Characterization of Shift Invariant Subspace of Matrix-valued Hardy Space

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***Abstract*— The characterization of the shift invariant subspace of the matrix-valued Hardy space is given. It is a matrix-valued generalization of the Beurling-Lax theorem. The Beurling-Lax theorem provides the one-sided representation of a shift invariant subspace by a unique inner function. Our characterization of the shift invariant subspace of the matrix-valued Hardy space is given by a two-sided representation of a shift invariant space by inner and co-inner functions.**

I. Introduction

The Beurling-Lax theorem is about the shift invariant subspace of the Hardy space. For a given subspace of the Hardy space, which is shift invariant, there exists an inner function to characterize the subspace [6].

Theorem 1.1: For a given shift invariant subspace $\mathcal{M}$ of the Hardy space on the unit disc $H_2(\mathbb{D})$, there exists a unique inner function $\theta(z)$ such that

$$\mathcal{M} = \theta(z)H_2(\mathbb{D}).$$

This theorem finds many applications to mathematical systems theory, e.g., the deterministic and stochastic realizations [2], [8], [19], [13], [24], the prediction theory [16], [21], [18], the positive real and bounded real interpolation [31], [22], [20], [7], [4], [9], the commutant lifting theorem [3], [32], the Nehari theory [25] and the Hankel operator [1], [15], [27], and model matching theory with the applications to control [11], [14], [10]. For these applications, the co-shift invariant space, which is the orthogonal complement of the shift invariant subspace, plays important roles [30], [33], [26].

This theorem is generalized to the vector-valued Hardy space [23] and to the operator setting with the elegant proof by wandering subspace [17].

Theorem 1.2: For a given shift invariant subspace $\mathcal{M}$ of the n-dimensional vector-valued Hardy space on the unit disc $H_2^n(\mathbb{D})$, there exists a unique inner function $\Theta(z)$ such that

$$\mathcal{M} = \Theta(z)H_2^n(\mathbb{D}).$$

A generalization for the indefinite metric space is also given in [5].

In this paper, we give the characterization of the shift invariant subspace of the matrix-valued Hardy space. It is a matrix-valued generalization of the Beurling-Lax theorem. Our characterization is given by a two-sided representation of a shift invariant space by inner and co-inner functions. The inner function to characterize the shift invariant space of the Beurling-Lax theorem is unique [29]. However, this is not true for our characterization of the shift invariant subspace, which is due to the property of matrix.

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Notations

Complex numbers are represented by $\mathbb{C}$. Denote by $\mathbb{C}^{j\times k}$ $j \times k$ complex matrices. $I_{m\times m}$ denotes $m \times m$ identity matrix. Denote by $\mathrm{tr}\,A$ the trace of a matrix A and by A^* the complex conjugate transpose of a matrix A. $\rho(A)$ is the spectrum of a matrix A. Denote by $\bar{\sigma}(A)$ the maximum singular value of a matrix A. The Frobenius norm of the vector space $\mathbb{C}^{j\times k}$ is defined by

$$||A||_F := \sqrt{\mathrm{tr}\,A^*A}.$$

for $A \in \mathbb{C}^{j\times k}$. Denote by $X_1 \oplus X_2$ the orthogonal direct sum of subspaces X_1 and X_2. $X_1 \ominus X_2$ is the orthogonal complement of X_2 in X_1, where X_2 is a subspace of a Hilbert space X_1.

Let $H_2(\mathbb{D})$ be the Hardy space of square integrable analytic functions in the unit disc $\mathbb{D}$ [12]. Then, $H_2^{m\times n}(\mathbb{D})$ are $m \times n$ matrix-valued functions whose entries belong to $H_2(\mathbb{D})$. The inner product of $H_2^{m\times n}(\mathbb{D})$ is defined by

$$\langle g, h\rangle := \mathrm{tr}\,\frac{1}{2\pi}\int_0^{2\pi} g(e^{i\theta})^* h(e^{i\theta})d\theta.$$

$f(z) \in H_2^{m\times n}(\mathbb{D})$ is represented as the expansion

$$f(e^{i\theta}) = \sum_{k=0}^{\infty} f_k e^{ik\theta}$$

with $f_k \in \mathbb{C}^{m\times n}$, and the norm is defined by

$$\begin{aligned} ||f||_2 \quad &:= \quad \sqrt{\mathrm{tr}\,\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})^* f(e^{i\theta})d\theta} \\ &= \quad \sqrt{\sum_{k=0}^{\infty} ||f_k||_F^2} < \infty. \end{aligned}$$

Denote by $L_2(\mathbb{T})$ the Lebesgue space of square integrable functions on the unit circle $\mathbb{T}$ and by $L_\infty(\mathbb{T})$ the Lebesgue space of essentially bounded functions on $\mathbb{T}$, of which vector-valued and matrix-valued Hardy spaces are similarly defined. The orthognal complement of $H_2(\mathbb{D})$ in $L_2(\mathbb{T})$ is defined by $H_2^\perp(\mathbb{D}) := L_2(\mathbb{T}) \ominus H_2(\mathbb{D})$.

Let $H_\infty(\mathbb{D})$ be the Hardy space of bounded analytic functions in $\mathbb{D}$. Then, $H_\infty^{m\times n}(\mathbb{D})$ are $m \times n$ matrix-valued functions whose entries belong to $H_\infty(\mathbb{D})$. The H_∞ norm of $g(z)$ is defined by

$$||g||_\infty := \mathrm{ess}\sup_{\theta\in[0,2\pi]} \bar{\sigma}(g(e^{i\theta})).$$

ISBN 978-963-311-370-7

The strictly bounded real function is the function $g \in H_\infty^{m\times p}(\mathbb{D})$ such that

$$||g||_\infty < 1.$$

A function $f \in H_\infty^{m\times \ell}(\mathbb{D})$ with $m \geq \ell$ is called inner if

$$f(e^{j\theta})^* f(e^{j\theta}) = I_\ell.$$

Similarly, a function $f \in H_\infty^{r\times n}(\mathbb{D})$ with $r \leq n$ is called co-inner if

$$f(e^{j\theta}) f(e^{j\theta})^* = I_r.$$

The projection operator from H_2 to a subspace A is defined by

$$P^A \quad : \quad H_2 \longrightarrow A$$

Let S be the shift operator on $H_2^{m\times n}(\mathbb{D})$, which is defined by

$$(Sf)(z) := zf(z). \tag{1}$$

Denote by U_Ξ a multiplication operator with the symbol $\Xi \in H_\infty^{m\times n}(\mathbb{D})$ on $H_2^{m\times n}(\mathbb{D})$, which is defined by

$$(U_\Xi f)(z) := \Xi(z) f(z) \tag{2}$$

for $f \in H_2^{m\times n}(\mathbb{D})$. The Hankel operator with the symbol $\Phi \in L_\infty^{m\times n}(\mathbb{T})$

$$\Gamma_\Phi \quad : \quad H_2 \longrightarrow H_2^\perp$$

is defined by

$$\Gamma_\Phi := P^{H_2^\perp}\Phi.$$

II. Main Result

We state the characterization of the shift invariant subspace of the matrix-valued Hardy space.

Theorem 2.1: Let $\mathcal{M}$ be a shift invariant subspace of $H_2^{m\times n}(\mathbb{D})$ with $m \geq n$. Then, there are an ℓ of $0 \leq \ell \leq n$ for an inner function $\Theta_\ell(z) \in H_\infty^{m\times \ell}(\mathbb{D})$ and an r of $0 \leq r \leq m$ for a co-inner function $\Theta_r(z) \in H_\infty^{r\times n}(\mathbb{D})$ such that

$$\mathcal{M} = \Theta_\ell(z) H_2^{\ell\times r}(\mathbb{D}) \Theta_r(z). \tag{3}$$

Conversely, the subspace, defined by (3), is shift invariant.

Proof: We prove the sufficiency. Clearly,

$$\begin{aligned} S\mathcal{M} &= z\Theta_\ell(z) H_2^{\ell\times r}(\mathbb{D})\Theta_r(z) \\ &= \Theta_\ell(z) z H_2^{\ell\times r}(\mathbb{D})\Theta_r(z) \\ &\subseteq \mathcal{M}. \end{aligned}$$

We prove the necessity. Let us define

$$\mathcal{V} := \mathcal{M} \ominus S\mathcal{M}. \tag{4}$$

Since $\mathcal{V} \subseteq \mathcal{M} \subseteq H_2^{m\times n}(\mathbb{D})$, the row size of $\mathcal{V}$ is at most m and the column size of $\mathcal{V}$ is at most n. Denote by ℓ with $0 \leq \ell \leq m$ the row size and by r with $0 \leq r \leq n$ the column size, respectively. By the Wold decomposition [28],

$$\mathcal{M} \quad = \quad \mathcal{V} \oplus S\mathcal{V} \oplus S^2\mathcal{V} \oplus \cdots . \tag{5}$$

Any function $f \in \mathcal{V}$ has a constant norm a.e., since $f \perp S^k f$, for $k \geq 1$, so that

$$\begin{aligned} &\frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} ||f(e^{i\theta})||_F^2 d\theta \\ &= \operatorname{tr} \frac{1}{2\pi}\int_0^{2\pi} e^{ik\theta} f(e^{i\theta})^* f(e^{i\theta}) d\theta \\ &= \langle f, S^k f\rangle \\ &= 0, \end{aligned}$$

and, by conjugation, we see that all the non-zero Fourier coefficients of $||f(e^{i\theta})||_F^2$ are zero.

We construct the isomorphism between $\mathcal{V}$ and $\mathbb{C}^{\ell\times r}$ by two steps. Let

$$i_\ell : \mathbb{C}^{\ell\times r} \longrightarrow \mathcal{V} \tag{6}$$

be an isometric embedding, and for $z \in \mathbb{D}$ define $\Theta_\ell(z) \in \mathbb{C}^{m\times \ell}$ by

$$\Theta_\ell(z) A = i_\ell(A)(z), \quad \forall A \in \mathbb{C}^{\ell\times r}.$$

Any function $f \in H_2^{\ell\times r}(\mathbb{D})$ can be written as

$$f(z) = \sum_{k=0}^{\infty} f_k z^k, \tag{7}$$

where $f_k \in \mathbb{C}^{\ell\times r}$ and $\sum_{k=0}^\infty ||f_k||^2 < \infty$. We see that $\Theta_\ell f$ is the same, pointwise, as $\sum_{k=0}^\infty S^k i_\ell(f_k)$. Moreover,

$$\begin{aligned} ||f||_2^2 &= \sum_{k=0}^{\infty} ||f_k||^2 \\ &= \sum_{k=0}^{\infty} ||S^k i_\ell(f_k)||^2 \\ &= ||\Theta_\ell f||_2^2 \end{aligned} \tag{8}$$

due to $i_\ell(f_k) \subset \mathcal{V}$ and (5). The identity (8) holds for all $f \in H_2^{\ell\times r}(\mathbb{D})$, which implies that $\Theta_\ell(z)$ is inner.

Let

$$i_r : \Theta_\ell \mathbb{C}^{\ell\times r} \longrightarrow \mathcal{V}$$

be an isometric isomorphism, and for $z \in \mathbb{D}$ define $\Theta_r(z) \in \mathbb{C}^{r\times n}$ by

$$\Theta_\ell(z) A \Theta_r(z) = i_r(A)(z), \quad \forall A \in \mathbb{C}^{\ell\times r}.$$

The size of Θ_r is $r \times n$ with $r \leq n$ since the size of $\Theta_\ell A$ is $m \times r$ with $m \geq r$ and $\mathcal{V} \subseteq \mathcal{M} \subseteq H_2^{m\times n}(\mathbb{D})$ with $m \geq n$. For $f \in H_2^{\ell\times r}(\mathbb{D})$ of (7), we see that $\Theta_\ell f \Theta_r$ is the same, pointwise, as $\sum_{k=0}^\infty S^k i_r(\Theta_\ell f_k)$. Moreover, since

$$\begin{aligned} ||f||_2^2 &= \sum_{k=0}^{\infty} ||f_k||_F^2 \\ &= \sum_{k=0}^{\infty} ||S^k i_r(\Theta_\ell f_k)||_F^2 \\ &= ||\Theta_\ell f \Theta_r||_2^2 \end{aligned}$$

by (8), we conclude that $\Theta_r(z)$ is co-inner since $\Theta_\ell(z)$ is inner. ■

The inner function and co-inner function to characterize the shift invariant subspace (3) are not unique, while the inner function is unique for the one-sided characterization of the shift invariant subspace [29]. This is also true for our case if the index r of (6) is equal to n in $Theorem$ 2.1.

Corollary 2.2: If the index r of (6) is equal to n, then, there is a unique inner function $\Theta_L(z) \in H_\infty^{m\times\ell}(\mathbb{D})$ such that

$$\mathcal{M} = \Theta_L(z)H_2^{\ell\times n}(\mathbb{D}). \quad (9)$$

We mention that the co-shift invariant subspaces derived by (3) and (9) are equivalent if the index r of (6) is equal to n.

Corollary 2.3: Suppose that the index r is equal to n. Then, the shift invariant subspace is characterized by

$$\mathcal{M} = \Theta_\ell(z)H_2^{\ell\times n}(\mathbb{D})\Theta_r(z) \quad (10)$$

and

$$\mathcal{M} = \Theta_L(z)H_2^{\ell\times n}(\mathbb{D}). \quad (11)$$

Let us define

$$H(\Theta_\ell,\Theta_r) := H_2^{m\times n}(\mathbb{D}) \ominus \Theta_\ell(z)H_2^{\ell\times n}(\mathbb{D})\Theta_r(z)$$

$$H(\Theta_L) := H_2^{m\times n}(\mathbb{D}) \ominus \Theta_L(z)H_2^{\ell\times n}(\mathbb{D}).$$

Then,

$$H(\Theta_\ell,\Theta_r) \simeq H(\Theta_L).$$

III. APPLICATIONS

A. Commutant Lifting Theorem

We state the application in the context of the commutant lifting theorem [31], [32]. We assume that all matrix functions below are square for the simplicity. For given inner function $\Theta_\ell(z)$ and co-inner function $\Theta_r(z)$, let us define the co-shift invariant subspace

$$\mathcal{K} := H_2^{n\times n}(\mathbb{D}) \ominus \Theta_\ell(z)H_2^{n\times n}(\mathbb{D})\Theta_r(z)$$

and the compressed shift

$$T := P^{\mathcal{K}}S\mid_{\mathcal{K}}.$$

Theorem 3.1: Let A be a contractive linear operator on $\mathcal{K}$, i.e., $||A|| < 1$. Suppose that A satisfies

$$AT = TA.$$

Then, there exists an operator B on $H_2^{n\times n}(\mathbb{D})$ such that

$$A = P^{\mathcal{K}}B|_{\mathcal{K}}$$

and

$$||A|| = ||B||.$$

Moreover, $B \in H_\infty^{n\times n}(\mathbb{D})$ and A is the compression of B.

Proof: A similar proof of the one-sided case is found in [26]. ■

B. Bounded Real Two-sided Residue Interpolation and Two-sided Nehari Problem

It is well known that the commutant lifting theorem in $Theorem$ 3.1 is the generalization of the bounded real interpolation theory. We state a bound real two-sided residue interpolation, for which the two-sided form of the shift invariant subspace naturally appears.

Given matrices $A_\ell \in \mathbb{C}^{n_\ell\times n_\ell}$, $X, Y \in \mathbb{C}^{n\times n_\ell}$ where (A_ℓ, X^*) is a reachable pair and $\rho(A_\ell) \subset \mathbb{D}$, and $A_r \in \mathbb{C}^{n_r\times n_r}$, $U, V \in \mathbb{C}^{n\times n_r}$, where (U, A_r) is an observable pair and $\rho(A_r) \subset \mathbb{D}$. Assume that $\rho(A_\ell) \cap \rho(A_r) = \phi$. The two-sided residue interpolation conditions on a strictly bounded real function $f(z) = \sum_{k=0}^\infty f_k z^k$ are given by

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{T}}(zI - A_\ell)^{-1}X^*f(z)dz &= Y^* \\ \frac{1}{2\pi i}\int_{\mathbb{T}} f(z)U(zI - A_r)^{-1}dz &= V. \end{aligned} \quad (12)$$

These interpolation conditions are equivalent to the conditions that the Fourier coefficients of $f(e^{i\theta})$ satisfy

$$\begin{aligned} \sum_{k=0}^\infty A_\ell^k X^* f_k &= Y^* \\ \sum_{k=0}^\infty f_k U A_r^k &= V. \end{aligned}$$

The bounded real two-sided residue interpolation problem is formulated as follows.

$Bounded\ Real\ Two\text{-}sided\ Residue\ Interpolation\ Problem$:

Parameterize bounded real functions satisfying the interpolation conditions (12) if it is solvable.

The bounded real two-sided residue interpolation problem has a natural connection to the two-sided Nehari problem. The inner function $\Theta_\ell(z)$ is associated with the left residue interpolation condition. The state-space realization of $\Theta_\ell(z)$ is given by

$$\Theta_\ell(z) = D_\ell + zX(I - zA_\ell^*)^{-1}B_\ell, \quad (13)$$

for which

$$\frac{1}{2\pi i}\int_{\mathbb{T}}(zI - A_\ell)^{-1}X^*\Theta_\ell(z)dz = 0 \quad (14)$$

holds and the matrices B_ℓ and D_ℓ are given such that $\Theta_\ell(z)$ is inner. Similarly, $\Theta_r(z)$ is associated with the right residue interpolation condition. The state-space realization of $\Theta_r(z)$ is given by

$$\Theta_r(z) = D_r + zC_r(I - zA_r^*)^{-1}U^*, \quad (15)$$

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\Theta_r(z)U(zI - A_r)^{-1}dz = 0 \quad (16)$$

holds and the matrices C_r and D_r are given such that $\Theta_r(z)$ is co-inner. The bounded real two-sided residue interpolation problem has a connection to the two-sided Nehari problem [4].

Theorem 3.2: For a solution $f(z)$ to the bounded real two-sided residue interpolation problem, there exists $R(z) \in H_\infty^{n\times n}(\mathbb{D})$ and $Q(z) \in H_\infty^{n\times n}(\mathbb{D})$ such that

$$f(z) = R(z) - \Theta_\ell(z)Q(z)\Theta_r(z). \tag{17}$$

Proof: See [4]. ■

The identities (17), (16) and (14) imply

$$\frac{1}{2\pi i}\int_{\mathbb{T}}[f(z) - R(z)]U(zI - A_r)^{-1}dz = 0$$
$$\frac{1}{2\pi i}\int_{\mathbb{T}}(zI - A_\ell)^{-1}X^*[f(z) - R(z)]dz = 0.$$

The equivalent formulation of the bounded real two-sided residue interpolation problem is given by the two-sided Nehari problem.

$Two\text{-}sided\ Nehari\ Problem:$

For a given $R(z)$, which satisfies

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}}(zI - A_\ell)^{-1}X^*R(z)dz &= Y^* \\ \frac{1}{2\pi i}\int_{\mathbb{T}}R(z)U(zI - A_r)^{-1}dz &= V,\end{aligned} \tag{18}$$

parameterize $Q(z)$ such that (17) is strictly bounded real.

We want to mention the connection to the commutant lifting theorem by $Theorem$ 3.1. The operator B corresponds to (17), and the operator A is the compression of $f(z)$, i.e., $A = R(T)$. An advantage of the commutant lifting theorem is that it deals with the non-square case, i.e., $\Theta_\ell(z)$ is tall and $\Theta_r(z)$ is flat.

C. Hankel Operator and Minimal Distance Problem

The two-sided Nehari formulation (17) has connections to the commutant lifting theorem, the Hankel operator and a minimal distance problem. The minimal distance problem gives the norm of the compression of $R(z)$.

Theorem 3.3: For given $R(z)$, $\Theta_\ell(z)$ and $\Theta_r(z)$,

$$\inf_{Q\in H_\infty^{n\times n}} ||R(z) - \Theta_\ell(z)Q(z)\Theta_r(z)||_\infty = ||P^{\mathcal{K}}U_R||. \tag{19}$$

Proof: A similar proof of the one-sided case is found in [26]. ■

The theorem is stated in terms of the Hankel operator.

Theorem 3.4: For given $R(z)$, $\Theta_\ell(z)$ and $\Theta_r(z)$, let us define

$$\Phi(z) := \Theta_\ell(z)^*R(z)\Theta_r(z)^*. \tag{20}$$

Then,

$$\inf_{Q\in H_\infty^{n\times n}} ||\Phi(z) - Q(z)||_\infty = ||\Gamma_\Phi||. \tag{21}$$

Proof: A similar proof of the one-sided case is found in [26]. ■

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