On a canonical QR decomposition and feedback control of discrete-time quantum dynamics

Francesco Ticozzi and Saverio Bolognani

Abstract— **We study feedback-controlled, discrete-time quantum Markovian dynamics focusing on pure-state stabilization problem. Assuming that the system is unitarily controllable, and accessible via a given quantum measurement, we explicitly construct a choice of control actions conditioned on the measurement outcome that globally stabilizes the target state for the averaged dynamics. A key step in deriving this result is the definition of a canonical QR decomposition for complex matrices.**

I. INTRODUCTION

Most of the proposed approaches to realize quantum information technology require the ability to perform sequences of a limited number of fundamental operations [8], [15], [19]. Two typical key tasks are concerned with the preparation of states of maximal information and engineering of protected realization of quantum information, i.e. the realization of information encodings that preserve the fragile quantum states from the action of noise. This paper, which contains and extends some of the results of [4], [**?**], focus on these issues, providing a design strategy for engineering stable quantum subspaces following and extending the ideas of [18], [17].

We conside discrete-time quantum dynamics described by sequences of trace-preserving quantum operations in Kraus representation [15]. One can then consider the discrete-time dynamical semigroup induced by iteration of a given TPCP map $\mathcal{T}$. The resulting discrete-time quantum system is described by $\rho(t+1) = \mathcal{T}[\rho(t)] = \sum_k M_k \rho(t) M_k^\dagger$, where the density operator ρ represents the state of the system. This class of models implies the Markovian character of the evolution [12], which, along with a forward composition law, ensures a semigroup structure.

After recalling the key concepts relative to *quantum subspaces* and dynamical stability, we will first focus on the analysis of the dynamics. Necessary and sufficient conditions on the dynamical model that ensure global stability of a certain quantum subspace are provided. We employ LaSalle's invariance principle, exploiting the linearity of the dynamics, as well as the convex character of the state manifold.

We then study the problem of designing a feedback control law capable of stabilizing a given subspace once a set of measurement operators is provided. The control scheme we employ follows the ideas of [14], [16], and is in fact an instance of the Markovian feedback models studied in e.g. [3], [10]. Assume that we are allowed to: (i) Perform a generalized quantum measurement on the system associated to measurement operators $\{M_k\}$; (ii) Unitarily control the state of the system, i.e. $\rho_c = U\rho U^\dagger$, $U \in \mathcal{U}(\mathcal{H}_I)$. We can then use the generalized measurement outcome k to condition the control choice, that is, a certain coherent transformation U_k is applied after the k-th output is recorded, $U(k) : k \mapsto U_k \in \mathcal{U}(\mathcal{H}_I)$. The measurement-control loop is then iterated: If we average over the measurement results at each step, this yields a different TPCP map, which depends on the design of the set of unitary controls $\{U_k\}$ and describes the evolution of the state *immediately after* each application of the controls, that is $\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger$.

The main tools we employ come from the stability theory of dynamical systems, namely LaSalle's Invariance principle [13], and linear algebra, namely the QR matrix decomposition [9]. We shall construct a "special form" of the QR decomposition: In particular, we prove that the upper triangular factor R can be rendered a *canonical form* with respect to the left action of the unitary matrix group. The synthesis results include a simple characterization of the controlled dynamics that can be enacted, and an algorithm that builds unitary control actions stabilizing a desired subspace. If such controls cannot be found, it is proven that no choice of controls can achieve the control task for the same measurement.

II. DISCRETE–TIME QUANTUM DYNAMICAL SEMIGROUPS

Let $\mathcal{I}$ denote the physical quantum system of interest. Consider the associated separable Hilbert space $\mathcal{H}_I$ over the complex field $\mathbb{C}$. In what follows, we consider finite-dimensional quantum systems, i.e. $\dim(\mathcal{H}_I) < \infty$. In Dirac's notation, vectors are represented by a *ket* $|\psi\rangle \in \mathcal{H}_I$, and linear functionals by a *bra*, $\langle\psi| \in \mathcal{H}_I^\dagger$ (the adjoint of $\mathcal{H}_I$), respectively. The inner product of $|\psi\rangle, |\varphi\rangle$ is then represented as $\langle\psi|\varphi\rangle$.

Let $\mathfrak{B}(\mathcal{H}_I)$ represent the set of linear bounded operators on $\mathcal{H}_I$, $\mathfrak{H}(\mathcal{H}_I)$ denoting the real subspace of hermitian operators, with $\mathbb{I}$ and $\mathbb{O}$ being the identity and the zero operator, respectively. Our (possibily uncertain) knowledge of the state of the quantum system is condensed in a density operator, or *state* ρ, with $\rho \geq 0$ and $\mathrm{Tr}\rho = 1$. Density operators form a convex set $\mathfrak{D}(\mathcal{H}_I) \subset \mathfrak{H}(\mathcal{H}_I)$, with one-dimensional projectors corresponding to extreme points (pure states, $\rho_{|\psi\rangle} = |\psi\rangle\langle\psi|$). Given an $X \in \mathfrak{H}(\mathcal{H}_I)$, we indicate

F. Ticozzi is with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy (ticozzi@dei.unipd.it).

S. Bolognani is with the Dipartimento di Ingegneria dell'Informazione, Università di Padova, via Gradenigo 6/B, 35131 Padova, Italy (saverio.bolognani@dei.unipd.it).

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with $\ker(X)$ its kernel (0-eigenspace) and with $\text{supp}(X) := \mathcal{H}_I \ominus \ker(X)$ its range, or *support*.

An effective tool to describe these dynamical systems is given by quantum operations [15], [11]. The most general, linear and physically admissible evolutions which take into account interacting quantum systems and measurements, are described by Completely Positive (CP) maps, that via the Kraus-Stinespring theorem [11] admit a representation of the form

$$\mathcal{T}[\rho] = \sum_k M_k \rho M_k^\dagger \tag{1}$$

(also known as operator-sum representation of $\mathcal{T}$), where ρ is a density operator and $\{M_k\}$ a family of operators such that the completeness relation

$$\sum_k M_k^\dagger M_k = I \tag{2}$$

is satisfied. Under this assumption the map is then Trace-Preserving and Completely-Positive (TPCP), and hence maps density operators to density operators. We refer the reader to e.g. [1], [15], [5], [7] for a detailed discussions of the properties of quantum operations and the physical meaning of the complete-positivity property.

One can then consider the discrete-time dynamical semigroup, acting on $\mathfrak{D}(\mathcal{H}_I)$, induced by iteration of a given TPCP map. The resulting discrete-time quantum system is described by

$$\rho(t+1) = \mathcal{T}[\rho(t)] = \sum_k M_k \rho(t) M_k^\dagger. \tag{3}$$

Given the initial conditions $\rho(0)$ for the system, we can then write $\rho(t) = \mathcal{T}^t[\rho(0)],\ \ t = 1, 2, \ldots$ where $\mathcal{T}^t[\cdot]$ indicates t applications of the TPCP map $\mathcal{T}[\cdot]$. Notice that while the dynamic map is linear, the "state space" $\mathfrak{D}(\mathcal{H}_I)$ is a convex, compact subset of the cone of the positive elements in $\mathfrak{H}(\mathcal{H}_I)$.

We now recall the relevant definitions of quantum dynamical invariance and attractivity. Consider an orthogonal decomposition of the system Hilbert space:

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R. \tag{4}$$

Let $n = \dim(\mathcal{H}_I)$, $m = \dim(\mathcal{H}_S)$, and $r = \dim(\mathcal{H}_R)$, and let $\{|\phi\rangle_j^S\}_{j=1}^m$, $\{|\phi\rangle_k^R\}_{k=1}^r$ denote orthonormal bases for $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively. Decomposition (4) is then naturally associated with the following basis for $\mathcal{H}_I$:

$$\{|\varphi_l\rangle\} = \{|\phi\rangle_j^S\}_{j=1}^m \cup \{|\phi\rangle_k^R\}_{k=1}^r.$$

This basis induces a block structure for matrices representing operators acting on $\mathcal{H}_I$:

$$X = \left[\begin{array}{c|c} X_S & X_P \\ \hline X_Q & X_R \end{array}\right].$$

In the rest of the paper the subscripts S, P, Q and R will follow this convention. Let Π_S and Π_R be the projection operators over the subspaces $\mathcal{H}_S$ and $\mathcal{H}_R$, respectively.

In this work we consider the case of pure state stabilization, i.e. $\dim(\mathcal{H}_S) = 1$. The more general case of $\dim(\mathcal{H}_S) = m \geq 1$ has been studied in [4].

III. Analysis of the Dynamics

Definition 1 (Invariance): Let $\mathcal{I}$ evolve under iterations of a TPCP map. The pure state $\rho_S = \Pi_S$ is invariant if

$$\rho_S = \mathcal{T}[\rho_S].$$

Definition 2 (Attractivity): Let $\mathcal{I}$ evolve under iterations of a TPCP map $\mathcal{T}$. The pure state $\rho_S = \Pi_S$ is attractive if $\forall \rho \in \mathcal{D}(\mathcal{H}_I)$ we have:

$$\lim_{t\to\infty} \left\| \mathcal{T}^t(\rho) - \Pi_S \mathcal{T}^t[\rho] \Pi_S \right\| = 0.$$

Definition 3 (Global asymptotic stability): Let $\mathcal{I}$ evolve under iterations of a TPCP map $\mathcal{T}$. The pure state $\rho_S = \Pi_S$ is *Globally Asymptotically Stable* (GAS) if it is invariant and attractive.

This section is devoted to recalling the necessary and sufficient conditions on the form of the TPCP map $\mathcal{T}$ for *a given* quantum subspace $\mathcal{S}$ to be GAS. We start by focusing on the invariance property.

Proposition 1: Let the TPCP transformation $\mathcal{T}$ be described by the Kraus map (1). Let the matrices M_k be expressed in their block form

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ M_{k,Q} & M_{k,R} \end{bmatrix}$$

according to the state space decomposition (4). Then the state $\rho_S = \Pi_S$ is invariant if and only if

$$M_{k,Q} = 0 \quad \forall k\,. \tag{5}$$

The proof can be found in [4].

The main tool we are going to use in deriving a characterization of TPCP maps that render a certain pure state GAS, is LaSalle's invariance principle, which we recall here in its discrete time form [13].

Theorem 1 (La Salle's theorem for discrete-time systems): Consider a discrete-time system $x(t+1) = \mathcal{T}[x(t)]$. Suppose V is a $\mathcal{C}^1$ function of $x \in \mathbb{R}^n$, bounded below and satisfying

$$\Delta V(x) = V(\mathcal{T}[x]) - V(x) \leq 0, \quad \forall x \tag{6}$$

i.e. $V(x)$ is non-increasing along forward trajectories of the plant dynamics. Then any bounded trajectory converges to the largest invariant subset W contained in the locus $E = \{x | \Delta V(x) = 0\}$.

Being any TPCP map a map from the compact set of density operators to itself, any trajectory is bounded. Let us then consider the function

$$V(\rho) = \text{Tr}(\Pi_R \rho) \geq 0. \tag{7}$$

The function $V(\rho)$ is $\mathcal{C}^1$ and bounded from below, and it is a natural candidate for a Lyapunov function for the system. In fact, it represents the probability of the event Π_R, that is, the probability that the system is found in the reminder subspace $\mathcal{H}_R$ after a measurement.

The variation of $V(\rho)$ along forward trajectories of the system (3) is

$$\Delta V(\rho) = \mathrm{Tr}\left[\Pi_R\left(\sum_k M_k\rho M_k^\dagger - \rho\right)\right] \qquad (8)$$

Notice that $\mathrm{Tr}(\sum_k M_k\rho M_k^\dagger - \rho) = 0$, and that $V(\rho_S) = 0$. If $\mathcal{H}_S$ is invariant, straightforward calculations show that

$$\Delta V(\rho) = \mathrm{Tr}\left[\sum_k M_{k,R}\rho_R M_{k,R}^\dagger - \rho_R\right], \qquad (9)$$

so that in order to get $\Delta V \leq 0$ the map $\mathcal{T}_R[\rho_R] := \sum_k M_{k,R}\rho_R M_{k,R}^\dagger$ has to be trace non-increasing. This condition is automatically verified, once $\mathcal{T}$ is a TPCP map.

This leaves us with determining when the pure state ρ_S is the largest invariant set in E. The following specialization of our result in [4] to pure states, provides a characterization of the dynamics that render a certain state GAS.

Theorem 2: Let the TPCP transformation $\mathcal{T}$ be described by the Kraus map (1). Consider an orthogonal subset decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$, with the pure state $\rho_S = \Pi_S$ being invariant. Let the matrices M_k be expressed in their block form

$$M_k = \begin{bmatrix} M_{k,S} & M_{k,P} \\ 0 & M_{k,R} \end{bmatrix}$$

according to the same state space decomposition. Then ρ_S is GAS if and only if there are no invariant states with support on $\bigcap_k \ker\left(M_{k,P}\right)$.

IV. A CANONICAL MATRIX FORM BASED ON THE QR DECOMPOSITION

In this section we will recall some technical results about QR decomposition that will allow us to develop a new algebraic tool, namely a canonical form with respect to the left action of the unitary matrix group. With this tool it will then be possible to move from the analysis results presented in the previous section to an algorithm for the synthesis of stabilizing control laws.

Definition 4 (QR decomposition [9]): A QR decomposition of a complex-valued square matrix A is a decomposition of A as

$$A = QR,$$

where Q is an orthogonal matrix (meaning that $Q^\dagger Q = I$) and R is an upper triangular matrix.

The QR decomposition of a given complex-valued square matrix A is not unique. In the case of non-singular matrix A, one can show that the upper triangular factors of any two QR decompositions of A differ only for the phase of their rows. When A is singular, on the other hand, this is not true.

However, introducing some conditions on the R matrix, it is possible to obtain a *canonical form* for the QR decomposition in a sense that will be explained later in this section. The following theorem characterizes the canonical QR decomposition and guarantees its existence.

Theorem 3: Given any (complex) square matrix A of dimension n, it is possible to derive a QR decomposition $A = QR$ such that

$$r_{ij} = 0 \quad \forall j \leq n, \forall i > \rho_j \qquad (10)$$

where ρ_j is the rank of the first j columns of A, and such that the first nonzero element of each row of R is real and positive.

The proof of this theorem is given in Appendix A, where a method to construct such a decomposition is also provided.

Moreover, we can prove that the R obtained in this way is a canonical form. We start by recalling what a matrix canonical form with respect to the action of some group action is. Let $\mathcal{G}$ be a group acting on $\mathbb{C}^{n\times n}$. Let $A, B \in \mathbb{C}^{n\times n}$. If there exists a $g \in \mathcal{G}$ such that $g(A) = B$, we say that A and B are $\mathcal{G}$-equivalent, and we write $A \sim_{\mathcal{G}} B$.

Definition 5: A canonical form with respect to $\mathcal{G}$ is a function $\mathcal{F} : \mathbb{C}^{n\times n} \to \mathbb{C}^{n\times n}$ such that for every $A, B \in \mathbb{C}^{n\times n}$:

i. $\mathcal{F}(A) \sim_{\mathcal{G}} A$;
ii. $\mathcal{F}(A) = \mathcal{F}(B)$ if and only if $A \sim_{\mathcal{G}} B$.

Let us consider the unitary matrix group $\mathcal{U}(n) \subset \mathbb{C}^{n\times n}$ and consider its action on $\mathbb{C}^{n\times n}$ through left-multiplication, that is, for any $U \in \mathcal{U}(n)$, $M \in \mathbb{C}^{n\times n}$:

$$U(M) = UM.$$

The following result has been proven in [4].

Theorem 4: Define $\mathcal{F}(A) = R$, with R the upper-triangular matrices obtained by the procedure described in the proof of Theorem 3. Then $\mathcal{F}$ is a canonical form with respect to $\mathcal{U}(n)$ (and its action on $\mathbb{C}^{n\times n}$ by left multiplication).

V. STABILIZING PURE STATES VIA CLOSED-LOOP CONTROL

In this section we deal with the problem of stabilization of a given quantum subspace by discrete-time measurements and unitary control. The control scheme we employ follows the ideas of [14], [16], and is in fact an instance of the Markovian feedback models studied in e.g. [3], [10]. Suppose that a generalized measurement operation can be performed on the system at times $t = 1, 2, \ldots$, resulting in an open system, discrete-time dynamics described by a given Kraus map, with associated Kraus operators $\{M_k\}$. Suppose moreover that we are allowed to unitarily control the state of the system, i.e. $\rho_{controlled} = U\rho U^\dagger$, $U \in \mathcal{U}(\mathcal{H}_I)$. We shall assume that the control is fast with respect to the measurement time scale, or the measurement and the control acts in distinct time slots.

We can then implement a Markovian feedback control, consisting in a map from the set of measurement outcomes to the set of unitary matrices, $U(k) : k \mapsto U_k \in \mathcal{U}(\mathcal{H}_I)$. The measurement-control loop is then iterated: If we average over the measurement results at each step, this yields a different TPCP map, which describes the evolution of the state *immediately after* each application of the controls:

$$\rho(t+1) = \sum_k U_k M_k \rho(t) M_k^\dagger U_k^\dagger.$$

Suppose that the operators $\{M_k\}$ are given, corresponding to a measurement that is performed on the quantum system, with corresponding outcomes $\{k\}$. We are then looking for a set of unitary transformations $\{U_k\}$ such that, once they are applied to the system, the resulting semigroup generator makes a given pure state ρ_S GAS. Let us introduce a preliminary, technical result that employs in a nontrivial way the structure of the canonical QR, the proof of which is given in [4].

Lemma 1: Let R be the upper triangular factor of a *canonical QR decomposition* in the form

$$R = \begin{bmatrix} R_S & R_P \\ 0 & R_R \end{bmatrix}$$

(according to the block structure induced by (4)) and suppose $R_P = 0$. Consider the matrix N obtained by left multiplying R by a unitary matrix V:

$$N = VR = \begin{bmatrix} V_S & V_P \\ V_Q & V_R \end{bmatrix} \begin{bmatrix} R_S & 0 \\ 0 & R_R \end{bmatrix} = \begin{bmatrix} N_S & N_P \\ N_Q & N_R \end{bmatrix}.$$

Then $N_Q = 0$ implies $N_P = 0$.

This result will be instrumental in proving the main theorem of the section, which provides an iterative control design procedure that renders the desired pure state asymptotically stable whenever it is possible.

Theorem 5: Consider a subspace orthogonal decomposition $\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_R$, $\dim(\mathcal{H}_S) = 1$, and a given generalized measurement associated to Kraus operators $\{M_k\}$. Let $\{R_k\}$ be the canonical R-factors associated to $\{M_k\}$. The task of achieving global asymptotic stability of Π_S by a feedback unitary control policy is feasible if and only if there exists a $\bar{k}$ such that:

$$[\Pi_S, R_{\bar{k}}] \neq 0. \qquad (11)$$

Proof: Let us first consider the case in which all the $R_{P,k} = 0$. Recall that each R_k has been put in canonical form, so it follows from Lemma 1 that any control choice that ensures invariance of the desired subspace, that is $N_k = U_k R_k$ with $N_{Q,k} = 0$, makes all N_k's block diagonal, since $N_{P,k} = 0$. Hence an invariant state with support on $\mathcal{H}_R$ always exists. This, via Theorem 2, precludes the existence of a control choice that renders Π_S GAS. Hence, necessity of (11) is proven.

On the other hand, if $R_{P,k} \neq 0$ for some k, one can devise a procedure to construct unitaries $\{U_k\}$ that "destabilize" any state with support on $\mathcal{H}_R$ only. This can be done in many different ways: an explicit algorithm is provided in Appendix B. The absence of stationary states with support in $\mathcal{H}_R$, through Theorem 2, is then sufficient to prove that Π_S is GAS. ■

While (11) resembles the condition emerging from the study of the Markovian feedback master equation in continuous-time [18], a remarkable difference is apparent: the structure of the R_k's also depends on the choice of target state, rendering the determination of the stabilizable pure-state manifold non trivial.

VI. Robustness of State-Preparation

A potential limitation to the implementation of this feedback strategy lays in the fact it requires strong control capabilities and perfect detection. That is, we assume that we know exactly the form of the measurement map, and that every measurement leads to a valid outcome.

In order to evaluate how critical this hypothesis is for the whole procedure, let us follow the approach of [18] and choose a suitable Hermitian basis in $\mathfrak{B}(\mathcal{H}_i) \approx \mathbb{C}^{d\times d}$. This can always be done for finite d, for example by employing the natural d-dimensional extension of the Pauli matrices [1], [2]. In such a basis, all density operators are represented by d^2-dimensional vectors $\bar{\rho} = (\rho_0, \rho_1, \ldots, \rho_{d^2-1})^T$, where the first component ρ_0, relative to $\frac{1}{\sqrt{d}}\mathbb{I}_d$, is invariant and equal to $\frac{1}{\sqrt{d}}$ for TP-dynamics. Let $\rho_v = (\rho_1, \ldots, \rho_{d^2-1})^T$. Hence any Kraus map $\mathcal{E}[\cdot]$, being a TP linear map, in this vectorized representation must take the form:

$$\bar{\rho}(t+1) = \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t+1) \end{bmatrix} = \left[\begin{array}{c|c} 1 & 0 \\ \hline C & D \end{array}\right] \begin{bmatrix} 1/\sqrt{d} \\ \rho_v(t) \end{bmatrix}. \qquad (12)$$

Assume that the dynamics has a unique attractive state $\bar{\rho}^{(0)}$. Thus $I - D$ must be invertible and we obtain:

$$\bar{\rho}^{(0)} = \frac{1}{\sqrt{d}} \begin{bmatrix} 1 \\ (I-D)^{-1}C \end{bmatrix}.$$

Consider now a small perturbation of the Kraus map, $\tilde{\mathcal{E}}[\cdot] = (1-\varepsilon)\mathcal{E}[\cdot] + \varepsilon\mathcal{E}'[\cdot]$ depending on the continuous parameter ε, and ε sufficiently small so that $(I - D - \varepsilon(D' - D))$ remains invertible. This may account for small detection errors, imperfect knowledge of the model and other non-idealities. The vectorized dynamics becomes:

$$\bar{\rho}(t+1) = \left((1-\varepsilon)\left[\begin{array}{c|c} 1 & 0 \\ \hline C & D \end{array}\right] + \varepsilon\left[\begin{array}{c|c} 1 & 0 \\ \hline C' & D' \end{array}\right]\right) \begin{bmatrix} 1/\sqrt{d} \\ \rho_v \end{bmatrix}, \qquad (13)$$

and the new attractive, unique equilibrium state is:

$$\bar{\rho}^{(\varepsilon)} = \frac{1}{\sqrt{d}} \begin{pmatrix} 1 \\ (I-(1-\varepsilon)D - \varepsilon D')^{-1}((1-\varepsilon)C + \varepsilon C') \end{pmatrix}.$$

Because $\bar{\rho}^{(\varepsilon)}$ is a continuous function of ε, we are guaranteed that for a sufficiently high detection efficiency the perturbed attractive state will be arbitrarily close to the desired one in trace norm. Therefore, if we relax our control task to a state preparation problem with sufficiently high fidelity, this may be accomplished with a sufficiently high detection efficiency, yet strictly less than 1.

VII. Examples

In this section we present some simple examples. For each of them, we study capabilities of feedback unitary control via the tools we just presented, and when possible we apply the algorithm proposed in Appendix A to design an effective control law.

A. Projective measurements

A particularly simple case is worth mentioning: When the M_k are rank one projectors, that is, *represent a non-degenerate von Neumann's measurement*, the stabilization of *any pure state* can be achieved. In fact, being a canonical form:

$$\mathcal{F}(M_k) = \mathcal{F}(U\Pi_k U^\dagger) = \mathcal{F}(\Pi_k U^\dagger) = R_k,$$

where Π_i is the rank one projector on the k-th basis element, and hence $\Pi_k U^\dagger$ is different from zero only in the k-th row, which is in turn the k-th column of U, u_k. Thus each R_k has only the first row different from zero, and it is proportional to $u_k^\dagger$. Being $\{u_k\}$ a basis, some $R_{P,k}$ has to be non-zero as it corresponds to the last $n-1$ components of the u_k's.

Physically, at any measurement step we obtain a known pure state, which can then be driven back to desired one. While the achieved "cyclic" stabilization may appear weak, the use of projective measurements renders it robust with respect unwanted noise effects: At each cycle a state of maximal information is deterministically determined by the measurement, virtually erasing any unwanted dynamics.

B. Entanglement Generation

We consider in this example a *two-qubit system*, defined on a Hilbert space $\mathcal{H}_I \simeq \mathbb{C}^2 \otimes \mathbb{C}^2$. Consider the task of stabilizing the maximally entangled state

$$\rho_d = \frac{1}{2}\left(|00\rangle + |11\rangle\right)\left(\langle 00| + \langle 11|\right). \tag{14}$$

In order to apply the proposed control design technique, let us consider a different basis $\mathcal{B}$ such that in the new representation $\rho_d^{\mathcal{B}} = \mathrm{diag}\left(\left[\begin{smallmatrix}1 & 0 & 0 & 0\end{smallmatrix}\right]\right)$. This can be achieved by considering the *Bell-basis*

$$\mathcal{B} = \left\{ \frac{|00\rangle + |11\rangle}{\sqrt{2}}, \frac{|00\rangle - |11\rangle}{\sqrt{2}}, \frac{|01\rangle + |10\rangle}{\sqrt{2}}, \frac{|01\rangle - |10\rangle}{\sqrt{2}} \right\}.$$

Suppose that the following generalized measurement is available

$$\mathcal{T}[\rho] = \sum_{k=1}^{3} M_k \rho M_k^\dagger$$

with operators (represented in the computational basis):

$$M_1 = \frac{1}{\sqrt{4}}\left(\sigma_+ \otimes I\right), \qquad M_2 = \frac{1}{\sqrt{4}}\left(I \otimes \sigma_+\right), \tag{15}$$

$$M_3 = \sqrt{I - M_1^\dagger M_1 - M_2^\dagger M_2}. \tag{16}$$

where $\sigma_+ = \left[\begin{smallmatrix}0 & 1\\ 0 & 0\end{smallmatrix}\right]$. These Kraus operators may be used to describe a discrete-time spontaneous emission process, where the event associated to $M_{1,2}$ corresponds to the decay of one qubit (with probability $\frac{1}{4}$ each), and we neglect the event of the two qubits decaying in the same time interval.

Let us move to the Bell basis, and then apply the algorithm. The canonical QR decomposition of the matrices $M_k^{\mathcal{B}}$ returns the following triangular factors (we do not report here the corresponding orthogonal matrices Q_k):

$$R_1 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \ R_2 = \begin{bmatrix} \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} 0.8660 & 0.2887 & 0 & 0 \\ 0 & 0.8165 & 0 & 0 \\ 0 & 0 & 0.8660 & 0 \\ 0 & 0 & 0 & 0.8660 \end{bmatrix}.$$

According to the proposed approach, by inspection of the upper triangular factors R_i we can decide about the feasibility of the stabilization task. Indeed, as the blocks $R_{P,k}$, $k = 1, \ldots, 3$ are non-zero blocks, hence the stabilization problem is feasible.

Moreover, notice that at this step no further transformation is needed on the matrices, as the obtained R factors are already decomposed according to

$$\mathcal{H}_I = \mathcal{H}_S \oplus \mathcal{H}_S^{(1)} \oplus \mathcal{H}_R^{(1)}.$$

where $\mathcal{H}_R^{(1)} = \bigcap_k \ker R_{P,k}$. Continuing with the iteration, we have then to determine the subspace $\mathcal{H}_R^{(2)} = \bigcap_k \ker R_{P,k}^{(1)}$. By inspection one can see that this space is empty, and therefore the iteration stops successfully.

It can be shown by direct computation that the Hamiltonians needed to implement the needed unitary transformation (using ideally unbounded control pulses in order to make the dissipation effect negligible on when the control is acting) form a 3-dimensional control algebra [6].

C. A non-stabilizable case

As a third example, we consider the case in which the problem of achieving global asymptotic stability of a given pure state by a feedback unitary control law is not feasible.

Consider a system of dimension $d = 2$, and consider the problem of stabilizing $\rho_0 = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]$. Suppose that the following set of two measurements is given:

$$M_1 = \begin{bmatrix} \sqrt{p} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_2 = \begin{bmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{bmatrix}.$$

Both M_1 and M_2 are already in the canonical upper triangular form prescribed by Theorem 3. Following Theorem 5, we check feasibility of the control problem by inspecting

$$[\rho_0, R_1] = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\sqrt{p} & 0\\ 0 & 0\end{smallmatrix}\right] - \left[\begin{smallmatrix}\sqrt{p} & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right]$$
$$[\rho_0, R_2] = \left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right]\left[\begin{smallmatrix}\sqrt{1-p} & 0\\ 0 & 1\end{smallmatrix}\right] - \left[\begin{smallmatrix}\sqrt{1-p} & 0\\ 0 & 1\end{smallmatrix}\right]\left[\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right] = \left[\begin{smallmatrix}0 & 0\\ 0 & 0\end{smallmatrix}\right].$$

As both the terms are zero, the problem is not feasible (there is no feedback unitary control that makes ρ_0 GAS).

VIII. Conclusions

Theorem 2 provides a characterization of the semigroup dynamics that render a certain pure state attractive, by employing LaSalle's invariance principle: in order to exploit this result in the design of stabilizing unitary feedback control strategies, we proved that a *canonical* QR decomposition can be derived, and that it allows us to establish the potential of the Markovian discrete-time feedback control scheme.

This suggests how the introduction of a single measurement can overcome some intrinsic limitations that pure open-loop strategies present. We believe that these results also represent a mathematical standpoint from which more challenging control problems can be tackled. Future research directions involve the effectiveness of the control in presence of imperfect detection, and the applicability of the theory to experimental systems, with a particular focus on state-preparation for optical and solid-state systems.

Appendix

A. Results on the QR decomposition

In order to provide a constructive proof for Theorem 3, we need the following lemma.

Lemma 2: Consider a QR decomposition of a square matrix A of dimension n, and an index $\bar{j}$ in $[1, n]$, such that

$$r_{ij} = 0 \quad \forall j \le \bar{j}, \forall i > \rho_j \tag{17}$$

where ρ_j is the rank of the first j columns of A. Let a_i and q_i, be the i-th column of A and Q respectively. Then

$$< a_1, \ldots, a_j > = < q_1, \ldots, q_{\rho_j} > \quad \forall j = 1, \ldots, \bar{j}.$$

Proof: Consider the expression for the j-th column of A, $a_j = Qr_j$. By the hypothesis, the last $n - \rho_j$ elements of r_j are zeros, hence it results $a_j \in < q_1, \ldots, q_{\rho_j} > \quad \forall j = 1, \ldots, \bar{j}$ and therefore $< a_1, \ldots, a_j > \subseteq < q_1, \ldots, q_{\rho_j} >$ $\forall j = 1, \ldots, \bar{j}$. As the rank of the first j columns is ρ_j, which is also the dimension of $< q_1, \ldots, q_{\rho_j} >$, equality of the two subspaces holds. ■

Note that the hypothesis 17 of Lemma 2 with $\bar{j} = n$ corresponds to the characterization 10 for the QR decomposition given in Theorem 3.

Proof: [Proof of Theorem 3] We explicitly construct the QR decomposition through a Gram-Schmidt orthonormalization process, fixing the degrees of freedom of the upper-triangular factor R column by column. We denote by A, Q, R the matrices, with a_i, q_i, r_i their i-th columns and with $a_{i,j}, q_{i,j}, r_{i,j}$ their elements, respectively. Let us start from the first non zero column of $A \in \mathbb{C}^{n \times n}$, a_{i_0}, and define

$$q_1 = \frac{a_{i_0}}{\|a_{i_0}\|}, \qquad r_{1,i_0} = \|a_{i_0}\|, \qquad r_{2,i_0} = \ldots = r_{n,i_0} = 0. \tag{18}$$

Also fix $r_j = 0$ for all $j < i_0$.

The next columns of Q, R are constructed by an iterative procedure. Define ρ_{i-1} as the rank of the first $i - 1$ columns of A. We can assume (by induction) to have the first ρ_{i-1} columns of Q and the first $i - 1$ columns of R constructed in such a way that $r_{k,j} = 0$ for $k > \rho_j$ and $j \le i - 1$.

Consider the next column of A, a_i. Assume as a first case that a_i is linearly dependent with the previous columns of A, that is $\rho_i = \rho_{i-1}$. Since Lemma 2 applies, a_i can be written as

$$a_i = \sum_{j=1}^{i-1} \alpha_j a_j = \sum_{j=1}^{i-1} \alpha_j \sum_{\ell=1}^{\rho_j} r_{\ell,j} q_\ell$$

and therefore, being a_i a linear combination of the columns $\{q_1, \ldots, q_{\rho_{i-1}}\}$, the elements of r_i are defined as

$$r_{\ell,i} = q_\ell^\dagger a_i\,, \qquad \text{for } \ell = 1, \ldots, \rho_i.$$

On the other hand, if the column a_i is linearly independent from the previous columns of A, then the rank $\rho_i = \rho_{i-1} + 1$. As before, the first ρ_{i-1} coefficients of r_i must be defined as

$$r_{\ell,i} = q_\ell^\dagger a_i\,, \qquad \text{for } \ell = 1, \ldots, \rho_i - 1.$$

Let us also introduce $\tilde{a}_i := a_i - \sum_{\ell=1}^{\rho_i} r_{\ell,i} q_\ell \neq 0$ and define $q_{\rho_i} = \frac{\tilde{a}_i}{\|\tilde{a}_i\|}$ $\quad r_{\rho_i,i} = \|\tilde{a}_i\|$. In both cases, let us set $r_{\ell,i} = 0$ for $\ell = \rho_i + 1, \ldots, n$. It is immediate to verify that the obtained q_{ρ_i} is orthonormal to the columns $q_1, \ldots, q_{\rho_i - 1}$, and that $a_i = Qr_{\rho_i}$.

After iterating until the last column of R is defined, we are left to choose the remaining columns of Q so that the set $\{q_1, \ldots, q_n\}$ is an orthonormal basis for $\mathbb{C}^{n \times n}$. By construction, $A = QR$. ■

B. Constructive Algorithm for the Control Design

Control design algorithm

Let R_k be the canonical R-factor of M_k. Define $\mathcal{H}_R^{(0)} = \mathcal{H}_R$, $R_{P,k}^{(0)} = R_{P,k}$, and assume that the control design problem is feasible (therefore there exists at least one k such that $R_{P,k} \neq 0$). Initialize $V^{(0)} = I$, $Z^{(0)} = I$, and consider the following iterative procedure, starting from $i = 0$:

1) Define $\mathcal{H}_R^{(i+1)} = \bigcap_k \ker R_{P,k}^{(i)}$:
If $\mathcal{H}_R^{(i+1)} = \{0\}$ then the iteration is successfully completed. Go to step 8).
If $\mathcal{H}_R^{(i+1)} \subsetneq \mathcal{H}_R^{(i)}$, define $\mathcal{H}_S^{(i+1)} = \mathcal{H}_R^{(i)} \ominus \mathcal{H}_R^{(i+1)}$ and $Y^{(i+1)} = I$.
If $\mathcal{H}_R^{(i+1)} = \mathcal{H}_R^{(i)}$ (i.e. $R_{P,k}^{(i)} = 0\ \forall k$) then, if $\dim(\mathcal{H}_R^{(i)}) \ge \dim(\mathcal{H}_S^{(i)})$:
   a) Choose a subspace $\mathcal{H}_S^{(i+1)} \subseteq \mathcal{H}_R^{(i)}$ of the same dimension of $\mathcal{H}_S^{(i)}$. (Re)-define $\mathcal{H}_R^{(i+1)} = \mathcal{H}_R^{(i)} \ominus \mathcal{H}_S^{(i+1)}$.
   b) Let $\mathcal{H}_T^{(i)} = \bigoplus_{j=0}^{i-1} \mathcal{H}_S^{(j)}$. Construct a unitary matrix Y with the following block form, according to a Hilbert space decomposition $\mathcal{H}_I = \mathcal{H}_T^{(i)} \oplus \mathcal{H}_S^{(i)} \oplus \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$:

$$Y^{(i+1)} = \left[\begin{array}{c|c|c|c} I & 0 & 0 & 0 \\ \hline 0 & 1/\sqrt{2}I & 1/\sqrt{2}I & 0 \\ \hline 0 & 1/\sqrt{2}I & -1/\sqrt{2}I & 0 \\ \hline 0 & 0 & 0 & I \end{array}\right]. \tag{19}$$

If instead $\dim(\mathcal{H}_R^{(i)}) < \dim(\mathcal{H}_S^{(i)})$:
   a) Choose a subspace $\mathcal{H}_S^{(i+1)} \subseteq \mathcal{H}_S^{(i)}$ of the same dimension of $\mathcal{H}_R^{(i)}$.
   b) Let $\mathcal{H}_T^{(i)} = \left(\bigoplus_{j=0}^{i-1} \mathcal{H}_S^{(j)}\right) \oplus \left(\mathcal{H}_S^{(i)} \ominus \mathcal{H}_S^{(i+1)}\right)$. Construct

a unitary matrix Y with the following block form, according to a Hilbert space decomposition $\mathcal{H}_I = \mathcal{H}_T^{(i)} \oplus \mathcal{H}_S^{(i+1)} \oplus \oplus \mathcal{H}_R^{(i+1)}$:

$$Y^{(i+1)} = \left[\begin{array}{c|c|c} I & 0 & 0 \\ \hline 0 & 1/\sqrt{2}I & 1/\sqrt{2}I \\ \hline 0 & 1/\sqrt{2}I & -1/\sqrt{2}I \end{array}\right]. \tag{20}$$

c) Define $Z^{(i+1)} = Z^{(i)} Y^{(i+1)}$ and go to step 8).

2) Define $Z^{(i+1)} = Z^{(i)} Y^{(i+1)}$.
3) Rewrite $\tilde{R}_{R,k}^{(i)} = W^{(i+1)} R_{R,k}^{(i)} W^{(i+1)\dagger}$ in a basis according to the $\mathcal{H}_R^{(i)} = \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$ decomposition.
4) Compute the canonical QR decomposition of $\tilde{R}_{R,k}^{(i)} = Q_k^{(i+1)} R_k^{(i+1)}$. Compute the matrix blocks $R_{P,k}^{(i+1)}, R_{R,k}^{(i+1)}$ of $R_k^{(i+1)}$, again according to the decomposition $\mathcal{H}_R^{(i)} = \mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$.
5) Define

$$U^{(i+1)} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & W^{(i+1)\dagger}\left(Q_k^{(i+1)}\right)^{\dagger} W^{(i+1)} \end{array}\right] U^{(i)}.$$

6) Define $V^{(i+1)} = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & W^{(i+1)} \end{array}\right] V^{(i)}$.
7) Increment the counter and go back to step 1).
8) Return the unitary controls $U_k = V^{(i)\dagger} Z^{(i)} V^{(i)} U_k^{(i)}$.

If the algorithm does not stop, then at each step of the iteration the dimension of $\mathcal{H}_R^{(i)}$ is reduced by at at least 1, hence the algorithm is completed in at most n steps. If the algorithm is successfully completed at a certain iteration j, we have built unitary controls $\{U_k^{(j)}\}$ and a unitary $V^{(j)}$ such that the controlled quantum operation element, under the change of basis $V^{(j)}$, is of the form:

$$\tilde{N}_k = V^{(j)} U_k M_k V^{(j)\dagger}$$
$$= Z^{(j)} \left[\begin{array}{c|c|c|c|c} R_{S,k}^{(0)} & \bar{R}_{P,k}^{(0)} & 0 & 0 & 0 \\ \hline 0 & R_{S,k}^{(1)} & \ddots & 0 & 0 \\ \hline 0 & 0 & \ddots & \bar{R}_{P,k}^{(j-1)} & 0 \\ \hline 0 & 0 & 0 & R_{S,k}^{(j)} & \bar{R}_{P,k}^{(j)} \\ \hline 0 & 0 & 0 & 0 & R_{R,k}^{(j)} \end{array}\right]$$

where the block structure is consistent with the decomposition $\bigoplus_{i=0}^{j+1} \mathcal{H}_S^{(i)}$ (where to simplify the notation we set $\mathcal{H}_S^{(j+1)} = \mathcal{H}_R^{(j)}$). Let $\bar{R}_k$ be the block matrix above and consider its upper-triangular part. The rows have the form $\left[\bar{R}_{P,k}^{(i)} \quad 0 \quad \dots \quad 0\right]$ because at each step of the iteration we choose a basis $W^{(i)}$ according to the decomposition $\mathcal{H}_S^{(i+1)} \oplus \mathcal{H}_R^{(i+1)}$, where $\mathcal{H}_R^{(i+1)} \subseteq \bigcap_k \ker R_{P,k}^{(i)}$, hence obtaining $R_{P,k}^{(i)} W^{(i)\dagger} = \left[\bar{R}_{P,k}^{(i)} \quad 0 \quad \dots \quad 0\right]$. It is easy to verify that the subsequent unitary transformations have no effects on the blocks $\bar{R}_{P,k}^{(i)}$.

The upper-triangular form of each $\bar{R}_k$ and the form of $Z^{(j)}$ and $V^{(j)}$, both block-diagonal with respect to the orthogonal decomposition $\mathcal{H}_S \oplus \mathcal{H}_R$, ensure invariance of $\mathcal{H}_S$.

By construction, for all $i = 0, \dots, j$, either $\bigcap_k \ker \bar{R}_{P,k}^{(i)} = \{0\}$ and $Y^{(i)} = I$, or $\bar{R}_{P,k}^{(i)} = 0$ for all k and $Y^{(i)}$ differs from the identity matrix and has the form (19) or (20).

The fact that no invariant state can have support on $\bigoplus_{i=1}^{j+1} \mathcal{H}_S^{(i)}$ can be proven by induction, following the reasoning in [4].

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