Martingale approach in quantum state estimation using indirect measurements

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***Abstract*— The aim of this work is to propose mathematically well grounded statistical methods for state estimation in the indirect measurement settings by using martingale theory, and to compare their efficiency to the usual direct approaches. The measurement scheme considered is the simplest possible discrete time case, where both the unknown and the measurement quantum systems are quantum bits. The repeated measurements performed on the measurement subsystem of the composite system enables us to construct an estimator for the initial state of the unknown system. An initial state relative method of detecting a stopping time is proposed where the final states are defined using a given distance from the unknown initial state. A simple estimator is proposed and used as an excellent initial point to build more complex and better estimation methods. The efficiency of the proposed procedure is investigated both analytically and experimentally using simulation in different settings of parameters. The possible generalizations of the proposed estimation methods are also outlined.**

I. Introduction

The problems related to quantum state estimation are fundamental for quantum information theory [8], while quantum state filtering [2], [3] plays a key role in quantum feedback control. For both the state estimation (in time-invariant case), and filtering (in the time varying situation) one needs to apply probabilistic methods because of the stochastic nature of any measurement applied to quantum systems.

Similarly to any realistic physical measurement, a quantum measurement is almost always realized by taking a measurement device that is put in interaction with the system to be measured, and then to "read" the meter on the measurement device. In the macroscopic measurement situation the measurement device is "small" compared to the system to be measured thus the measurement back-action, i.e. the disturbance caused by the measurement is negligible, but that is not the case in the quantum setting. In quantum state estimation the above measurement configuration, when the 'unknown' quantum system is coupled with a 'measurement' (also called 'probe' or 'ancilla') system and the measurements are only applied on the measurement system [6] is termed an *indirect measurement scheme*.

Indirect measurements can be realized by coupling the investigated system to its environment in a "loose" way, when the so-called *weak measurements* [12], [5] do not demolish the system state completely, i.e. the post-measurement state still contains information about the original one, but on the price of a decreased information gained from the measurement. However, it is intuitively clear, that one must make a compromise between the information gained in a measurement and the disturbance or demolition caused by it. The general impossibility of determining the state of a single quantum system is proved in [4] whatever measurement scheme is used. This indicates that the efficiency or precision provided by an indirect measurement scheme is necessarily smaller than that of a scheme that uses von Neumann measurements.

The aim of our work is to develop well grounded statistical methods for state estimation in the indirect measurement setting applying martingale theory. The idea of using martingale methods for state estimation and filtering for quantum systems is not new, the possibility emerged by the description of the quantum state evolution under measurement using quantum stochastic differential equations in the continuous time case [2], and quantum stochastic difference equations in discrete time [3]. The state evolution driven by measurements has also been modelled by quantum martingales in [1].

The outline of the paper is as follows. The investigated measurement scheme is described in the next section, that is followed by the introduction of our martingale approach. Then two sections deal with the effect of the measurement scheme parameters on the estimation quality, and with possible generalizations. Finally, conclusions are drawn.

II. An indirect measurement scheme

In this paper the simplest possible discrete time case of indirect measurements is considered, where both the unknown and the measurement quantum systems are quantum bits. We concentrate on the resulting stochastic process, therefore we only give here a short introduction of the measurement scheme we are using [10], [11].

A. Measurement setup

Throughout the paper the Bloch-vector representation of the states of quantum bits is used, i.e.

$$\rho = \frac{1}{2}(I + \theta_1\sigma_1 + \theta_2\sigma_2 + \theta_3\sigma_3), \qquad (1)$$

where σ_i stands for the Pauli-matrices, $i = 1, 2, 3$, and $\theta = [\theta_1, \theta_2, \theta_3]^T$ is the Bloch-vector. This way, the states of a quantum bit can be described with three dimensional real vectors of maximal length 1, i.e. $\theta \in \mathbb{R}^3, \quad ||\theta|| \leq 1$. Thus, the state space of the system is the unit ball in $\mathbb{R}^3$.

The system to be investigated consists of two qubits in interaction: the unknown system and the measurement qubit.

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Let us denote the Bloch representation of the unknown system and the probe (measurement device) as

$$\rho_S(k) = \frac{1}{2}(I + \theta_S(k)\sigma^S), \quad \rho_M(k) = \frac{1}{2}(I + \theta_M(k)\sigma^M)$$

where θ_S and θ_M are 3 dimensional real vectors, σ^S and σ^M are symbolic vectors constructed from the Pauli operators acting on the Hilbert spaces $\mathcal{H}^S$ and $\mathcal{H}^M$.

The state of the composite system is represented as a 4×4 density matrix $\rho_{S+M}(k)$. The state of the composite system after an interaction is given by

$$\rho_{S+M}(k+1) = U_{S+M}\rho_{S+M}(k)U^*_{S+M}$$

where U_{S+M} is the overall system evolution unitary. Suppose in the sequel that the qubits are interacting only in the y direction for time h (sampling time) [7]:

$$U_{S+M} = e^{-\mathrm{i}h(a_y\sigma_2^S\otimes\sigma_2^M)},$$

where a_y is the coupling parameter. For our purposes choose h in such a way, that $a_y h = \pi/2$.

Afterwards, an indirect measurement is performed, in particularly the von Neumann measurement $I\otimes\sigma_x$ is applied on the composite system. Then the probabilities of the different outcomes of this measurement will be

$$\begin{aligned} \mathrm{Prob}(+1) &= \tfrac{1}{2}(1+\theta_{S2}\theta_{M3}) \\ \mathrm{Prob}(-1) &= \tfrac{1}{2}(1-\theta_{S2}\theta_{M3}). \end{aligned} \tag{2}$$

Since we are interested in the dynamical change of the system S, the first reduced density matrix should only be considered:

$$\rho_S(k+1) = \mathrm{Tr}_M\rho^*_{S+M}(k+1),$$

where $\rho^*_{S+M}(k+1)$ is the composite state after measurement. Doing the computations we get that the post-measurement states are

$$\theta_S(\pm 1) = \begin{bmatrix} \frac{\theta_{S3}\theta_{M2}\pm\theta_{S1}\theta_{M1}}{1\pm\theta_{S2}\theta_{M3}} \\ \\ \frac{\theta_{S2}\pm\theta_{M3}}{1\pm\theta_{S2}\theta_{M3}} \\ \\ \frac{\theta_{S3}\pm\theta_{M1}-\theta_{S1}\theta_{M2}}{1\pm\theta_{S2}\theta_{M3}} \end{bmatrix} \tag{3}$$

depending of the measurement outcome that can be ± 1.

This measurement setup is useful for unknown state estimation since the probabilities and the new states depend on both θ_S and θ_M. This means, that we both gain information from the measurements and retrieve information in the new states after the measurement.

B. Properties of repeated indirect measurements

After performing the above measurement process we do not obtain a completely destroyed system just a little bit changed, hence we can resume the procedure repeatedly. It is clear from (2) and (3) that we should concentrate on the estimate of the second unknown state coordinate, i.e. we describe the change of θ_{S2} (notation: $x = \theta_{S2}$) during the measurements. If we set the relevant measurement parameter to a constant $c = \theta_{M3}$ $(0 < c < 1)$, we obtain the following discrete-time Markov process:

$$x(k+1) = \left\{ \begin{array}{l} \frac{x(k)+c}{1+cx(k)}, \text{with } \frac{1+cx(k)}{2} \text{ probability} \\ \\ \frac{x(k)-c}{1-cx(k)}, \text{with } \frac{1-cx(k)}{2} \text{ probability} \end{array} \right\} \tag{4}$$

Now let us assume that we have the initial state x, and we measure first a $+1$, and thereafter a -1. After the first measurement, x changes to

$$x^{+1} = \frac{x+c}{1+cx}$$

then from x^{+1} it turns to be

$$x^{+1,-1} = \frac{x^{+1}-c}{1-cx^{+1}} = \frac{\frac{x+c}{1+cx}-c}{1-c\frac{x+c}{1+cx}} = x,$$

so the initial state x will not change after this two measurements.

The probability of these outcomes can be calculated too. First from x it will be $x^{+1} = \frac{x+c}{1+cx}$ with probability:

$$P_1 = \frac{1}{2}(1+cx)$$

then from x^{+1} it will be x with the conditional probability: $P_2 = \frac{1}{2}(1-cx^{+1}) = \frac{1}{2}(1-c\frac{x+c}{1+cx}) = \frac{1}{2}\frac{1-c^2}{1+cx}$. So the joint probability of these outcomes is

$$P_1\cdot P_2 = \frac{1-c^2}{4}.$$

Notice that this probability does not depend on the initial state x.

Both the above properties remain true when we first measure -1 and then $+1$. Therefore the measurement process can be consider as a random walk on discrete values:

$$\ldots \rightleftarrows x_{-2} \rightleftarrows x_{-1} \rightleftarrows x_0 \rightleftarrows x_1 \rightleftarrows x_2 \rightleftarrows \ldots$$

where x_0=$x(0)$ and x_i $(i\in\mathbb{Z})$ is a determined function of x_0.

Let p_n be the probability that from n measurements all outcomes are $+1$s, so we ended up in x_n after n steps. With this notation, the probability of each outcome sequence with ℓ_+ times $+1$ and ℓ_- times -1 outcomes (assuming $\ell_+ > \ell_-$) can be computed as:

$$\left(\frac{1-c^2}{4}\right)^{\ell_-}\cdot p_d$$

where $d = \ell_+ - \ell_-$ and the state after this sequence of outcomes will be x_d.

Next we will show a recursion method how to calculate x_d and p_d efficiently. Let

$$p_k := \frac{q_k}{2^k} \quad \text{and} \quad x_k := \frac{y_k}{q_k}. \tag{5}$$

If $k=0$, then $q_0 = p_0 = 1$ and $y_0 = x_0$. Next let us suppose that both q_k and y_k are known. Then

$$p_{k+1} = p_k\cdot\frac{1}{2}(1+cx_k) = \frac{1}{2^{k+1}}\cdot(q_k+cy_k).$$

From definition $p_{k+1} = \frac{q_{k+1}}{2^{k+1}}$, so

$$q_{k+1} = q_k + cy_k. \tag{6}$$

Furthermore,

$$x_{k+1} = \frac{x_k + c}{1 + cx_k} = \frac{\frac{y_k}{q_k} + c}{1 + c\frac{y_k}{q_k}} = \frac{y_k + cq_k}{q_k + cy_k} = \frac{y_k + cq_k}{q_{k+1}}.$$

On the other hand $x_{k+1} = \frac{y_{k+1}}{q_{k+1}}$, therefore

$$y_{k+1} = y_k + cq_k. \tag{7}$$

So if we are interested in the value of x_d as a function of x_0, we can use the following linear recursion:

$$(y_d, q_d) \to (y_d + cq_d, q_d + cy_d) =: (y_{d+1}, q_{d+1}), \tag{8}$$

where $y_0 = x_0$ and $q_0 = 1$. It follows from construction that y_d and q_d are linear in x_0. Moreover if $y_d = a_d + b_d x_0$, then $q_d = b_d + a_d x_0$. Thus

$$x_d = \frac{a_d + b_d x_0}{b_d + a_d x_0} \tag{9}$$

Similarly we get:

$$x_{-d} = \frac{-a_d + b_d x_0}{b_d - a_d x_0} \tag{10}$$

where a_d and b_d are the same as in x_d. They can be calculated using the recursion (8), and they are polynomials of c.

We can conclude from the above properties that if we have the *measurement record*, that is, the observed measurement outcomes (+1s and -1s) and know the *parameter* c then we can calculate the state of the unknown qubit $x(k)$ $(= x_{\ell_+ - \ell_-})$ at any time instance from the initial state x_0. Therefore, ***the only meaningful problem statement is to estimate the initial state*** x_0 ***from the measurement record and from*** c.

C. Standard estimations on the initial value x_0

In the special case $c = 1$ we have:

$$Prob(\pm 1) = \frac{1}{2}(1 \pm x)\ , \quad \text{and} \quad x(1) = \pm 1 \tag{11}$$

so we get the so-called (direct) standard measurement scheme.

If we have multiple copies of identically prepared qubits and on each we perform this measurement then we can easily construct an estimate of the initial state x_0. If we denote the relative frequency of the $+1$ measurements by ν_+, then

$$\tilde{x}_0 = 2\nu_+ - 1 \tag{12}$$

will be an unbiased and efficient estimator [8], with variance

$$Var(\tilde{x}_0) = \frac{1 - x_0^2}{N}, \tag{13}$$

if the number of the used quantum bits is N.

It is easy to see from (4) that this type of measurement would be a totally invasive measurement, i.e. the information about the true state would be lost after doing it, thus we assume from now on that $c < 1$.

Of course, in general case we can also make an estimation only from the first measurement by using multiple copies of qubit pairs. Accordingly to the probability of measuring $+1$ from (4), we can derive estimator:

$$\tilde{x}_0^* = \frac{1}{c}(2\nu_+ - 1) \tag{14}$$

In this case this will be an unbiased and efficient estimator. Its variance will be:

$$Var(\tilde{x}_0^*) = \frac{1}{c^2}\frac{1 - x_0^2}{N}, \tag{15}$$

which is quite good when c is close to 1, but if c is small (close to 0) then its value is much larger than that of the standard estimator (13). Therefore more sophisticated methods are needed for the general case.

To describe the efficiency of the indirect estimation methods we are comparing the mean square error (MSE) of their estimates ($\hat{x}_0$) with the theoretical variance of the direct standard estimation method from (13):

$$\eta = \frac{MSE(\hat{x}_0)}{Var(\tilde{x}_0)}. \tag{16}$$

III. The martingale approach

The process described in (4) is also a martingale, because

$$E(x(k+1)) = x(k).$$

We can make use of this property to make a new kind of estimation method.

A. Theoretical results

We fix the values u, v $(u < x_0 < v)$ and then we start the process from x_0 by performing indirect measurements until we reach either below u or above v. In that case we stop the process and note the final state.

In order to get a valid estimation method we assume that

- $|u| < 1$, $|v| < 1$
- x_0 is in the interval $[u, v]$.

Let assume that we stop the procedure at time instance T, in that case T will be a stopping time, and accordingly to Doob's optional stopping theorem [13]: $E(x(T)) = x_0$

We can simplify the situation if we assume that the final state is exactly u or v, with probabilities p and q $(= 1 - p)$. In that case

$$E(x(T)) = p\ u + (1 - p)\ v = x_0 \tag{17}$$

and one obtains after rearranging:

$$p = \frac{v - x_0}{v - u}$$

We can use (17) for state estimation if we replace the probabilities $(p, 1 - p)$ with relative frequencies after N repeated settings using different unknown qubits. Let us denote $\nu_u = N_u/N, \nu_v = N_v/N$, where N_i $(i \in u, v)$ is the number of experiments with the final state i. Then

$$\hat{x}_0 = \nu_u\ u + \nu_v\ v = \nu_u\ u + (1 - \nu_u)\ v = v + (u - v)\nu_u$$

The variance of this estimate is:

$$Var(\hat{x}_0) = (u-v)^2 \; Var(\nu_u) = (u-v)^2 \; \frac{p(1-p)}{N}$$

because N_u has a binomial distribution with parameters (N, p). After substitution we obtain:

$$Var(\hat{x}_0) = \frac{1}{N}(v - x_0)(x_0 - u)$$

Note if we have $u = -1$ and $v = 1$ then we get an equivalent method to the direct standard method (11)-(13).

B. Stopping relative to the initial state

The above described variance is not in agrement with the known facts, that is because it is not feasible, we do not know neither the state x_k nor x_0, so we can not know when we reach the stopping time. Therefore, an initial state relative method of detecting a stopping time is proposed where the final states are defined using a given distance from the unknown initial state.

Let be $d(k)$ the number of $+1$ measurements minus the number of -1 measurements after the first k measurements. Let us repeat the measurements until $d(k)$ reaches $\pm D$, where D is a given integer, and let be in this case $\tau = k$.

Then τ is a stopping time, and the final state will be x_+ or x_-, accordingly, if $d(\tau)$ is equal to $+D$ or $-D$. We can calculate x_+ and x_- from the properties of the process described in subsection II-B:

$$x_+ = \frac{a_D + b_D x_0}{b_D + a_D x_0}$$

and

$$x_- = \frac{-a_D + b_D x_0}{b_D - a_D x_0}.$$

We can use the results from subsection III-A to estimate the initial state x_0 using the $[x_-, x_+]$ interval. The probability that the process ends up in x_+ is

$$p_+ = \frac{x_0 - x_-}{x_+ - x_-} = \frac{b_D + a_D x_0}{2b_D} = \frac{1}{2}\left(1 + \frac{a_D}{b_D}\; x_0\right), \quad (18)$$

that depends on the ratio of a_D and b_D, so let us use the notation $\gamma_D = \frac{b_D}{a_D}$. From the previous probability we can easily get an estimate on x_0. Denote μ_+ the relative frequency of the x_+ outcome, then

$$\hat{x}_0 = \gamma_D(2\mu_+ - 1) \quad (19)$$

Note that the estimate (19) is unbiased, because (18) is linear in x_0. The variance of (19) is

$$Var(\hat{x}_0) = 4\gamma_D^2 Var(\mu_+) = \frac{\gamma_D^2 - x_0^2}{N} \quad (20)$$

Let us also remark that if k goes to infinity, then $x(k)$ will converge to ± 1, so the process will terminate with probability 1. As D goes to infinity γ_D decreasingly converges to 1, x_+ and x_- will be close to ± 1, and so it will converge to the direct standard estimator.

C. Repeated estimations

At the stopping time the initial state is not completely destroyed, so we can try to get to the initial state again. That means we can run the process further until we reach $d(k) = 0$.

Let us suppose that we are in x_+ and continue the measurements. Once again we can use the results of III-A with initial state x_+ and with interval $[x_0, 1]$.

The probability that we return to x_0 is:

$$p_{return+} = \frac{1 - x_+}{1 - x_0} = \frac{b_D - a_D}{b_D + a_D x_0} = \frac{\frac{b_D}{a_D} - 1}{\frac{b_D}{a_D} + x_0}$$

Similarly the probability that we return to x_0 from x_- is

$$p_{return-} = \frac{1 - x_+}{1 - x_0} = \frac{b_D - a_D}{b_D - a_D x_0} = \frac{\frac{b_D}{a_D} - 1}{\frac{b_D}{a_D} - x_0}$$

So the probability we return to x_0 is

$$p_{return} = p_+ \cdot p_{return+} + (1 - p_+) \cdot p_{return-} = 1 - \frac{1}{\gamma_D}$$

This gives the natural $p_{return} = 1$, if $\gamma_D = \infty$, when the change is infinitesimal. Another example is $p_{return} = 0$, if $\gamma_D = 1$, when we have a von Neumann-measurement as in the direct standard method.

Of course, if the state returns to the initial state we can repeat the whole procedure to extract more information from a quantum bit. The number of returns to the initial state (R) will be geometrical distributed with parameter p_{return}. In the long run the average number of returns will be the mean value of R:

$$E(R) = \frac{1}{1 - p_{return}} = \gamma_D$$

This means that in average γ_D returns can be observed on one unknown qubit. The number of returns are independent for different qubits therefore the asymptotic variance will be:

$$\frac{Var(\hat{x}_0)}{E(R)} = \frac{1}{N}\frac{\gamma_D^2 - x_0^2}{\gamma_D}$$

This will be minimal if $\gamma_D = 1$.

IV. Simulation investigations

The effect of the measurement scheme parameters on the estimation quality is investigated in this section using computer simulations. In particularly the properties of the estimator given in (19) were examined. For each measurement settings we repeated the whole estimation procedure 1000 times and calculated the empirical mean values and empirical mean square errors (MSEs) of estimators, which are presented in the summarizing figures.

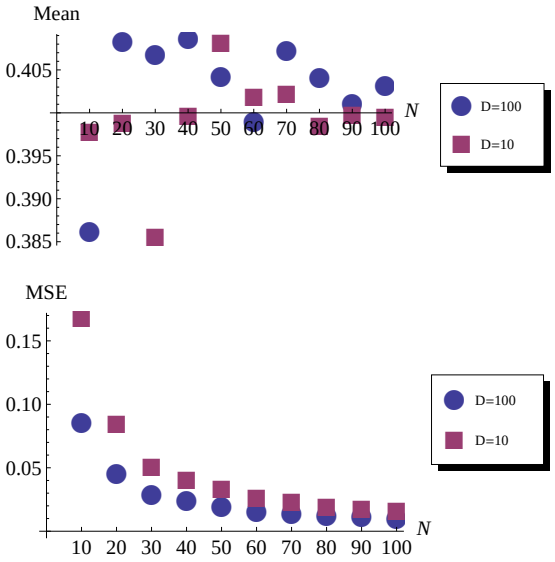

Fig. 1. The mean and the MSE as a function of the number of used qubits

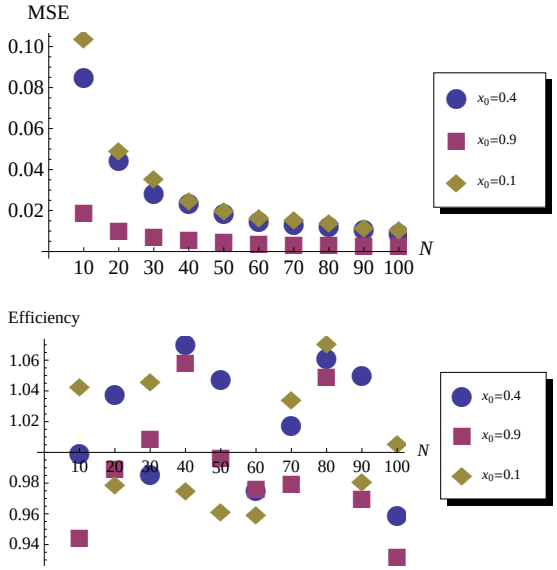

Fig. 2. Comparison of MSE and efficiency for different x_0 values (0.1: diamond, 0.4: circle, 0.9: square) as a function of the used qubits

A. *The effect of the parameter* D

In the following let us assume that $c = 0.1$ and $x_0 = 0.4$. The simulations were performed for two different D values: first for $D = 10$ and then for $D = 100$. For these numbers we can easily calculate the a_D and b_D values, and from that we get $\gamma_{10} \approx 1.3106$ and $\gamma_{100} \approx 1$.

For $D = 10$ we observed that the mean is unbiased, the empirical values are close to the real initial state $x_0 = 0.4$ (upper sub-figure of Fig. 1.). On the lower sub-figure it is seen that the variance converges to the zero. The order of the convergence is $1/N$, where N is the number of used qubits.

For $D = 100$ the mean is also unbiased (Fig. 1.), and the variance is converging to zero with the order of $1/N$, but a smaller coefficient is obtained than previously, which is in good agreement with (20). Note that this case is practically equivalent to the direct standard measurement, since $\gamma_{100} \approx 1$.

B. *The effect of the initial state* x_0

Here the measurement settings have been investigated by using different initial states $x_0 = 0.1; 0.4; 0.9$ but with the same parameter values $c = 0.1$ and $D = 100$. The simulation results are shown in Fig. 2. where the obtained values are denoted by a full diamond for $x_0 = 0.1$, a circle for 0.4, and a square for 0.9.

The upper sub-figure of Fig. 2. shows the variation of the MSE with the number of used qubit pairs. The graphs show a similar shape with an initial state dependent scaling coefficient. It is not surprising because we expect from (20) that the coefficient is $1 - x_0^2$ (because $\gamma_{100} \approx 1$). If we plot the efficiency η from (16), we can see (lower sub-figure of Figure 2) that efficiency is always close to 1, there is no trivial dependence on the value of x_0.

C. *The effect of coupling parameter* c

Here we used different values of parameter c with the same initial state value $x_0 = 0.4$ and $D = 100$. The simulation results are shown in Fig. 3. where the obtained values are denoted by a full circle for $x_0 = 0.1$, a square for 0.2, and a diamond for 0.05. Note that one needs more measurements until the stopping points for $c = 0.05$ compared to the standard $c = 0.1$ case, and one has less with $c = 0.2$.

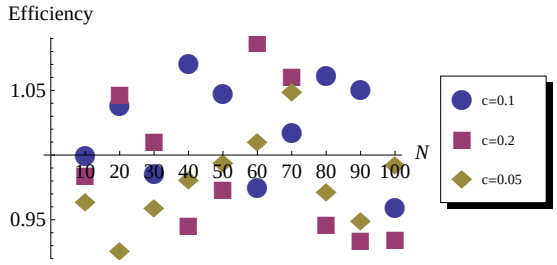

Fig. 3. Comparison of efficiency for different c values (0.05: diamond, 0.1: circle, 0.2: square) as a function of the used qubits

It can be seen from Fig. 3. that the efficiency does not depend strongly on the value of c in the investigated parameter domain. This is a consequence of the fact that the same $x_0 = 0.4$ was used, and the γ_{100} is very close to 1 for these c parameter values with $D = 100$. With further decreased c a greater value of D should be choose to achieve the same efficiency. In conclusion we can say that a greater value of c does not affect very negatively on the quality of the estimate but a smaller number of measurements is needed in this case, which is a very useful property in practical situations.

V. Possible generalizations

Until now the single parameter simple case described in subsection II-A was used to estimate selectively one of the components of the unknown qubit's Bloch vector, similarly to the so called standard measurement scheme [9] for single qubits. A straightforward modification of the measurement setup, that is, the change of the interaction direction and the observable together with the initial state of the measurement qubit, leads to the estimators of the other two Bloch vector components. This way we can get an efficient complete state tomography method for weak measurements.

A. Generalization to n-level quantum systems

Consider a k-level quantum system with density matrix ρ acting on the Hilbert space $\mathcal{H}$. Then the state of n identically prepared quantum systems is described by $\rho_n := \rho^{n\otimes}$ acting on the n-fold tensor product Hilbert space $\mathcal{H}_n$. When $\dim\mathcal{H} = k$, we can identify the operators of $\mathcal{H}_n$ with matrices of $k^n \times k^n$. Since the density matrices are self-adjoint matrices with unit trace, they can be characterized by $m = k^2 - 1$ real parameters forming the parameter vector θ, the generalized Bloch vector.

Denote by E_{ij} the $k \times k$ matrix units and set

$$\begin{array}{rcll} Z_{ii} & := & E_{ii} & (1 \le i < k), \\ X_{ij} & := & E_{ij} + E_{ji} & (i < j), \\ Y_{ij} & := & -\mathrm{i}E_{ij} + \mathrm{i}E_{ji} & (i < j). \end{array}$$

The spectrum of Z_{ii} is $\{0, 1\}$ and the spectrum of X_{ij} and Y_{ij} is $\{-1, 0, 1\}$. These observables can be used to estimate the k^2-1 real parameter of the $k\times k$ density matrix selectively. If each observables are measured r times, then $n = r(k^2 - 1)$ copies of the quantum system are used.

Using the above state parametrization and observables, one can construct estimators that can be applied in the indirect setting similarly to the ones described in the qubit case above.

B. Other measurement schemes

One of the key properties of the repeated weak quantum measurements is that the expectation value of the measurements is equal to the initial state. Therefore the martingale property holds for any other measurement strategy, too. The key element of using the martingale method is the knowledge of "stopping" probability, which is usually a determined function of the initial state: $p = f(x_0)$. From this we can easily construct an estimator: if the relative frequency of stopping in that individual point is ν, then we get

$$\hat{x}_0 = f^{-1}(\nu).$$

It can be calculated for many possible scenarios, for example using continuous measurements, so the basic idea can be quite useful in quantum state estimation theory.

VI. Conclusions and future works

The simplest possible discrete time indirect measurement scheme has been investigated here, where both the unknown and the measurement quantum systems are quantum bits. The measurements applied on the measurement qubit were chosen to be the classical von Neumann measurements using the Pauli matrices as observables. The repeated measurements performed on the measurement subsystem of the composite system were used to construct estimators of the initial state of the unknown system.

An initial state relative method of detecting a stopping time is advised where the final states are defined using a given distance from the unknown initial state. Using martingale property a simple estimator is proposed, which can be an excellent initial point to build more complex and better estimation methods. It has been shown that the proposed estimators are unbiased and almost as efficient as the direct standard measurement.

The effect of the parameters of the estimation scheme on the quality of the estimate has also been investigated by simulation, and turned out that it has the necessary robustness to be a successful candidate for more complex settings. Finally possible generalizations are also mentioned: estimating multiple parameters, higher dimensional systems and different measurement schemes.

The main use of weak measurements is the area of continuous measurements. Although we can achieve this by taking the limit of discrete time models, a thorough analysis of the applicability of martingale method would be recommended in that case too.

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