Bifurcation Control in an Infectious Disease Model

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Abstract— **In this work we examine a basic mathematical model describing the spread of a class of infectious diseases. A system of four integral equations represents the SEIRS model, where individuals go through stages of being susceptible (S), exposed (E), infective (I), and recovered (R) for constant periods of time intervals. Transcritical bifurcation of steady state solutions can be observed in the system as the basic reproduction number increases. The eigenvalue analysis of the linearized equations provides local stability results. A stable numerical algorithm is developed that demonstrate the theoretical results.**

I. INTRODUCTION

In recent years our understanding of epidemiology through mathematical modeling has increased greatly. While many new infectious diseases emerged, mathematical modeling and computational mathematics have become important tools in the fight against these diseases. It is hard to find an area of mathematics that has not been used to treat mathematical models of epidemiology. Some of these areas include network theory, stochastic processes considering nonconstant population, fuzzy systems and virtual games. It seems to be the case that mathematical models describing the spread of diseases themselves became endemic.

In this paper we examine a deterministic *SEIRS* model describing the spread of a class of infectious diseases with constant population. Individuals go through stages of being susceptible (S), exposed (E), infective (I), and recovered (R) for constant periods of time intervals (see Fig. 1). The mathematical model is written as a system of four integral equations

$$S(t) = 1 - r\int_{t-m-\sigma-\omega}^{t} I(x)\,S(x)\;dx, \qquad (1)$$

$$I(t) = r\int_{t-\sigma-m}^{t-m} I(x)\,S(x)\;dx, \qquad (2)$$

$$E(t) = r\int_{t-m}^{t} I(x)\,S(x)\;dx, \qquad (3)$$

$$R(t) = r\int_{t-\sigma-\omega-m}^{t-\sigma-m} I(x)\,S(x)\;dx. \qquad (4)$$

In the above system t represents time; $r > 0$ is a constant contact rate; $m > 0$ is the constant time it takes an individual exposed to the infection to become infective; $\sigma > 0$ is the constant time it takes an infective individual to recover; and $\omega > 0$ is the constant time it takes an individual to become susceptible to the infection again after recovery. The basic assumption about the spread of infection is that the

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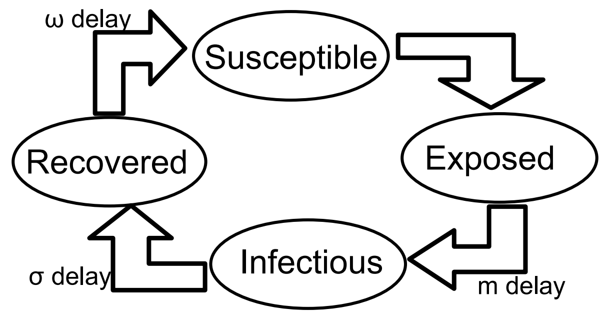

Fig. 1. Population groups

number of individuals becoming exposed to the infection at any moment is proportional to both the number of infected individuals and to the number of susceptible individuals: $\delta E(t) = rI(t)S(t)$. System (1)–(4) is the simplified version of the general model (see [14], [15]).

$$\int_{\tau(t)}^{t} [\rho_1(x) + \rho_2(x)\,I(x)]\,dx = m, \qquad (5)$$
$$\tau(t) = 0, \quad t \le t_0$$

$$S(t) = I_1(t) + S_0 - \int_{\tau(t-\sigma-\omega)}^{t} r(x)\,I(x)\,S(x)\;dx, \qquad (6)$$

$$I(t) = I_0(t) + \int_{\tau(t-\sigma)}^{\tau(t)} r(x)\,I(x)\,S(x)\;dx, \qquad (7)$$

$$E(t) = \int_{\tau(t)}^{t} r(x)\,I(x)\,S(x)\;dx, \qquad (8)$$

$$R(t) = I_2(t) + \int_{\tau(t-\sigma-\omega)}^{\tau(t-\sigma)} r(x)\,I(x)\,S(x)\;dx. \qquad (9)$$

The initial conditions above are given using functions $I_1(t)$ and $I_2(t)$ in terms of $I_0(t)$ by

$$I_1(t) = \begin{cases} 0, & t \le \omega, \\ I_0 - I_0(t-\omega), & t \ge \omega, \end{cases} \qquad (10)$$

$$I_2(t) = \begin{cases} I_0 - I_0(t), & t \le \omega, \\ I_0(t-\omega) - I_0(t), & t \ge \omega. \end{cases} \qquad (11)$$

System (1)–(4) is obtained from system (5)–(11) by substituting $\rho_2 \equiv 0$ and $\rho_1 \equiv 1$ (resulting in $\tau(t) = t - m$), by normalizing the total population to one, and by considering only large values of time t, for which all initial conditions disappear. The other special case $\rho_2 \equiv 1$ and $\rho_1 \equiv 0$ was considered in [16]. It is known that system (5)–(9) is well posed (unique, nonnegative solution exists which depends continuously on the parameters and initial conditions) based on some simple assumptions on I_0, see, e.g., [14], [15].

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Remark 1: We omit the initial conditions here, because we are interested in the long time behaviour of solutions, but we note that special conditions are required on them in order to obtain biologically meaningful solutions.
Solutions of system (1)–(4) can be completely determined by restricting ourselves to its first two equations

$$S(t) = 1 - r \int_{t-m-\sigma-\omega}^{t} I(x) S(x)\, dx, \tag{12}$$

$$I(t) = r \int_{t-\sigma-m}^{t-m} I(x) S(x)\, dx. \tag{13}$$

Once we know the functions S and I, the other two functions E and R can be calculated from (3) and (4).

The main results of the paper are presented in Section II. We obtain a transcritical bifurcation of steady state solutions as the basic reproduction number $R_0 = r\sigma$ increases above the value 1. In order to prove our main result we derive stability properties of the trivial and non–trivial steady state solutions in Sections III and IV respectively. Numerical simulations are provided in Section V.

II. Main Results

The steady state solutions are obtained by substituting constants $S(t) \equiv \hat{S}$ =constant and $I(t) \equiv \hat{I}$ =constant into system (12)–(13). We obtain the equations

$$\hat{S} = 1 - r\hat{I}\hat{S}(m + \sigma + \omega), \tag{14}$$

$$\hat{I} = r\hat{I}\hat{S}\sigma. \tag{15}$$

This nonlinear system of equations has two sets of solutions: the trivial steady state solutions

$$S_{00} = 1, \quad I_{00} = 0 \tag{16}$$

and the non–trivial, endemic steady state solutions

$$S_{01} = \frac{1}{r\sigma}, \quad I_{01} = \frac{r\sigma - 1}{r(m + \sigma + \omega)}. \tag{17}$$

The trivial steady state solutions (16) imply $E(t) \equiv 0$ and $R(t) \equiv 0$. In this case the whole population consists of only *susceptible individuals*, with no infection present. The non–trivial steady state solutions (17) imply

$$E(t) \equiv \frac{m(r\sigma - 1)}{r\sigma(m + \sigma + \omega)},$$

$$R(t) \equiv \frac{\omega(r\sigma - 1)}{r\sigma(m + \sigma + \omega)}.$$

While this case coincides with the trivial case for $r\sigma = 1$, in general the infection is endemic with all four population groups having nonzero size. Furthermore, for $r\sigma < 1$ the non–trivial steady state solutions (17) are non-biological, that is $S_{01} > 1$ and $I_{01} < 0$. In the case of $r\sigma > 1$ the non–trivial steady state solutions are realistic, that is $0 < S_{01}, I_{01}, R_{01}, E_{01} < 1$. Our main result is summarized in the following theorem:

Theorem 1: The system of integral equations (12)–(13) goes through transcritical bifurcation as the basic reproduction number $R_0 = r\sigma$ increases. Namely, at the value $r\sigma = 1$ the trivial steady state solutions (16) change from being globally asymptotically stable to unstable and the non–trivial steady state solutions (17) change from being unstable to locally asymptotically stable (LAS).

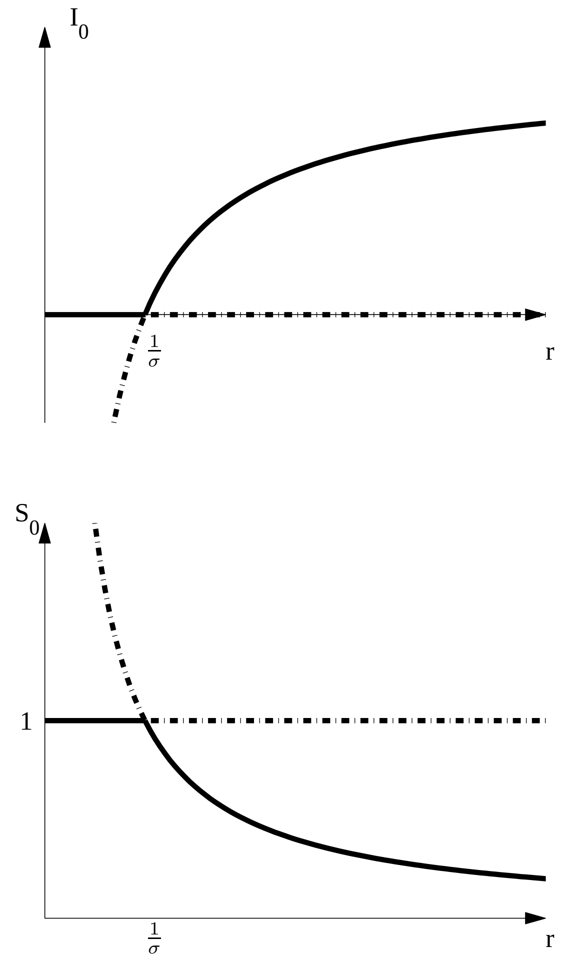

Fig. 2. Bifurcation Diagrams: – stable and - - unstable steady states.

Fig. 2 shows the bifurcation diagram of the steady state solutions.

Remark 2: The stability change of the disease–free equilibrium as the basic reproduction number increases is true for most epidemic models. In many other models Hopf bifurcation and periodic solutions about the endemic equilibrium are possible. In our model we did not observe periodic solutions neither in the theoretical analysis nor in numerically simulations.
We prove Theorem 1 in several steps. First the stability of the trivial steady state solution is considered and then the stability of the non–trivial steady state solutions.

III. Stability of the Trivial Steady State Solutions

The stability of the trivial steady state solutions (16) can be examined using relatively simple analysis.

Theorem 2: The trivial steady–state solutions $S_{00} = 1$ and $I_{00} = 0$ are globally asymptotically stable when $r\sigma < 1$.

Proof: The proof is broken into several steps.

Step 1: *The function I is bounded by another function M.* Note that $0 \le S(t) \le 1$ for all $t \ge 0$ according to [14], and it follows that

$$\begin{aligned} I(t) &= r\int_{t-\sigma-m}^{t-m} I(x)\,S(x)\,dx \\ &\le r\int_{t-\sigma-m}^{t-m} I(x)\,dx \\ &\le r\sigma \max_{x\in[t-\sigma-m,\,t-m]} I(x) \end{aligned} \tag{18}$$

for all $t \ge 0$. Using the notation

$$M(t) = \max_{x\in[t-\sigma-m,\,t-m]} I(x) \tag{19}$$

note that

$$\begin{aligned} I(t) &\le \sigma r \max_{x\in[t-\sigma-m,\,t-m]} I(x) \\ &< \max_{x\in[t-\sigma-m,\,t-m]} I(x) \\ &= M(t), \end{aligned}$$

therefore

$$I(t) < M(t) \tag{20}$$

for all $t > 0$.

Step 2: *The function I is bounded by a constant. More precisely: Let $t_1 > 0$ be an arbitrary number. Then $I(t) < M(t_1)$ for all $t > t_1$.* To show this using contradiction we assume that $I(t) \ge M(t_1)$ for some $t > t_1$. Denote by t_2 the first such t. Then

$$I(t_2) = M(t_1)$$

and

$$I(t) < M(t_1) \text{ for all } t < t_2. \tag{21}$$

Furthermore,

$$\begin{aligned} I(t_2) &< M(t_2) \\ &= \max_{t\in[t_2-\sigma-m,\,t_2-m]} I(t) \\ &< \max_{t\in[t_2-\sigma-m,\,t_2-m]} M(t_1) \\ &= M(t_1) \end{aligned} \tag{22}$$

According to (22) we obtain $I(t_2) < M(t_1)$ that contradicts $I(t_2) = M(t_1)$. This proves that $I(t) < M(t_1)$ for all $t > t_1$.

Step 3: *$M(t_1 + n2(\sigma+m))$ converges to zero as n approaches infinity.* We will use induction to show that $M(t_1 + n2(\sigma+m)) < (\sigma r)^n M(t_1)$ for all $n \in \mathbb{N}$. Since $\sigma r < 1$, this inequality proves the statement. For $n = 1$, we have

$$\begin{aligned} M(t_1 + 2(\sigma+m)) &= \max_{x\in[t_1+\sigma+m,\,t_1+2\sigma+m]} I(x) \\ &\le \sigma r \max_{x\in[t_1+\sigma+m,\,t_1+2\sigma+m]}\ \max_{s\in[x-\sigma-m,\,x-m]} I(s) \\ &= \sigma r \max_{s\in[t_1,\,t_1+2\sigma]} I(s) \\ &< \sigma r M(t_1) \end{aligned} \tag{23}$$

where after the definition of M we used (18), then we combined the two maximum expressions, and finally (21) was used. Assuming $M(t_1 + n2(\sigma+m)) < (\sigma r)^n M(t_1)$ for all $t_1 > 0$ for an $n \in \mathbb{N}$ we obtain for $n+1$ that

$$\begin{aligned} &M(t_1 + 2(n+1)(\sigma+m)) \\ &\quad = M(t_1 + 2(\sigma+m) + 2n(\sigma+m)) \\ &\quad < (\sigma r)^n M(t_1 + 2(\sigma+m)) \\ &\quad < (\sigma r)^{n+1} M(t_1) \end{aligned}$$

where we used the induction hypothesis with t_1 replaced by $t_1 + 2(\sigma+m)$, and then inequality (23) was used.

Step 4: *$I(t)$ converges to zero as t approaches infinity.* Because the choice of t_1 was arbitrary, Step 3 implies that $M(t)$ converges to zero as t approaches infinity. Since $I(t) < M(t)$ for all t, we obtain that $I(t)$ converges to zero as well, and the convergence is exponential fast.

Step 5: Finally we obtain that $\lim_{t\to\infty} S(t) \le 1$ and

$$\begin{aligned} \lim_{t\to\infty} S(t) &= 1 - \lim_{t\to\infty} r\int_{t-m-\sigma-\omega}^{t} I(x)\,S(x)\,dx \\ &\ge 1 - r(m+\sigma+\omega)\lim_{t\to\infty}\max_{x\in[t-m-\sigma-\omega,t]} I(x) \\ &= 1 \end{aligned}$$

■

Remark 3: We also obtain in the case of $r\sigma < 1$ from Theorem 2 and Equations (3)–(4) that $\lim_{t\to\infty} R(t) = \lim_{t\to\infty} E(t) = 0$, that is the infection dies out.

Theorem 3: The trivial steady state solutions $S_{00} = 1$ and $I_{00} = 0$ are unstable when $r > 1/\sigma$.

Proof: We are going to examine the eigenvalues of the linearized equations. Let us denote perturbations of the trivial steady state solutions (16) by

$$S = 1 + u \qquad \text{and} \qquad I = v \tag{24}$$

respectively. Substituting (24) into (12) and (13) we obtain the perturbation equations

$$1 + u = 1 - r\int_{t-m-\sigma-\omega}^{t} (1+u)\,v\,dx \tag{25}$$

$$v = r\int_{t-m-\sigma}^{t-m} (1+u)\,v\,dx, \tag{26}$$

which can be linearized to get

$$u = -r\int_{t-m-\sigma-\omega}^{t} v\,dx \tag{27}$$

$$v = r\int_{t-m-\sigma}^{t-m} v\,dx \tag{28}$$

and rearranged as

$$0 = u + r\int_{t-m-\sigma-\omega}^{t} v\,dx \tag{29}$$

$$0 = v - r\int_{t-m-\sigma}^{t-m} v\,dx. \tag{30}$$

The characteristic equation corresponding to (29) and (30) is calculated by substituting exponential functions

$$u = ae^{\lambda t}, \quad v = be^{\lambda t} \tag{31}$$

with arbitrary constants a and b. We obtain

$$\begin{aligned} 0 &= ae^{\lambda t} + rb\int_{t-m-\sigma-\omega}^{t} e^{\lambda x}\,dx \\ &= ae^{\lambda t} + rb\frac{e^{\lambda t}\left(1-e^{-\lambda(m+\sigma+\omega)}\right)}{\lambda} \end{aligned} \tag{32}$$

and

$$\begin{aligned} 0 &= be^{\lambda t} - rb\int_{t-m-\sigma}^{t-m} e^{\lambda x}\,dx \\ &= be^{\lambda t} - rb\frac{e^{\lambda(t-m)}\left(1-e^{-\lambda\sigma}\right)}{\lambda} \end{aligned} \tag{33}$$

respectively. The characteristic equation of the above system of two equations is

$$0 = e^{\lambda t}\left(e^{\lambda t} - r\frac{e^{\lambda(t-m)}\left(1-e^{-\lambda\sigma}\right)}{\lambda}\right), \tag{34}$$

which can be simplified to obtain

$$0 = 1 - r\frac{e^{-\lambda m}\left(1-e^{-\lambda\sigma}\right)}{\lambda}. \tag{35}$$

Note that

$$\lim_{\lambda\to 0}\left(1 - r\frac{e^{-\lambda m}\left(1-e^{-\lambda\sigma}\right)}{\lambda}\right) = 1 - r\sigma \neq 0, \tag{36}$$

hence $\lambda = 0$ is not a root of the characteristic equation (35). Rearranging (35) we now look for nonzero solutions of equation

$$\lambda - re^{-\lambda m} + re^{-\lambda(\sigma+m)} = 0. \tag{37}$$

Fig. 3 shows two typical graphs of the function $f(\lambda) = \lambda - re^{-\lambda m} + re^{-\lambda(\sigma+m)}$ (the left hand side of the simplified characteristic equation (37)) for the cases of $r\sigma < 1$ (left hand side picture) and $r\sigma > 1$ (right and side picture). Note that $f(0) = 0$ and $\lim_{\lambda\to\infty} f(\lambda) = \infty$. Furthermore,

$$\begin{aligned} f'(\lambda) &= 1 + rme^{-\lambda m} - r(\sigma+m)e^{-\lambda(\sigma+m)}, \quad (38) \\ f'(0) &= 1 - r\sigma. \quad (39) \end{aligned}$$

As a consequence, we obtain that for $r\sigma > 1$ there is a positive real root of the characteristic equation (35), which completes the proof of the theorem. ■

IV. Stability of the Non–Trivial Steady State Solutions

In this section we examine the stability properties of the non–trivial steady state solutions (17).

Theorem 4: The non–trivial steady state solutions (17) are locally asymptotically stable if and only if $r > 1/\sigma$.

Proof: As in the proof of Theorem 3, we start by examining the eigenvalues of the linearized equations. Let us

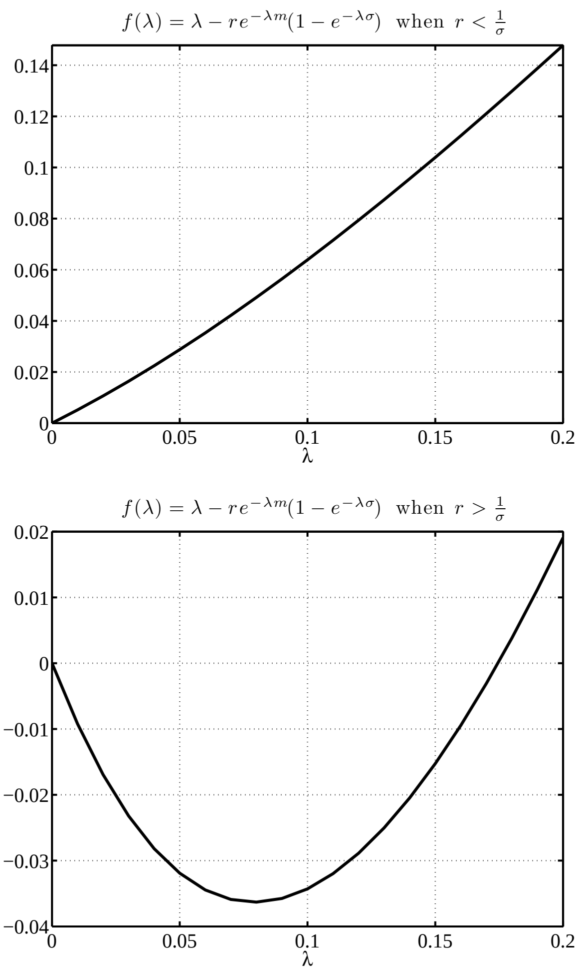

Fig. 3. Two cases for the modified characteristic function (37).

denote perturbations of the non–trivial steady state solutions (17) by

$$S = \frac{1}{r\sigma} + u \qquad \text{and} \qquad I = \frac{r\sigma - 1}{r(m+\sigma+\omega)} + v \tag{40}$$

respectively. Substituting (40) into (12) and (13) we obtain the perturbation equations

$$\begin{aligned} &\frac{1}{r\sigma} + u \\ &= 1 - r\int_{t-m-\sigma-\omega}^{t}\left(\frac{1}{r\sigma}+u\right)\left(\frac{r\sigma-1}{r(m+\sigma+\omega)}+v\right)dx, \end{aligned}$$

$$\begin{aligned} &\frac{r\sigma-1}{r(m+\sigma+\omega)} + v \\ &\qquad = r\int_{t-m-\sigma}^{t-m}\left(\frac{1}{r\sigma}+u\right)\left(\frac{r\sigma-1}{r(m+\sigma+\omega)}+v\right)dx, \end{aligned}$$

which can be linearized to obtain

$$u = -\frac{1}{\sigma}\int_{t-m-\sigma-\omega}^{t} v\,dx - \frac{r\sigma-1}{(m+\sigma+\omega)}\int_{t-m-\sigma-\omega}^{t} u\,dx \quad (41)$$

$$v = \frac{1}{\sigma}\int_{t-m-\sigma}^{t-m} v\,dx + \frac{r\sigma-1}{(m+\sigma+\omega)}\int_{t-m-\sigma}^{t-m} u\,dx. \quad (42)$$

The characteristic equations corresponding to (41) and (42) are calculated as in Section III and after simplification we obtain

$$\left(1 - \frac{(1-r\sigma)\left(1-e^{-\lambda(m+\sigma+\omega)}\right)}{\lambda(m+\sigma+\omega)}\right)a + \frac{1-e^{-\lambda(m+\sigma+\omega)}}{\lambda\sigma}a = 0 \quad (43)$$

$$\frac{(1-r\sigma)e^{-\lambda m}\left(1-e^{-\lambda\sigma}\right)}{\lambda(m+\sigma+\omega)}a + \frac{e^{-\lambda(m+\sigma)}-e^{-\lambda m}-\lambda\sigma}{\lambda\sigma}b = 0 \quad (44)$$

The characteristic function of system (43)-(44) is

$$D(\lambda) = \frac{\sigma(\lambda(m+\sigma+\omega)-(1-r\sigma))}{\lambda\sigma(m+\sigma+\omega)} + \frac{\sigma(1-r\sigma)e^{-\lambda(m+\sigma+\omega)}+(m+\sigma+\omega)e^{-\lambda m}\left(1-e^{-\lambda\sigma}\right)}{\lambda\sigma(m+\sigma+\omega)}$$

Note that $\lim_{\lambda\to 0} D(\lambda) = -1 - r\sigma$, hence $\lambda = 0$ is not a root of the characteristic equation. In order to find nonzero solutions of the characteristic equation we simplify the equation $D(\lambda) = 0$ to obtain

$$\begin{aligned} D^*(\lambda) &\equiv \sigma(1-r\sigma)\left(e^{-\lambda(m+\sigma+\omega)}-1\right) \\ &\quad + (m+\sigma+\omega)\left(e^{-\lambda m}\left(1-e^{-\lambda\sigma}\right)+\sigma\lambda\right) \\ &= 0 \end{aligned} \quad (45)$$

The modified characteristic equation (45) is equivalent (up to multiplication by λ) to the characteristic equation of a delay differential equation that we get by differentiating (41)–(42):

$$\begin{aligned} \dot{u} &= \frac{1}{\sigma}\Big(v(t-m-\sigma-\omega)-v(t)\Big) \\ &\quad + \frac{r\sigma-1}{(m+\sigma+\omega)}\Big(u(t-m-\sigma-\omega)-u(t)\Big), \end{aligned} \quad (46)$$

$$\begin{aligned} \dot{v} &= \frac{1}{\sigma}\Big(v(t-m)-v(t-m-\sigma)\Big) \\ &\quad + \frac{r\sigma-1}{(m+\sigma+\omega)}\Big(u(t-m)-u(t-m-\sigma)\Big). \end{aligned} \quad (47)$$

We now invoke Theorem 2.19 of [18], which states that the equilibrium solution of system (46)–(47) is exponentially asymptotically stable if and only if

$$S(\rho_k) \neq 0, \quad k = 1,\ldots,r-1, \quad \text{and} \quad (48)$$

$$\sum_{k=1}^{r}(-1)^k \operatorname{sgn} S(\rho_k) = -1, \quad (49)$$

where

$$R(\omega) = \operatorname{Re} i\omega D^*(i\omega), \quad S(\omega) = \operatorname{Im} i\omega D^*(i\omega), \quad (50)$$

and $\rho_1 \geq \ldots \geq \rho_r \geq 0$ are the nonnegative real zeros of R. In our case

$$\begin{aligned} S(x) &= x\sigma(1-r\sigma)\Big(\cos(x(m+\sigma+\omega))-1\Big) \\ &\quad + x(m+\sigma+\omega)\Big(\cos(xm)-\cos(x(m+\sigma))\Big) \end{aligned}$$

and

$$\begin{aligned} R(x) &= -x\sigma(1-r\sigma)\sin(x(m+\sigma+\omega)) \\ &\quad - x(m+\sigma+\omega)\Big(\sin(xm)-\sin(x(m+\sigma))+\sigma x\Big). \end{aligned}$$

Note that $R(0) = S(0) = S'(0) = S''(0) = 0$, and while $S(x)$ oscillates between negative and positive values as $x \to \infty$, for $R(x)$ we have $\lim_{x\to\infty} R(x) = -\infty$. Furthermore,

$$R'(0) = -\sigma(1-r\sigma)(m+\sigma+\omega). \quad (51)$$

The key expression above is equation (51) which, together with $\lim_{x\to\infty} R(x) = -\infty$, determines the number of positive zeros of $R(x)$: it is odd if $r > 1/\sigma$ and even otherwise. From here, applying (49) the statement of the theorem follows. ■

V. Numerical Computations

It is tempting to convert the integral equations (12)–(13) into delay differential equations, since most mathematical software packages have built in solvers for the later one, while not so for the first one. It turns out that, due to the nonlinearity in the system, it is more stable numerically to work with the original integral equation formulation than with a differential equation formulation. Only the integral equation approach is presented below, without comparison to the differential equation settings.

Equations (12)–(13) are the equations describing the long time behavior of the model without initial conditions. The complete model that includes initial conditions is system (6)–(7) with the assumption that $\tau(t) = t - m$, and $r = constant$, that is

$$S(t) = I_1(t) + S_0 - r\int_{t-\sigma-\omega-m}^{t} I(x)S(x)\,dx, \quad (52)$$

$$I(t) = I_0(t) + r\int_{t-\sigma-m}^{t-m} I(x)S(x)\,dx. \quad (53)$$

The initial function I_1 is defined as

$$I_1(t) = \begin{cases} 0, & t \leq \omega, \\ I_0 - I_0(t-\omega), & t \geq \omega, \end{cases} \quad (54)$$

where I_0 satisfies the assumptions

$$I_0(t) = \begin{cases} \text{monotone increasing} & \text{for } -\sigma \leq t \leq 0, \\ I_0(0) - I_0(t-\sigma) & \text{for } 0 \leq t \leq \sigma, \\ 0 & \text{for } \sigma < t, \end{cases} \quad (55)$$

and $I_0(0) + S_0 = 1$. For our simulations we chose the initial function

$$I_0(t) = \left(\frac{t+\sigma}{1.2\sigma}\right)^2 \quad \text{for } -\sigma \le t \le 0. \tag{56}$$

Further constants are $m = 2$, $\omega = 1.5$, and $\sigma = 1$. By choosing $\sigma = 1$ we set the unit time to the time length an individual remains infective. For integrations we used Simpson's rule with equally distributed grid points. Solving (52)–(53) is done over intervals of length σ. First (53) is solved using explicit scheme. Once values of I are obtained, a discretized version of (52) gives us an implicit formula for $S(t)$. The numerical code was programmed in MATLAB. With the number of grid points between 100 and 200 per unit time and with time interval $[0, 40]$, our simulation takes only a few seconds on a 2GHz computer with 2GB memory. The numerical algorithm could be optimized, if needed, by reusing integration data from previous steps. Fig. 4 shows simulation results for two values of the contact rate parameter. The left hand side picture in Fig. 4 with $r = 1.8 > 1 = 1/\sigma$ corresponds to the endemic case with the proportion of infectives and susceptibles converging to levels between zero and one. The right hand side picture in Fig. 4 corresponds to the case $r = 0.8 < 1 = 1/\sigma$, when the infection dies out exponentially fast.

VI. Conclusions

In the present paper we examined the bifurcation of a mathematical model for the spread of an infectious disease. The model is represented by a system of integral equations. We chose the contact rate as the bifurcation parameter, and we prove that system goes through a transcritical bifurcation as the contact rate increases. The theoretical results are demonstrated through numerical simulations.

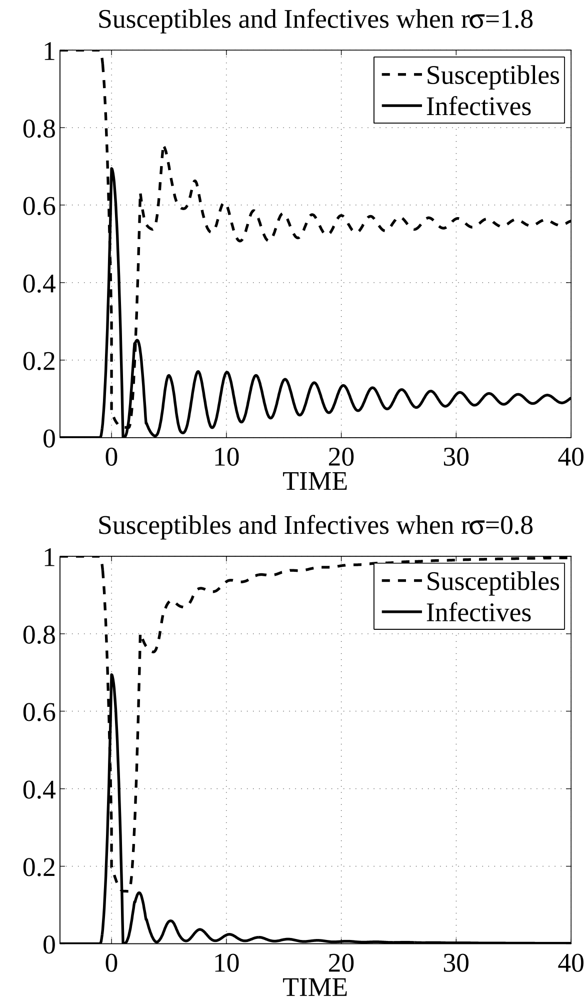

Fig. 4. Numerical simulations. **Top:** solutions converging to the nontrivial steady states. **Bottom:** solutions converging to the trivial steady states.

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