Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

Column distances for 2D-convolutional codes

Diego Napp Avelli, Carmen Perea and Raquel Pinto

***Abstract*— In this work we introduce the concept of column distance for delay-free two dimensional (2D) finite support convolutional codes. We present its principal properties and an upper and lower bound for the column distances.**

I. INTRODUCTION

2D convolutional codes are the higher dimensional generalization of 1D convolutional codes. These codes may prove useful in transmission of 2D pictures, the storage of digital information etc. While 1D convolutional codes have been thoroughly understood, the literature about 2D convolutional codes is quite limited. Fornasini and Valcher [2], introduced the general theory for the study of two-dimensional (2D) convolutional codes constituted by sequences indexed on $\mathbb{Z}^2$, discussing issues such as the characterization of such codes in terms of their internal properties and input-output representations. In [6] the same authors consider 2D convolutional codes in which the codewords have compact support in $\mathbb{Z}^2$, and present several properties of their encoders and syndrome formers (parity-check matrices) under different hypotheses on the code structure. They also introduced the dual codes of such codes. However crucial aspects of the theory of 2D convolutional codes such as the construction of codes with designed free distance or decoding algorithms, have not been deeply studied. In order to construct "good" 2D convolutional codes it is important to have some distance measures. In this paper, we consider 2D convolutional codes over a finite field $\mathbb{F}$, constituted by 2D trajectories with values in $\mathbb{F}^n$,

$$v : \mathbb{N}^2 \to \mathbb{F}^n : (i,j) \mapsto v(i,j) \tag{1}$$

with finite support in $\mathbb{N}^2$, i.e., which are zero in all points (i,j) outside a finite subset of $\mathbb{N}^2$. We generalize of one of the most important distance measures for 1D convolutional codes to 2D convolutional codes, namely, the column distance. We present its principal properties and we also derive an upper and lower bound.

II. PRELIMINARIES

In all this paper we consider the following definition of 2D finite support convolutional codes.

Definition 1 *[6], [7] A 2D finite support convolutional code $\mathcal{C}$ of rate $\frac{k}{n}$ is a free $\mathbb{F}[z_1,z_2]$-submodule of $\mathbb{F}[z_1,z_2]^n$, where k is the dimension of $\mathcal{C}$. A full column rank matrix $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ whose columns constitute a basis for $\mathcal{C}$, i.e., such that*

$$\begin{aligned} \mathcal{C} &= \mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1,z_2) \\ &= \{\hat{v}(z_1,z_2) \in \mathbb{F}^n[z_1,z_2] : \exists\, \hat{u}(z_1,z_2) \in \mathbb{F}^k[z_1,z_2] \\ &\quad\ \hat{v}(z_1,z_2) = G(z_1,z_2)\hat{u}(z_1,z_2)\}, \end{aligned} \tag{2}$$

is called an encoder *of $\mathcal{C}$. The elements of $\mathcal{C}$ are called codewords.*

Two full column rank matrices $G(z_1,z_2), \bar{G}(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ are *equivalent encoders* if they generate the same 2D finite support convolutional code, i.e., if $\mathrm{Im}_{\mathbb{F}[z_1,z_2]} G(z_1,z_2) = \mathrm{Im}_{\mathbb{F}[z_1,z_2]} \bar{G}(z_1,z_2)$, which happens if and only if there exists a unimodular matrix $U(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{k\times k}$ such that $G(z_1,z_2)U(z_1,z_2) = \bar{G}(z_1,z_2)$ [6], [7]. Moreover, in all that follows, we consider the particular case in that $G(z_1,z_2) \in \mathbb{F}[z_1,z_2]^{n\times k}$ satisfies that $G(0,0)$ is full column rank.

Following the ideas in [1] we call separation sets the sets in $\mathbb{Z}\times\mathbb{Z}$ defined as

$$C_l = \{(i,j) \in \mathbb{Z}\times\mathbb{Z}, i+j=l\}, \qquad l = 0,1,\dots. \tag{3}$$

We write the (encoder) matrix $G(z_1,z_2)$ of $\mathcal{C}$ as,

$$G(z_1,z_2) := \sum_{l=0}^{\nu} \sum_{i+j=l} G_{ij} z_1^i z_2^j \ \in \mathbb{F}[z_1,z_2]^{n\times k}, \tag{4}$$

and the corresponding parity check matrix as

$$H = \sum_{l=0}^{\mu} \sum_{i+j=l} H_{ij} z_1^i z_2^j \ \in \mathbb{F}[z_1,z_2]^{n\times(n-k)}. \tag{5}$$

Obviously, $H(0,0)$ has full (row) rank. For every $l \in \mathbb{N}$ the 2D truncated generator matrices $G^c_{\mathcal{C}_l} \in \mathbb{F}^{\frac{(l+1)(l+2)}{2}n\times\frac{(l+1)(l+2)}{2}k}$ is defined by $G^c_{\mathcal{C}_l} :=$

$$\begin{bmatrix} G_{00} \\ G_{10} & G_{00} \\ G_{01} & 0 & G_{00} \\ G_{20} & G_{10} & 0 & G_{00} \\ G_{11} & G_{10} & G_{01} & 0 & G_{00} \\ G_{02} & 0 & G_{01} & 0 & 0 & G_{00} \\ \vdots & & & & & \ddots & \ddots \\ \vdots & & & & & & \ddots & \ddots \\ G_{l0} & G_{(l-1)0} & 0 & G_{(l-2)0} & 0 & 0 & \dots & G_{10} & 0 & 0 & \dots & G_{00} \\ \vdots \\ G_{0l} & 0 & G_{0(l-1)} & 0 & 0 & G_{0(l-2)} & 0 & \dots & 0 & 0 & \dots & \dots & G_{00} \end{bmatrix},$$

and the parity-check matrices $H^c_{\mathcal{C}_l} \in \mathbb{F}^{\frac{(l+1)(l+2)}{2}(n-k)\times\frac{(l+1)(l+2)}{2}k}$ is defined in analogous way. Further, if we identify $u = \sum_{ij} u_{ij} z_1^i z_2^j$ with its

D. Napp and R. Pinto are is with the Department Mathematics, University of Aveiro, Portugal `diego@ua.pt`, `raquel@ua.pt`

C. Perea is with Center of Operation Research, Miguel Hernndez University, Orihuela, Ctra. de Beniel km 3,2, C.P. 03312, Spain `perea@umh.es`

ISBN 978-963-311-370-7

corresponding series $(u_{00}, u_{10}, ...)$, then we denote by $u_{\mathcal{C}_i}$ the vector associated to u in the separation set $\mathcal{C}_i$ and

$$u_{[\mathcal{C}_0,\mathcal{C}_l]} := (u_{\mathcal{C}_0}, u_{\mathcal{C}_1}, .., u_{\mathcal{C}_l}).$$

The identity $H^T G = 0$ and the full ranks of G_{00} and H_{00} immediately imply the full rank of the truncated matrices as well as the identities

$$\{G^c_{\mathcal{C}_l} u,\ u \in \mathbb{F}^{\frac{(l+1)(l+2)}{2}k}\} = \{v \in \mathbb{F}^{\frac{(l+1)(l+2)}{2}n},\ (H^c_{\mathcal{C}_l})v = 0\}$$

for all $l \in \mathbb{N}$.

An important measure of robustness of a code is its distance. We define the notion of distance as in [7]. The weight of $\hat{v}(z_1, z_2) = \sum_{(i,j)\in\mathbb{N}^2} v(i,j) z_1^i z_2^j \in \mathbb{F}[z_1, z_2]^n$, with $v(i,j) \in \mathbb{F}^n$ for $(i,j) \in \mathbb{N}^2$, is given by $\mathrm{wt}(\hat{v}) = \sum_{(i,j)\in\mathbb{N}^2} \mathrm{wt}(v(i,j))$, where $\mathrm{wt}(v(i,j))$ is the number of nonzero elements of $v(i,j)$. The distance between $\hat{v}_1(z_1, z_2), \hat{v}_2(z_1, z_2) \in \mathbb{F}[z_1, z_2]^n$, is $\mathrm{dist}(\hat{v}_1, \hat{v}_2) = \mathrm{wt}(\hat{v}_1 - \hat{v}_2)$.

Definition 2 *Given a 2D finite support convolutional code $\mathcal{C}$, the* distance *of $\mathcal{C}$ is defined as*

$$\begin{aligned} \mathrm{dist}(\mathcal{C}) \;=\; & \min\{\mathrm{dist}(\hat{v}_1, \hat{v}_2) : \hat{v}_1(z_1, z_2), \hat{v}_2(z_1, z_2) \in \mathcal{C}, \\ & \hat{v}_1(z_1, z_2) \neq \hat{v}_2(z_1, z_2)\}. \end{aligned}$$

Note that the linearity of $\mathcal{C}$ implies that

$$\mathrm{dist}(\mathcal{C}) = \min\{\mathrm{wt}(\hat{v}) : \hat{v}(z_1, z_2) \in \mathcal{C},\ \hat{v}(z_1, z_2) \neq 0\}$$

.

III. COLUMN DISTANCES

In this section we introduce the concept and the main results of this paper. Since 2D convolutional codes is the higher dimensional (nontrivial) generalizations of 1D convolutional codes, the definitions and results introduced here for 2D convolutional codes can be also considered as a generalizations of the concepts and properties of column distances for 1D convolutional codes. Considering 2D information sequences where $u_{\mathcal{C}_0} \neq 0$ and using the notation introduced in the previous section, we present, as a generalization of the concept of column distances for one dimensional convolutional codes (see [3], chapter 3) the following definition:

Definition 3 *The l-th separation set column distance of the two dimensional convolutional code $\mathcal{C}$ is given as*

$$\begin{aligned} d^c_{\mathcal{C}_l} &= \min_{u_{\mathcal{C}_0}\neq 0}\{\mathrm{weight}\left(G^c_{\mathcal{C}_l} u_{[\mathcal{C}_0,\mathcal{C}_l]}\right)\} && (6)\\ &= \min_{v_{\mathcal{C}_0}\neq 0}\{\mathrm{weight}\left(v_{[\mathcal{C}_0,\mathcal{C}_l]}\right), / \left(H^c_{\mathcal{C}_l}\right)^T v_{[\mathcal{C}_0,\mathcal{C}_l]} = 0\} && (7) \end{aligned}$$

where $G^c_{\mathcal{C}_l}$ is the truncated matrix of an encoder matrix $G(z_1, z_2)$ of $\mathcal{C}$.

Note that, the l-separation set column distance is invariant over the class of equivalent encoding matrices because we are considering the particular case in that $G(0,0)$ is full column rank. In these case the minimization over $u_{\mathcal{C}_0} \neq 0$ in (7) is the minimization over the set of sequences $v_{[\mathcal{C}_0,\mathcal{C}_l]}$ of codewords situated in the first quarter plane.

Proposition 1 *The separation set column distances of a code $\mathcal{C}$ satisfy*

$$d^c_{\mathcal{C}_0} \leq d^c_{\mathcal{C}_1} \leq d^c_{\mathcal{C}_2} \cdots \leq \mathrm{dist}^c(\mathcal{C}), \tag{8}$$

where $\mathrm{dist}^c(\mathcal{C})$ denotes the separation set column distance upper bound.

Equation (7) implies the following fact, which can be regarded as the analogous result of Proposition 2.1 of ([4]) for separation set column distance for 2D convolutional codes.

Proposition 2 *Let $d \in \mathbb{N}$. The following properties are equivalent:*

1) $d^c_{\mathcal{C}_l} = d$
2) *none of the first n rows of $H^c_{\mathcal{C}_l}$ is contained in the span of any other d-2 rows and one of the first n rows of $H^c_{\mathcal{C}_l}$ is in the span of some other $d-1$ rows of that matrix.*

The following upper bound on the separation set column distance is an immediate consequence of the previous result.

Theorem 1 *For every $l \in \mathbb{N}_0$ we have*

$$d^c_{\mathcal{C}_l} \leq (n-k)\frac{(l+2)(l+1)}{2} + 1$$

On the other hand, if we define

$$\begin{gathered} \mathcal{C}_1 = \{\hat{v}(z_1, 0) : \hat{v}(z_1, z_2) \in \mathcal{C}\} \\ \text{and} \\ \mathcal{C}_2 = \{\hat{v}(0, z_2) : \hat{v}(z_1, z_2) \in \mathcal{C}\} \end{gathered} \tag{9}$$

of a 2D finite support convolutional $\mathcal{C}$, it is easy to check that $\mathcal{C}_i$ is a (free) submodule of $\mathbb{F}^n[z_i]$, $i = 1, 2$, and therefore a 1D finite support convolutional code [5].

Let $G(z_1, z_2) \in \mathbb{F}[z_1, z_2]^{n\times k}$ be an encoder of $\mathcal{C}$ and define the polynomial matrices

$$G(z_1, 0) \in \mathbb{F}[z]^{n\times k} \ \text{ and } \ G(0, z_2) \in \mathbb{F}[z]^{n\times k}. \tag{10}$$

Hence, if $G(z_1, z_2)$ is an encoder of $\mathcal{C}$ then $\mathcal{C}_1 = \mathrm{Im}_{\mathbb{F}[z]}\, G_1(z)$ and $\mathcal{C}_2 = \mathrm{Im}_{\mathbb{F}[z]}\, G_2(z)$. Note, however the fact that $G(z_1, z_2)$ is an encoder of $\mathcal{C}$ does not imply, in general, that $G_1(z)$ and $G_2(z)$ are also encoders of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively. But, if $G(z_1, z_2)$ is factor prime then $G_1(z)$ and $G_2(z)$ are encoders. Taking into account these considerations we furnish the following lower bound on the separation set column distance.

Theorem 2 *Let $\mathcal{C}$ be a 2D finite support convolutional code with a right factor prime encoder matrix $G(z_1, z_2)$. Then For every $l \in \mathbb{N}_0$ we have*

$$d^c_{\mathcal{C}_l} \geq d^c_l(\mathcal{C}_1) + d^c_l(\mathcal{C}_2) - d^c_{\mathcal{C}_0}$$

where $d^c_l(\mathcal{C}_i)$ is the l-column distance of the code $\mathcal{C}_i$, $i = 1, 2$.

Note that $d^c_{\mathcal{C}_0} = d^c_0(\mathcal{C}_1) = d^c_0(\mathcal{C}_2)$.

IV. CONCLUSIONS AND FUTURE WORKS

We think that this paper is a first step in the study of distance measures of 2D convolutional codes. It would be nice to obtain also an upper bound over the distance of a 2D convolutional codes of fixed parameters, in a similar way that the Generalized Singleton bound. On the other hand, using the results presented in this paper, we are working in the construction of 2D convolutional codes with large column distances.

REFERENCES

[1] E. Fornasini, G- Marchesini *Properties of pairs of matrices and state models for two-dimensional systems. II. Models structure and realization problems*. Multivariate analysis: future directions (University Park, PA, 1992), 155–180, North-Holland Ser. Statist. Probab., 5, North-Holland, Amsterdam, 1993.

[2] E. Fornasini, M.E. Valcher *algebraic aspects of two-dimensional convolutional codes*. IEEE Tans. Inf. Th., 40(4):1068-1082, 1994.

[3] R. Johannesson K. Zigangirov *Fundamentals of Convlutional Coding*. IEEE Serios on Digital and Mobile Comunication. John B. Anderson, Series Editor, 1999.

[4] H. Gluesin-Luerssen, J. Rosenthal and R. Smarandache Strongly-MDS Convolutional Codes *IEEE Transactions on Information Theory*, 52(2): 584-599 (2006).

[5] J. Rosenthal, J.M. Schumacher, and E.V. York On Behaviors and convolutional codes. *IEEE Transactions on Information Theory*, 42: 1881-1891, (1996).

[6] M.E. Vlacher, E. Fornasini *On 2D finite supoort convolutional codes: an algebraic approach.* Multidimensional Systems and Signal Processing, 5::231-243, 1994.

[7] p. Weiner *Multidimensional Convolutional Codes.* PhD Dissertation, University of Notre Dame, USA 1998.