result and refer the reader to [6]. In fact, for the remainder of this section, we will recall some of the key results from that work and then move on to the new contributions in the next section.

Theorem 2.1 allows us to characterize controllability purely in terms of the network's graph topology, i.e., it does not rely on any rank tests. Moreover, in [6], it was shown that the difference between states within cells in $\text{dom}(\pi^\star)$ is uncontrollable and if G is connected they decay asymptotically due to the fact that A in Equation 5 is positive definite if the graph is connected, i.e.,

Theorem 2.2: If G is connected, with $\pi^\star$ being its maximal, leader-invariant EEP, then for all $C_i \in \text{dom}(\pi^\star)$

$$\lim_{t\to\infty}(x_k(t) - x_\ell(t)) = 0,\ \forall k, \ell \in \pi^{\star -1}(C_i). \qquad (10)$$

But we can do even better than this in that we can characterize what the dimension of the controllable subspace is, which tells us that for connected graphs, the previous theorem completely characterizes what we cannot control. (Again, we refer to [6] for the proof of this result.)

Theorem 2.3: Let (A, B) be given in Equation 6 and let Γ be the controllability matrix. Then

$$\textit{rank}(\Gamma) = |\textit{dom}(\pi^\star)| - 1. \tag{11}$$

These results, although elegant, do not tell us if we can in fact give the controllable subspace a graph theoretic interpretation, i.e., if there is a network structure associated with the controllable subspace, which is the topic of the next section.

III. QUOTIENT GRAPH DYNAMICS

In order to establish a graph theoretic interpretation of the controllable subspace, we need the notion of a *quotient graph*.

Definition 3.1: Given a graph G together with an EEP π, the quotient graph *$G\backslash\pi = (V_\pi, E_\pi, w_\pi)$ is the weighted and directed graph whose node set is $V_\pi = \textit{dom}(\pi)$, the edge set is the set of ordered pairs such that $(C_i, C_j) \in E_\pi \Leftrightarrow \textit{deg}_\pi(C_i, C_j) \neq 0,\ C_i \neq C_j$, and the weight between cells is given by the cell-to-cell degree, i.e., $w_\pi(C_i, C_j) = \textit{deg}_\pi(C_i, C_j),\ C_i \neq C_j$.*

We note that, in general, $w_\pi(C_i, C_j) \neq w_\pi(C_j, C_i)$ and an example of this is seen in Figure 3. We will also use the notation $C_j \in \mathcal{N}_{\pi, C_j}$ to denote the fact that nodes in $\pi^{-1}(C_i)$ are adjacent to at least some nodes in $\pi^{-1}(C_j)$.

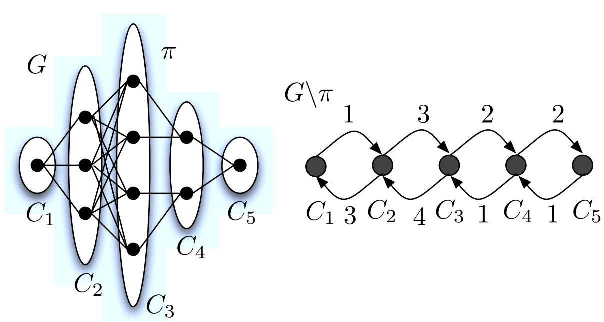

Fig. 3. A graph G with an EEP π (left) and the resulting weighted and directed quotient graph $G\backslash\pi$ (right). For this quotient graph, we have $w_\pi(C_i, C_j) \neq w_\pi(C_j, C_i)$, i.e., the edge weights are different along different directions.

Lemma 3.1: Let G be connected, π be an EEP, and $G\backslash\pi$ be the quotient graph. Then, for all $(C_i, C_j) \in E_\pi$,

$$\frac{|\pi^{-1}(C_i)|}{|\pi^{-1}(C_j)|} = \frac{w_\pi(C_i, C_j)}{w_\pi(C_j, C_i)}. \tag{12}$$

Proof: The total number of edges going in to nodes in $\pi^{-1}(C_i)$ from C_j (in the original graph G) is $w_\pi(i, j)|\pi^{-1}(C_i)|$. But since G is undirected, this is also the number of edges going to nodes in $\pi^{-1}(C_j)$ from C_i, and as such

$$w_\pi(C_j, C_i) = \frac{w_\pi(C_i, C_j)|\pi^{-1}(C_i)|}{|\pi^{-1}(C_j)|}, \tag{13}$$

and the lemma follows. ∎

As $V_{\pi^\star} = \text{dom}(\pi^\star)$ we should be able to endow the quotient graph with a dynamics such that the resulting system is completely controllable. And, as the difference between state values inside a cell in the EEP vanishes asymptotically, what we can in fact control should thus be the average inside a cell. For this, we let ξ_i be the average state value of a cell $C_i \in \text{dom}(\pi^\star))$, i.e.,

$$\xi_i = \frac{1}{|\pi^{\star-1}(C_i)|} \sum_{j \in \pi^{\star-1}(C_i)} x_j. \tag{14}$$

Taking the derivative of ξ_i, where we assume that cell C_i does not correspond to the cell containing the input node, i.e., $\pi^{\star-1}(C_i) \neq \{N\}$, we get that

$$\dot{\xi}_i = \frac{1}{|\pi^{\star-1}(C_i)|} \sum_{j \in \pi^{\star-1}(C_i)} \sum_{k \in \mathcal{N}_j} (x_k - x_j), \tag{15}$$

which we can rewrite as

$$\begin{aligned} \dot{\xi}_i = \tfrac{1}{|\pi^{\star-1}(C_i)|} \Big(&\textstyle\sum_{C_j \in \mathcal{N}_{\pi^\star, C_i}} \Big[\sum_{k \in \pi^{\star-1}(C_j)} w_{j,i} x_k \\ &- \textstyle\sum_{\ell \in \pi^{\star-1}(C_i)} w_{i,j} x_\ell \Big] \\ &+ \textstyle\sum_{\ell \in \pi^{\star-1}(C_i)} \Big[\sum_{n \in \pi^{\star-1}(C_i),\ n \in \mathcal{N}_\ell} (x_n - x_\ell) \Big] \Big). \end{aligned} \tag{16}$$

Here the first term corresponds to nodes that are adjacent to nodes in cell C_i but that are not in that cell. The second term corresponds to nodes that are in cell C_i, and together these two terms describes the dynamics between nodes in different cells. Finally, the last term captures the effect on the dynamics in terms of nodes that are in the same cell, and it is straightforward to show that this term is 0.

Now, using Lemma 3.1, this expression simplifies to

$$\begin{aligned} \dot{x}_i = \textstyle\sum_{C_j \in \mathcal{N}_{\pi^\star, C_i}} w_{i,j} \Big(&\tfrac{1}{|\pi^{\star-1}(C_j)|} \textstyle\sum_{k \in \pi^{\star-1}(C_j)} x_k \\ &- \tfrac{1}{|\pi^{\star-1}(C_i)|} \textstyle\sum_{\ell \in \pi^{\star-1}(C_i)} x_\ell \Big), \end{aligned} \tag{17}$$

which in turn is equal to

$$\dot{x}_i = \sum_{C_j \in \mathcal{N}_{\pi, C_i}} w_{i,j}(\xi_j - \xi_i). \tag{18}$$

As such, we have proved the following theorem

Theorem 3.1: Let G be a connected network whose node dynamics satisfies Equation 6. Let $\pi^\star$ be the maximal, leader-invariant EEP associated with that network, with $G\backslash\pi^\star$ being the corresponding quotient graph. Let the dynamics associated with the quotient graph be

$$\dot{\xi}_i = -\sum_{C_j \in \mathcal{N}_{\pi^\star, C_i}} w_{i,j}(\xi_i - \xi_j), \tag{19}$$

for all i *such that* $\pi^{\star-1}(C_i) \neq \{N\}$*, i.e., cell* i *does not contain the input node, and let*

$$\xi_i = u, \tag{20}$$

if $\pi^{\star-1}(C_i) = \{N\}$.

This dynamics is completely controllable. Moreover, ξ_i *satisfies*

$$\xi_i(t) = \frac{1}{|\pi^{\star-1}(C_i)|} \sum_{j \in \pi^{\star-1}(C_i)} x_j(t) \tag{21}$$

as long as

$$\xi_i(0) = \frac{1}{|\pi^{\star-1}(C_i)|} \sum_{j \in \pi^{\star-1}(C_i)} x_j(0). \tag{22}$$

IV. DISCUSSION

What Theorem 3.1 tells us is that given a network, what we can control is in fact another smaller network, given by the quotient graph. The equivalent dynamics over the quotient graph is given in terms of the average state values inside cells in the EEP. And, as the differences between state values inside the cells vanish asymptotically, this is not only characterizing the controllable subspace of the system, it also describes the behavior of the actual states in the original system as t approaches infinity.

The reason why it is beneficial to be able to view the controllable subspace as a network is that this allows control designers to focus directly on smaller structures with a physical interpretation. It also allows for the network design to be done in such a way that the desired quotient graphs are obtained. An example of this is shown in Figure 4, in which different edges are removed from the graph in order to produce different quotient graphs.

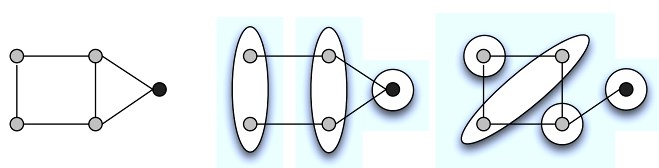

Fig. 4. An original graph (left) together with two graphs obtained through the removal of edges. As a result, the corresponding minimal, leader-invariant EEPs (leader node in black) lead to different quotient graphs (middle and right).

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