Stopping problems of Markov processes with discontinuous functionals

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***Abstract*— The paper summarizes recent results on optimal stopping of Feller Markov processes with time or space discontinuous functionals. We characterize value functions and their potential discontinuity points for various cost functionals: finite time horizon, first exit from an open set horizon and infinite horizon. Formulae for optimal or ϵ optimal stopping times are also given.**

I. INTRODUCTION

We assume that the state process $(x(s))$ is a right continuous Markov on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ with values in a locally compact separable metric space E. It has a transition operator P_t such that $P_t C_0 \subset C_0$, where C_0 is the space of continuous functions on E vanishing at infinity. The above assumption is satisfied by a wide variety of Markov processes including all Levy processes (see Theorem 3.1.9 of [1]). We shall consider the following three discounted cost functionals:

$$J^1(s,x,\tau) = E_{sx}\left\{\int_0^\tau e^{-\alpha u} f(s+u,x(u))du \right. \tag{1}$$
$$+1_{\tau<T-s} e^{-\alpha\tau} G(s+\tau, x(\tau)) +$$
$$\left. 1_{\tau=T-s} e^{-\alpha(T-s)} H(T, x(T-s))\right\}$$

$$J^2(s,x,\tau) = E_{sx}\Big\{ \int_0^{\tau\wedge\tau_\mathcal{O}} e^{-\alpha u} f\big(s+u,x(u)\big)du$$
$$+1_{\tau<\tau_\mathcal{O}} e^{-\alpha\tau} G\big(s+\tau, x(\tau)\big)$$
$$+1_{\tau\geq\tau_\mathcal{O}} e^{-\alpha\tau_\mathcal{O}} H\big(s+\tau_\mathcal{O}, x(\tau_\mathcal{O})\big)\Big\}, \tag{2}$$

$$J^3(s,x,\tau) = E_{sx}\Big\{ \int_0^{\tau} e^{-\alpha u} f\big(s+u,x(u)\big)du +$$
$$e^{-\alpha\tau} F\big(s+\tau, x(\tau)\big)\Big\}. \tag{3}$$

where $\alpha > 0$ is a fixed discount rate, $\mathcal{O}$ is an open set in E, $\tau_\mathcal{O} = \{t : x(t) \notin \mathcal{O}\}$, $F(s,x) = G(s,x)$ for $x \in \mathcal{O}$ and $F(s,x) = H(s,x)$ for $x \in \mathcal{O}^c$, and G, H and also f are bounded continuous functions. Let

$$w^1(s,x) = \sup_{\tau\leq T-s} J^1(s,x,\tau) \tag{4}$$

$$w^2(s,x) = \sup_{\tau} J^2(s,x,\tau) \tag{5}$$

$$w^3(s,x) = \sup_{\tau} J^3(s,x,\tau). \tag{6}$$

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This functions are value functions corresponding respectively to finite horizon stopping problem (w^1), optimal stopping till the first exit from the open set $\mathcal{O}$ (function w^2) and infinite horizon stopping with discontinuities of the cost functional at the boundary of the set $\mathcal{O}$ (function w^3). Notice furthermore that the value functions w^1 and w^2 correspond to optimal stopping with either discontinuities at the terminal time T or at the exit from the set $\mathcal{O}$. Such kinds of discontinuities appear in a natural way in the problem of impulse control in the presence of execution delay and decision lag and has many application in finance and decision-making process (regulatory delays, delayed data availability, liquidity risk, real-options (see [7], and also [2], [3]). We study regularity (continuity) of the above value functions and the form of optimal or ϵ - optimal strategies. The paper summarizes the results of the following papers [7], [8] and [9]. Notice moreover that for the purpose of impulsive control with delays it is important to have a regularity results with respect to parameters also, which is neglected in this note, and for which we refer to [7], where discretization approach was considered. In this paper we restrict ourselves to penalty method approach.

II. PENALTY METHOD

Among various methods to approximate value functions w^1, w^2 and w^3 penalty method introduced in a general setting in [10] seems to be very powerful. We have to find solutions to the following equations for $\beta > 0$

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-\alpha u}\left(f(s+u,x(u)) + \right.\right.$$
$$\beta(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^+\big)\, du +$$
$$\left. e^{-\alpha(T-s)} H(T, x(T-s))\right\}, \tag{7}$$

$$w^{2,\beta}(s,x) = E_{sx}\Big\{ \int_0^{\tau_\mathcal{O}} e^{-\alpha u}\Big[f\big(s+u,x(u)\big) +$$
$$\beta\Big(G\big(s+u,x(u)\big) - w^{2,\beta}\big(s+u,x(u)\big)\Big)^+\Big]du +$$
$$e^{-\alpha\tau_\mathcal{O}} H\big(s+\tau_\mathcal{O}, x(\tau_\mathcal{O})\big)\Big\}, \tag{8}$$

$$w^{3,\beta}(s,x) = E_{sx}\Big\{ \int_0^{\infty} e^{-\alpha u}\Big[f\big(s+u,x(u)\big) +$$
$$\beta\big(F - w^{3,\beta}\big)^+\big(s+u,x(u)\big)\Big]du\Big\}. \tag{9}$$

We need the following transformation Lemma (see Lemma 1 of [9] and Lemma 2.1 of [8])

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Lemma 1: Assuming that d, g and h are bounded functions for any bounded progressively measurable process $(b(t))$, the following formulae are equivalent

$$z(s,x) = E_{sx}\left\{\int_0^{T-s} e^{\int_0^u d(s+t,x(t))dt} g(s+u,x(u))du + e^{\int_0^{T-s} d(s+t,x(t))dt} h(T,x(T-s))\right\}, \tag{10}$$

$$z(s,x) = E_{sx}\left\{\int_0^{T-s} e^{\int_0^u (d(s+t,x(t))-b(s+t))dt}\left[g(s+u,x(u)) + b(s+u)z(s+u,x(u))\right]du + e^{\int_0^{T-s}(d(s+t,x(t))-b(s+t))dt} h(T,x(T-s))\right\} \tag{11}$$

in the following sense: z defined in (10) is a solution to (11); and any solution to (11) is of the form (10). The Lemma is also true if we replace $T-s$ by $\tau_{\mathcal{O}}$.

Basing on the above Lemma we can write down the equation (7) in an equivalent form

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-(\alpha+\beta)u}\left(f(s+u,x(u)) + \beta\left[(G(s+u,x(u)) - w^{1,\beta}(s+u,x(u)))^+ + w^{1,\beta}(s+u,x(u))\right]\right)du + e^{-(\alpha+\beta)(T-s)}H(T,x(T-s))\right\} \tag{12}$$

One can easily notice that the right hand side of the equation (12) forms a contractive operator within the class of bounded functions with sup norm. Consequently there is a unique bounded solution $w^{1,\beta}$ to the equation (7). Using a version of Lemma 1 with random terminal time $\tau_{\mathcal{O}}$ we obtain the existence of a unique bounded solution $w^{2,\beta}$ to the equation (8). Finally using a version of Lemma 1 for $T=\infty$ we obtain the existence of a unique bounded solution $w^{3,\beta}$ to the equation (9). Notice furthermore that under above assumed Feller property the right hand side of the equation transforms the set of continuous bounded functions into itself and consequently $w^{1,\beta}$ is continuous bounded. If we additionally assume that

(A1) The stopped semigroup

$$P_t^{\tau_{\mathcal{O}}} h(x) = E_x\{1_{t<\tau_{\mathcal{O}}} h(x(t))\}$$

maps the space of continuous bounded functions h into itself,

then we can show (for details see [8]) that $w^{2,\beta}$ is also a continuous bounded function. Assuming now

(A2) $(x(t))$ is strongly Feller, i.e., the mapping $x \mapsto E_x\{h(x(t))\}$ is continuous for any measurable bounded function h and $t>0$.

we have that $w^{3,\beta}$ is also a continuous bounded function (see [8] for details). The following two lemmas, written in term of the function $w^{2,\beta}$ characterize the properties of the solutions to the penalty equation

Lemma 2: The function $w^{2,\beta}$ has the following equivalent representation:

$$w^{2,\beta}(s,x) = \sup_{b\in M_\beta} E_x\Big\{\int_0^{\tau_{\mathcal{O}}} e^{-\alpha u - \int_0^u b(t)dt}\Big[f\big(s+u,x(u)\big) + b(u)G\big(s+u,x(u)\big)\Big]du + e^{-\alpha\tau_{\mathcal{O}} - \int_0^{\tau_{\mathcal{O}}} b(t)dt} H\big(\tau_{\mathcal{O}}, x(\tau_{\mathcal{O}})\big)\Big\}, \tag{13}$$

where M_β is the class of progressively measurable processes with values from $[0,\beta]$.

Lemma 3: Under assumption (A1) the function $w^{2,\beta}$ has the following equivalent representation:

$$w^{2,\beta}(s,x) = \sup_\tau \Big\{J^2(s,x,\tau) - E_x\Big\{1_{\tau<\tau_{\mathcal{O}}} e^{-\alpha\tau}\big(G - w^{2,\beta}\big)^+\big(s+\tau,x(\tau)\big)\Big\}\Big\}. \tag{14}$$

The proofs of these lemmas can be found in [8] or [7]. A similar versions of Lemmas 2 and 3 hold for $w^{1,\beta}$ and $w^{3,\beta}$. We have the following consequences of these lemmas. First of all it is clear that the function $w^{i,\beta}$ for $i=1,2,3$ are nondecreasing in β. Furthermore they do not exceed the functions w^i for $i=1,2,3$ respectively. Lemma 2 in fact shows that to prove the convergence of $w^{i,\beta}$ to w^i we have to estimate the value of $\big(G - w^{i,\beta}\big)^+\big(s+\tau,x(\tau)\big)$. We have the following

Proposition 1: The functions $w^{1,\beta}(s,x)$ increase pointwise to $w^1(s,x)$ and under (A1) also the functions $w^{2,\beta}(s,x)$ increase pointwise to $w^2(s,x)$, as $\beta\to\infty$.

Proof: We sketch the proof in the case of $w^{2,\beta}$ only. It is clear that there is a limit $\lim_{\beta\to\infty} w^{2,\beta}(s,x) = w^{2,\infty}(s,x)$ and $w^{2,\infty}(s,x) \le w^2(s,x)$. From (13) for $b(u) = \beta 1_{u\le\eta}$ we have

$$\begin{aligned} w^{2,\beta}(s,x) &\ge E_x\Big\{\int_0^{\tau_{\mathcal{O}}} e^{-\alpha u - \beta u\wedge\eta} f\big(s+u,x(u)\big)du \\ &+ \int_0^{\tau_{\mathcal{O}}\wedge\eta} e^{-(\alpha+\beta)u}\beta G\big(s+u,x(u)\big)du \\ &+ e^{-\alpha\tau_{\mathcal{O}} - \beta\eta\wedge\tau_{\mathcal{O}}} H\big(\tau_{\mathcal{O}}, x(\tau_{\mathcal{O}})\big)\Big\} \\ &= E_x\big\{(I) + (II) + (III)\big\}. \end{aligned} \tag{15}$$

By continuity of G whenever $x\in\mathcal{O}$ we have that $w^{2,\infty}(s,x) \ge G(s,x)$. By Lemma 3 we have for any stopping time τ

$$w^{2,\beta}(s,x) \ge J^2(s,x,\tau) - E_x\big\{1_{\tau<\tau_{\mathcal{O}}} e^{-\alpha\tau}\big(G - w^{2,\beta}\big)^+\big(s+\tau,x(\tau)\big)\big\}. \tag{16}$$

Letting $\beta\to\infty$ we obtain $w^{2,\infty}(s,x) \ge J^2(s,x,\tau)$ for $x\in\mathcal{O}$ and any stopping time τ. Whenever $x\in\partial\mathcal{O}$ we have $w^{2,\beta}(s,x) = H(s,x) = w^2(s,x)$. Consequently $w^{2,\infty} = w^2$, which completes the proof. ■

III. Properties of the value functions

We consider first the case of the value function w^1. Consider now the pair time - state process $(s+t, x(t))$ starting from (s,x). It is Markov with the transition semigroup $(\mathcal{P}_t)$ such that for bounded Borel measurable function ψ on $[0,\infty)\times E$ we have

$$\mathcal{P}_t\psi(s,x) = E_{sx}\left\{\psi(s+t, x(t))\right\}.$$

Let $C_0([0,\infty)\times E)$ be the space of continuous bounded functions on $[0,\infty)\times E$ vanishing at infinity i.e. such functions ψ that for each $\epsilon>0$ there is a compact set $K(\epsilon)$ and $N>0$ such that for $x\notin K(\epsilon)$ and for $t\geq N$ and any $x\in E$ we have $|\psi(t,x)|\leq\epsilon$. By Theorem T1, Chapter XIII in [6] the semigroup $(\mathcal{P}_t)$ is continuous in the class $C_0([0,\infty)\times E)$ i.e. for $\psi\in C_0([0,\infty)\times E)$, the mapping $[0,\infty)\ni t\mapsto\mathcal{P}_t\psi(s,x)$ is continuous in the supremum norm.

Notice that the above mentioned continuity of the semigroup by section 1.3B of [4] implies that the domain of the infinitesimal operator $\mathcal{A}$ of the semigroup $(\mathcal{P}_t)$ is dense in $C_0([0,T]\times E)$. If G is in that domain then

$$G(s,x) = E_{sx}\left\{\int_0^\infty e^{-\alpha u}\psi(s+u,x(u))du\right\},$$

with $\psi(s,x)=\alpha G(s,x)-\mathcal{A}G(s,x)$. Consequently

$$G(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-\alpha u}\psi(s+u,x(u))du + e^{-\alpha(T-s)}G(T,x(T-s))\right\}$$

Using this form we can rewrite (7) for $\bar{w}^{1,\beta}(s,x) = w^{1,\beta}(s,x)-G(s,x)$ as

$$\bar{w}^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{T-s} e^{-\alpha u}\left(f(s+u,x(u)) - \psi(s+u,x(u)) + \beta(0-\bar{w}^{1,\beta}(s+u,x(u)))^+\right)du + e^{-\alpha(T-s)}(H-G)(T,x(T-s))\right\}, \tag{17}$$

Notice that $\bar{w}^{1,\beta}$ is a solution of the penalty equation corresponding to the optimal stopping problem with cost functional (1) where f is replaced by $f-\psi$, G by 0 and H by $H-G$. Consequently by a version of Lemma 2 for $w^{1,\beta}$ we obtain

$$w^{1,\beta}(s,x) = \sup_{b\in M_\beta} E_{s,x}\Big\{\int_0^{T-s} e^{-\alpha u-\int_0^u b(t)dt}\Big[f\big(s+u,x(u)\big) - \psi\big(s+u,x(u)\big)\Big]du + e^{-\alpha(T-s)-\int_0^{T-s}b(t)dt}(H-G)\big(T,x(T-s)\big)\Big\} \geq -\frac{\|f-\psi\|}{\alpha+\beta} \tag{18}$$

whenever $H\geq G$, with $\|\cdot\|$ standing for the supremum norm. Consequently by a $w^{1,\beta}$ version of (14) we have

$$w^1(s,x) - w^{1,\beta}(s,x) \leq \frac{\|f-\psi\|}{\alpha+\beta}, \tag{19}$$

from which taking into account that $w^1(s,x)\geq w^{1,\beta}(s,x)$ we have uniform convergence of $w^{1,\beta}$ to w^1 as $\beta\to\infty$. We can now formulate the main result for w^1

Theorem 1: If $H\geq G$ then $w^{1,\beta}(s,x)$ converges uniformly on compact subsets to $w^1(s,x)$, which is a continuous function. Furthermore there is an optimal stopping time $\hat{\tau}^1$ for $w^1(s,x)$ of the form

$$\hat{\tau}^1(s) = \inf\left\{t\geq 0: w^1(s+t,x(t)) = G(s+t,x(t))\right\}\wedge(T-s). \tag{20}$$

Under (A1) we have that $w^{2,\beta}(s,x)$ converges uniformly on compact subsets to $w^2(s,x)$, which is also a continuous function and the stopping time $\hat{\tau}^2(s)$ of the form

$$\hat{\tau}^2(s) = \inf\left\{t\geq 0: w^2(s+t,x(t)) = G(s+t,x(t)) \ or \ x(t)\notin\mathcal{O}\right\} \tag{21}$$

is an optimal stopping time for $w^2(s,x)$.

Proof: We have proved above the uniform convergence of $w^{1,\beta}$ to w^1 in the case when G is in the domain of the infinitesimal generator $\mathcal{A}$. Since the domain of $\mathcal{A}$ is dense in $C_0([0,T]\times E)$ by the form of the cost functional J^1 we have a uniform approximation of the penalty equation also in the case when $G\in C_0([0,T]\times E)$. For general continuous bounded function G we have to use the following useful property of the transition operator P_t: let

$$\gamma_T(x,R) = P_x\left\{\exists_{s\in[0,T]}\ \rho\big(x,x(s)\big)\geq R\right\}, \tag{22}$$

where ρ is a metric compatible with the topology of E. We have (see Proposition 2.1 of [7])

Lemma 4: For any compact set $K\subseteq E$

$$\sup_{x\in K}\gamma_T(x,R)\to 0 \tag{23}$$

as $R\to\infty$.

Using a number of technical calculations we obtain the uniform convergence on compact subsets of $w^{1,\beta}(s,x)$ to $w^1(s,x)$ as $\beta\to\infty$. Therefore we have that w^1 as a uniform limit on compact sets of continuous functions is also a continuous function. We now use penalty equation to determine an optimal stopping time. Let for $\epsilon>0$ and $s\in[0,T]$

$$\tau_\epsilon(s) = \inf\left\{t\geq 0: t<T-s, w^1(s+t,x(t))\leq G(s+t,x(t))+\epsilon\right\},$$

$$\tau^\beta(s) = \inf\left\{t\geq 0: t<T-s, w^{1,\beta}(s+t,x(t))\leq G(s+t,x(t))\right\},$$

with the values $T-s$ when we have an infimum over an empty set. By the penalty equation (7) we then have

$$w^{1,\beta}(s,x) = E_{sx}\left\{\int_0^{\tau_\epsilon(s)\wedge\tau^\beta(s)} e^{-\alpha u} f(s+u,x(u))du + e^{-\alpha\tau_\epsilon(s)\wedge\tau^\beta(s)} w^{1,\beta}(s+\tau_\epsilon(s)\wedge\tau^\beta(s), x(\tau_\epsilon(s)\wedge\tau^\beta(s)))\right\}. \quad (24)$$

Since $w^{1,\beta}$ converges to w^1 uniformly on compact subsets we have $P_{sx}\left\{\tau^\beta(s)\le\tau_\epsilon(s)\right\}\to 0$ as $\beta\to\infty$. Therefore letting $\beta\to\infty$ in (24) we obtain (using Lemma 4)

$$w(s,x) = E_{sx}\left\{\int_0^{\tau_\epsilon(s)} e^{-\alpha u} f(s+u,x(u))du + e^{-\alpha\tau_\epsilon(s)}\ w(s+\tau_\epsilon(s), x(\tau_\epsilon(s)))\right\}.$$

It is clear that both $\tau_\epsilon(s)$ are increasing as ϵ decreases to 0. Let $\bar\tau(s) = lim_{\epsilon\to 0}\tau_\epsilon(s)$. By the quasi leftcontinuity of the Feller Markov process $(x(t))$ (see Theorem 3.13 of [4]) we have that $x(\tau_\epsilon(s))\to x(\bar\tau(s))$ $P_{s,x}$ a.e. as $\epsilon\to 0$. Therefore

$$w(s,x) = E_{sx}\left\{\int_0^{\bar\tau(s)} e^{-\alpha u} f(s+u,x(u))du + e^{-\alpha\bar\tau(s)}\ w^1(s+\bar\tau(s), x(\bar\tau(s)))\right\}. \quad (25)$$

Consequently by continuity of w, G and H we obtain that $\bar\tau(s)$ coincides with $\hat\tau^1(s)$, and from (25) optimality of $\hat\tau^1(s)$ immediately follows. The case if $w^{2,\beta}$ and w^2 can be studied in as similar way replacing $T-s$ by $\tau_{\mathcal{O}}$ (see [8] for details). ■

The following example (from [8], which is in fact a slight modification of an example from [11]) shows that the assumption $P_t C_0 \subseteq C_0$ can not be replaced by $P_t C\subseteq C$, where C is the set of continuous bounded functions.

Example. Let $E = E_0\cup E_1$, with $E_0 = \{(0,1),(0,\frac{1}{2}),\ldots,(0,\frac{1}{n}),\ldots,(0,0)\}$, $E_1 = \{(1,0),(2,0),\ \ldots(n,0),\ldots\}$ with the topology induced by R^2. Define a Markov process in the following fashion. The state $(0,0)$ is absorbing. The process starting from $(0,\frac{1}{n})$, after an independent exponentially distributed time with parameter 1, is shifted to the state $(n,0)$ and then after an independent exponentially distributed time with parameter n^2 is shifted to $(0,\frac{1}{n+1})$. One can check that such a process is Markov with a transition operator P_t satisfying $P_t C\subseteq C$. Let $f(s,x)=0$ for $x\in E_0$ and $f(s,x)=1$ for $x\in E_1$. Then $w^1(s,(0,\frac{1}{n})) = 1-e^{-(T-s)}$ and $w^1(s,(0,0))=0$, which means that the value function is discontinuous in $(0,0)$.

In what follows we shall also need the following assumption

(A3) for $\epsilon>0$ we have, uniformly in x from compact sets, $\lim_{t\to 0} P_x\left\{\sup_{s\in[0,t]}\rho(x,x(s))\ge\epsilon\right\} = 0$.

The above assumption is satisfied by a wide variety of diffusion and jump diffusion models.

It was important in the proof of the convergence of penalty method to the value function that $G\le H$. In the theorem below we get a general result under more restrictive assumptions.

Theorem 2: Under (A2) and (A3) the function $w^2(s,x)$ is continuous on $(s,x)\in[0,T]\times\mathcal{O}$. Assuming additionally (A1) we have that the penalized functions $w^{2,\beta}(s,x)$ are continuous and converge to $w^2(s,x)$ uniformly on compact subsets of $\mathcal{O}$. An $\epsilon>0$-optimal stopping time is given by the formula

$$\tau^\epsilon(s) = \inf\{t\ge 0 : w^2(s+t,x(t))\le G(s+t,x(t))+\epsilon\ or\ x(t)\notin\mathcal{O}\}. \quad (26)$$

Proof: Under (A2) Markov process $(x(t))$ is strongly Feller. Therefore for $h>0$ the function

$$w_h^2(s,x) = E_{sx}\{e^{-\alpha h} w^2(s+h,x(h))\}. \quad (27)$$

is continuous (to guarantee the continuity also with respect to the time coordinate we use here additionally Lemma 4.4 and 4.5). Let $\tau_{\mathcal{O}}^h = \inf\{t\ge h : x(t)\notin\mathcal{O}\}$ and

$$J^{2,h}(s,x,\tau) = E_{sx}\Big\{\int_0^{\tau\wedge\tau_{\mathcal{O}}^h} e^{-\alpha u} f\big(s+u,x(u)\big)du + 1_{\tau<\tau_{\mathcal{O}}^h} e^{-\alpha\tau} G\big(s+\tau,x(\tau)\big) + 1_{\tau\ge\tau_{\mathcal{O}}^h} e^{-\alpha\tau_{\mathcal{O}}^h} H\big(s+\tau_{\mathcal{O}}, x(\tau_{\mathcal{O}}^h)\big)\Big\}.$$

By Theorem 3b of [5] applied to the Markov process consisting of the pair $(s+t,x(t))$ we have

$$\sup_{\tau\ge h} J^{2,h}(s,x,\tau) = w_h^2(s,x) + E_{sx}\Big\{\int_0^h e^{-\alpha u} f\big(s+u,x(u)\big)du\Big\}.$$

Consider an auxiliary value function

$$\tilde w_h^2(s,x) = \sup_{\tau\ge h} J^2(s,x,\tau).$$

We have the following inequalities

$$|w_h(s,x) + E_{sx}\Big\{\int_0^h e^{-\alpha u} f\big(s+u,x(u)\big)du\Big\} - \tilde w_h^2(s,x)| \le \sup_{\tau\ge h}|J^{2,h}(s,x,\tau) - J^2(s,x,\tau)| \le C\,P_{sx}\big\{\tau_{\mathcal{O}}<h\big\} \quad (28)$$

and

$$|w^2(s,x) - \tilde w_h^2(s,x)| \le \sup_\tau |J^2(s,x,\tau) - J^2(s,x,\tau_h)| =: I_h(x), \quad (29)$$

where $\tau_h = \tau\vee h$. Assumption (A3) and the continuity of G and H imply that $\lim_{h\to 0} P_{sx}\{\tau_{\mathcal{O}}<h\} = 0$ and $\lim_{h\to 0} I_h(x) = 0$ uniformly in x from compact subsets of $\mathcal{O}$. Hence, $w_h^2(s,x)$ converges to $w^2(s,x)$ as $h\to 0$ uniformly in $(s,x)\in[0,T]\times K$ for any compact $K\subseteq E$. Consequently, w^2 is continuous in $\mathcal{O}$. For the remaining part of the proof see the proof of Theorem 4.6 of [8]. ■

In Theorem we obtain continuity of w^2 in $[0,T]\times\mathcal{O}$. It is also clear that w^2 is continuous on $E\setminus\mathcal{O}$ because on this set w^2 coincides with H. However, if there is a downward jump on the boundary of $\mathcal{O}$ ($G(s,x)>H(s,x)$ for some $x\in\partial\mathcal{O}$) the function w^2 will have a discontinuity in this point. This follows from the observation that $w^2\geq G$ on the set $\mathcal{O}$ and $w^2=H$ on $E\setminus\mathcal{O}$. Therefore, the statement of the above theorem cannot be strengthened. The same observation implies that an optimal stopping time might not exist.

In the case of infinite horizon problem, i.e. the problem of $w^{3,\beta}$ and w^3 we have the following result (see [8] for details)

Theorem 3: Under (A2) and (A3) the mapping $(s,x)\mapsto w^3(s,x)$ is continuous on $(s,x)\in[0,\infty)\times(E\setminus\partial\mathcal{O})$. If $G\geq H$ on $\partial\mathcal{O}$ we have that the mapping $(s,x)\mapsto w^3(s,x)$ is l.s.c. (lowersemicontinuous) and $\lim_{\beta\to\infty}w^{3,\beta}(s,x):=w^{3,\infty}(s,x)$ is also l.s.c. and $w^{3,\infty}(s,x)\geq w^3(s,x)$. If F is upper semicontinuous, i.e. $G\leq H$ on $\partial\mathcal{O}$, and

$$E_x\Big\{\int_0^\infty e^{-\alpha u-\beta\int_0^u 1_{x(t)\in\mathcal{O}}dt}du\Big\}\to 0$$
$$E_x\Big\{\int_0^\infty e^{-\alpha u-\beta\int_0^u 1_{x(t)\in E\setminus\mathcal{O}}dt}du\Big\}\to 0 \quad (30)$$

whenever $\beta\to\infty$, uniformly on compact subsets then $w^{3,\beta}(s,x)$ converges to w^3 uniformly on compact subsets of $[0,\infty)\times(E\setminus\partial\mathcal{O})$ and the mapping $(s,x)\mapsto w^3(s,x)$ is continuous.

Proof: We sketch the proof of continuity of w^3 only. Let $\tilde{w}_h^3(s,x)=\sup_{\tau\geq h}J^3(s,x,\tau)$. Theorem 3b of [5] implies

$$\tilde{w}_h^3(s,x)=E_{sx}\big\{e^{-\alpha h}w^\infty(s+h,x(h))+ \int_0^h e^{-\alpha u}f\big(s+u,x(u)\big)du\big\}.$$

By Lemma 4.4 and Lemma 8.1 from [8], for each $h>0$ the function $\tilde{w}_h^\infty$ is continuous. Clearly $\tilde{w}_h^\infty(s,x)\uparrow\tilde{w}^\infty(s,x)$ as $h\to 0$, where $\tilde{w}^3(s,x)=\sup_{\tau>0}J^3(s,x,\tau)$. Therefore by Theorem 3b of [5] $w^3(s,x)=\max\big\{\tilde{w}^3(s,x),F(s,x)\big\}$, and whenever F is l.s.c. ,i.e. $G\geq H$ on $\partial\mathcal{O}$ the mapping $(s,x)\mapsto w^3(s,x)$ is l.s.c as maximum of two l.s.c. functions. Under (A3) in the same way as in Theorem 2 we have that $\tilde{w}_h^3(s,x)\to w^3(s,x)$ uniformly on compact subsets of $[0,\infty)\times(E\setminus\partial\mathcal{O})$.

By suitable version of Lemma 2, $w^{3,\beta}$ is increasing in β. When $G\geq H$ on $\partial\mathcal{O}$ similarly as in the proof of Propositon 1 we obtain that $w^{3,\infty}(s,x)\geq F(s,x)$. Therefore by suitable version of Lemma 3 for any stopping time τ

$$w^{3,\beta}(s,x)\geq\Big\{J^3(s,x,\tau)- E_{sx}\big\{e^{-\alpha\tau}\big(F-w^\beta\big)^+(s+\tau,x(\tau))\big\}\Big\},$$

and letting $\beta\to\infty$ we obtain $w^{3,\infty}(s,x)\geq J^3(s,x,\tau)$, which completes the proof that $w^{3,\infty}(s,x)\geq w^3(s,x)$ in the case when $G\geq H$ on $\partial\mathcal{O}$.

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