Scalable decentralized control and the Davis-Wielandt shell

Ioannis Lestas

Department of Engineering
University of Cambridge,
Cambridge CB2 1PZ, UK
icl20@cam.ac.uk

Abstract— We consider a large scale network comprised of heterogeneous dynamical agents. We derive scalable stability certificates that involve the input/output properties of individual subsystems and corresponding properties of the interconnection matrix. The stability conditions presented are based on the Davis-Wielandt shell, a higher dimensional version of the numerical range, which allows to relax normality or symmetry assumptions on the interconnection matrix. The conditions derived include small gain and passivity approaches as special cases, and generalize many results within the areas of consensus protocols and Internet congestion control.

I. Introduction

We consider a large scale interconnection of heterogeneous dynamical agents. The paper addresses the problem of deriving conditions on the input/output properties of individual subsystems and the interconnection matrix such that the interconnection is guaranteed to be stable. This is a problem that has received considerable attention by the control community from an early stage (e.g. the dissipativity approaches in [1], [2]) with a renewed interest in recent years due to the significance in many important applications such as data network protocols and group coordination problems [3], [4], [5], [6], [7], [8].

We derive in the paper stability conditions that follow from $\mathcal{L}_2$ graph separation arguments, a relaxation that can be used if additional homotopy arguments are employed, as was shown in [9]. In particular, the graph separation conditions considered are shown to be equivalent to corresponding Davis-Wielandt shells being disjoint. The latter is a higher dimensional generalization of the numerical range with important convexity properties that are of significance in its characterization. This higher dimensional convexification allows to relax normality and symmetry assumptions on the interconnection matrix and also includes known approaches (such as ones relying on passivity or small gain) as part of a common generalized framework.

II. Preliminaries

A. Notation

Real/complex numbers are denoted by $\mathbb{R}/\mathbb{C}$ respectively, $\mathbb{R}_+$ denotes the nonnegative reals, and for $x \in \mathbb{C}^n$, its Euclidean norm is denoted by $\|x\|$. For a matrix $M \in \mathbb{C}^{n \times m}$ its conjugate transpose is denoted by M^*, and its Moore-Penrose pseudoinverse by M^+. For $M \in \mathbb{C}^{n \times n}$ it spectrum is denoted by $\sigma(M)$, and $I_n \in \mathbb{R}^{n \times n}$ denotes the $n \times n$ identity matrix. $\mathcal{L}_2^l[0, \infty)$ is the Hilbert space of functions $f : [0, \infty) \to \mathbb{R}^l$ with finite energy $\|f\|^2 = \int_0^\infty f^*(t)f(t)dt$. This is a subspace of $\mathcal{L}_{2e}^l[0, \infty)$ whose elements need to be square integrable on finite intervals. The fourier transform of $f \in \mathcal{L}_2^l[0, \infty)$ is denoted by $\hat{f}(j\omega) = \int_0^\infty e^{-j\omega t} f(t)dt$. The Kronecker product is denoted by $\otimes$, and for $\Pi : \mathbb{C} \to \mathbb{C}^{n \times m}$, $A \in \mathbb{C}^{p \times q}$ we denote $\Pi \otimes A$ the function $\Pi \otimes A : \mathbb{C} \to \mathbb{C}^{np \times mq}$, $(\Pi \otimes A)(s) = \Pi(s) \otimes A$. The direct sum of operators Δ_i $i = 1, \ldots n$ is denoted by $\oplus_{i=1}^n \Delta_i$, and for a set A in a real vector space, $Co(A)$ denotes its convex hull.

We say that convex sets $A, B \subset \mathbb{R}^n$ are ϵ-*strictly separated* for some $\epsilon > 0$ if there exists a $v \in \mathbb{R}^n$ and $a \in \mathbb{R}$ such that $x^*v \geq \epsilon + a$ $\forall v \in A$ and $x^*v \leq a$ $\forall x \in B$. We also say in this case that the hyperplane $\{x : x \in \mathbb{R}^n, x^*v = a\}$ strictly separates A and B. It is known from the *separating hyperplane theorem* that if $A, B \subset \mathbb{R}^n$ are non-empty, closed, convex sets with at least one compact, then A, B being disjoint implies that they are strictly separated by a hyperplane (e.g. [10]).

We will be using the algebra of transfer functions introduced in [11] (this has properties analogous to those of proper real rational transfer functions). $\mathcal{A}$ denotes the class of transfer functions obtained as the Laplace transform of impulse response functions $h = h_a + \sum_{i=0}^\infty k_i \delta_{t_i}$, where $h_a : \mathbb{R} \to \mathbb{C}$ satisfies $\int_0^\infty |h_a(t)|dt < \infty$ and $h_a(t) = 0$ for $t \leq 0$, $t_0 = 0$, $t_i > 0$ for $i > 0$, $k_i \in \mathbb{C}$ s.t. $\sum_{i=0}^\infty |k_i| < 0$ and δ_{t_i} denotes the t_i shifted Dirac delta distribution.

B. Quadratic graph separation

We will be considering an IQC framework for verifying stability of the interconnection [9]. This requires that the graphs of the interconnected operators be separated in $\mathcal{L}_2[0, \infty)$, with stability guaranteed by means of an additional homotopy argument that ensures there is no transition from stability to instability.

For an operator $\Delta : \mathcal{L}_{2e}^m[0, \infty) \to \mathcal{L}_{2e}^l[0, \infty)$ its graph is defined as $\mathcal{G}_\Delta := \{(\Delta(v), v) : v \in \mathcal{L}_2^l[0, \infty), \Delta(v) \in \mathcal{L}_2^m[0, \infty)\}$ and its inverse graph as $\overline{\mathcal{G}}_\Delta := \{(v, \Delta(v)) : v \in \mathcal{L}_2^l[0, \infty), \Delta(v) \in \mathcal{L}_2^m[0, \infty)\}$.

We will be using quadratic functionals of the form

$$\sigma_\Pi(h) := \int_{-\infty}^\infty \hat{h}(j\omega)^* \Pi(j\omega) \hat{h}(j\omega) d\omega \qquad (1)$$

where $\Pi : j\mathbb{R} \to \mathbb{C}^n$ is a Hermitian valued function.

Definition 1 (quadratic graph separation): We say that the graphs of operators $G : \mathcal{L}_{2e}^l[0,\infty) \to \mathcal{L}_{2e}^m[0,\infty)$ and $\Delta : \mathcal{L}_{2e}^m[0,\infty) \to \mathcal{L}_{2e}^l[0,\infty)$ are ϵ-quadratically separated by the functional $\sigma_\Pi : \mathcal{L}_2^{l+m}[0,\infty) \to \mathbb{R}$ if $\epsilon > 0$ and

$$\sigma_\Pi(g) \leq -\epsilon\|g\|^2 \qquad \forall g \in \mathcal{G}_G$$
$$\sigma_\Pi(h) \geq 0 \qquad \forall h \in \overline{\mathcal{G}}_\Delta$$

C. The Davis-Wielandt shell

Let $B(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$ with inner product $\langle g, h \rangle$ for $g, h \in \mathcal{H}$. The *numerical range* of $A \in B(\mathcal{H})$ is defined by $W(A) := \{\langle Ax, x \rangle : x \in \mathcal{H}, \langle x, x \rangle = 1\}$.

The *Davis-Wielandt shell* [12], [13] of $A \in B(\mathcal{H})$ is a higher dimensional generalization of the numerical range which is defined by

$$DW(A) := \big\{(\Re\langle Ax, x\rangle, \Im\langle Ax, x\rangle, \langle Ax, Ax\rangle) :$$
$$x \in \mathcal{H}, \langle x, x\rangle = 1\big\}$$

The shell captures more information about the operator relative to the numerical range as indicated in the Lemma below.

Lemma 1: Let $A \in B(\mathcal{H})$ with spectrum $\sigma(A)$, and $\dim(\mathcal{H})$ finite. Then $DW(A) = Co\{(\Re(\lambda), \Im(\lambda), |\lambda|^2) : \lambda \in \sigma(A)\}$ iff A is normal.
It should be noted that if the sets in the Lemma are projected to their first two coordinates (i.e. the numerical range was used instead of the Davis-Wielandt shell), the 'only if' part would not be true.

It is known from the Toeplitz-Hausdorff theorem that the numerical range of $A \in B(\mathcal{H})$ is convex. This is, however, not always the case for the Davis-Wielandt shell. The convexity properties of $DW(A)$ depend on $n = \dim(\mathcal{H})$ and can be deduced from corresponding results on the joint range of m-tuples of Hermitian forms.

Lemma 2: Let $A \in B(\mathcal{H})$. Then $DW(A)$ is convex if $\dim(\mathcal{H}) \neq 2$ or A is normal, and an ellipsoid if $\dim(\mathcal{H}) = 2$.

The Davis-Wielandt shell is therefore either a convex set or encloses a convex set. It can hence be efficiently characterized by means of techniques analogous to the ones used for the numerical range [14]. That is, points/support hyperplanes can easily be generated on its boundary, and these can then be used to derive lower/upper bounds for the shell with arbitrarily high precision.

For a matrix $A \in \mathbb{C}^{n \times n}$ with Moore-Penrose pseudoinverse A^+ we denote the shell of A^+ restricted to range(A) as $\tilde{DW}(A^+) := \{(\Re(x^*A^+x), \Im(x^*A^+x), \|A^+x\|^2) : x \in \mathbb{C}^n, \|x\| = 1, x \in \text{range}(A)\}$ and define

$$\overline{DW}(A) := \begin{cases} \tilde{DW}(A^+) & \text{if rank}(A) = n \\ \tilde{DW}(A^+) + \{0, 0, \mathbb{R}_+\} & \text{if rank}(A) < n \end{cases}$$

i.e. $\overline{DW}(A) = DW(A^{-1})$ if A is invertible. If A is normal then $\tilde{DW}(A^+)$ simplifies in an analogous way to $DW(A)$ as shown in the Lemma below.

Lemma 3: Let $A \in \mathbb{C}^{n \times n}$ be normal. Then $\tilde{DW}(A^+) = Co\left\{\left(\Re\frac{1}{\lambda}, \Im\frac{1}{\lambda}, \frac{1}{|\lambda|^2}\right) : \lambda \neq 0, \lambda \in \sigma(A)\right\}$.

III. Main results

We consider the interconnection

$$v = (\oplus_{i=1}^n g_i)(w) + f$$
$$w = A(v) + e \qquad (2)$$

where $f, e \in \mathcal{L}_{2e}^n[0,\infty)$ represent disturbances, $\oplus_{i=1}^n g_i$ the direct sum of linear operators g_i with transfer functions $\hat{g}_i \in \mathcal{A}$ and linear operator A has transfer function $\hat{A} \in \mathcal{A}^{n \times n}$.

This can be seen as a network comprised of heterogeneous agents with dynamics g_i, and with the interconnections between them determined by means of the operator A. The aim is to determine conditions on the input-output properties of the individual operators g_i and the interconnection operator A such the network is stable.

It is shown below that such conditions on g_i and A can be derived by means of the Davis-Wielandt shell. These lead to a quadratic separation of the graphs of G and A (Theorem 1) and stability can then be guaranteed by employing appropriate homotopy arguments.

Theorem 1: The following are equivalent for operators g_i with transfer functions $\hat{g}_i \in \mathcal{A}$ for $i = 1, \ldots, n$ and an operator A with transfer function $\hat{A} \in \mathcal{A}^{n \times n}$:

(i) $\exists$ functional $\sigma_\Pi : \mathcal{L}_2^2[0,\infty) \to \mathbb{R}$ as defined in (1) and $\epsilon > 0$ s.t.

$$\sigma_\Pi(h) \leq -\epsilon\|h\|^2 \quad \forall h \in \mathcal{G}_{g_i}, \forall i = 1, \ldots, n \quad \text{and}$$
$$\sigma_{\Pi \otimes I_n}(k) \geq 0 \qquad \forall k \in \overline{\mathcal{G}}_A.$$

(ii) $\exists$ functional $\sigma_{\Pi \otimes I_n} : \mathcal{L}_2^{2n}[0,\infty) \to \mathbb{R}$ as defined in (1) and $\epsilon > 0$ s.t. the graphs of $G = \oplus_{i=1}^n g_i$ and A are ϵ-quadratically separated by $\sigma_{\Pi \otimes I_n}$.

(iii) $\exists \epsilon > 0$ s.t. for almost all $\omega \in \mathbb{R}_+$

$$Co\big(\cup_i DW(\hat{g}_i(j\omega))\big) \text{ and } Co\big(\overline{DW}(\hat{A}(j\omega))\big)$$

are ϵ-strictly separated.

(iv) $\exists \epsilon > 0$ s.t. for almost all $\omega \in \mathbb{R}_+$

$$DW(\oplus_{i=1}^n \hat{g}_i(j\omega))) \text{ and } Co\big(\overline{DW}(\hat{A}(j\omega))\big)$$

are ϵ-strictly separated.

(v) $\exists \epsilon > 0$ s.t. for almost all $\omega \in \mathbb{R}_+$

$$\overline{DW}(\oplus_{i=1}^n \hat{g}_i(j\omega))) \text{ and } Co\big(DW(\hat{A}(j\omega))\big)$$

are ϵ-strictly separated.

Quadratic separation of the graphs of operators $\oplus_{i=1}^n g_i$ and A is not sufficient to prove stability of the interconnection in (2), but it does ensure that the network remains stable as the dynamics are continuously perturbed from a stable interconnection. Stability can therefore be deduced if such a homotopy argument is employed. Such a homotopy associated with the interconnection matrix is used in the theorem below. Analogous continuity arguments associated with the system dynamics can also be employed to deduce stability (included in an extended version of the paper).

Theorem 2: The interconnection in (2) is stable if $\exists \epsilon > 0$ s.t. for all $\tau \in [0, 1]$ and all $\omega \in \mathbb{R}_+$

$$Co\big(\cup_i DW(\hat{g}_i(j\omega))\big) \text{ and } Co\big(\overline{DW}(\tau\hat{A}(j\omega))\big) \quad (3)$$

are ϵ-strictly separated.

Remark 1: In the case $\hat{A}(j\omega)$ is normal then the stability condition involves only the spectrum of $\hat{A}(j\omega)$ (Lemmas 1, 3). This recovers the conditions in [7], [5] and also earlier results on Internet congestion control [15], [16].

It should be noted that the conditions derived involve only individual subsystems. Connections with conditions involving also neighbouring dyanmics as in [4], [17] is part of ongoing work.

IV. CONCLUSIONS

Stability conditions for large scale networks have been derived that involve the input/ouptut properties of individual subsystems and those of the interconnection matrix. These are based on the Davis-Wielandt shell, a higher dimensional generalization of the numerical range with important convexity properties, which has been shown to be related to corresponding graph separation arguments.

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