Finite Time System Operator and Balancing for Model Reduction and Decoupling

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***Abstract*— In this paper we explore the operator mapping a finite time segment of the input signal to the output over the same interval. The properties of this operator are compared to the finite time Hankel operator that was useful in sliding interval balancing (SIB). Potential applications for model reduction and decoupling of systems are discussed.**

I. Introduction

The past decades have seen an explosive growth of research on model reduction for LTI systems involving SVD-based and Krylov-based approaches. See for instance [1]. One interesting approach is the model reduction based on balanced truncation, first developed by Moore [2]. Extensions for time varying systems have been obtained [3], and one of them, sliding interval balancing (SIB) led to an approach for model reduction of nonlinear systems [4]. However, SIB is a topic that still generates interesting discoveries in the LTI case. This paper explores some of these properties related to the LTI case. These are of interest in order to better understand their application towards nonlinear balancing and model reduction.

Thus let us consider the single -input, single-output (SISO) LTI system of dimension n

$$\dot{x} = Ax + Bu \quad (1)$$

$$y = Cx \quad (2)$$

In order to avoid complications from the beginning, we shall assume that the realization (A, B, C) is minimal, hence reachable and observable, and its transfer function, $G(s)$, exists as an irreducible rational function, with denominator degree equal to n. The impulse response of the system is

$$h(t) = C\mathrm{e}^{At}B, \quad t \geq 0 \quad (3)$$

and zero otherwise. As is well-known, $h(t) = \mathcal{L}^{-1}(G(s))$, where $\mathcal{L}^{-1}$ is the inverse Laplace transform.

II. Model Reduction by Projection

Define a canonical projection in $\mathbb{R}^n \rightarrow \mathbb{R}^k$ by the matrix $P = [I_k, 0]$ with $k < n$, the dimension of the full order system. By *Projection of Dynamics* (POD) of the system (A, B, C) we shall mean the system $(\hat{A}, \hat{B}, \hat{C}) = (PAP^T, PB, CP^T)$. By itself this may not be very exciting, but combined with a preliminary similarity transformation T, we obtain a reduced model $(\hat{A}_T, \hat{B}_T, \hat{C}_T) = (W^T AV, W^T B, CV)$ where $W^T = PT$ and $V = T^{-1}P^T$.

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Now a continuum of k-th order models result. Conversely, assume that for the original system, the states x associated to some interesting nominal behavior lie approximately in the column span of a matrix V of rank k, i.e., $x \approx Vz$ for some $z \in \mathbb{R}^k$, then $\dot{x} \approx V\dot{z}$ and $Ax + Bu \approx AVz + Bu$. Define the mismatch as $V\dot{z} - AVz - Bu$. Forcing this mismatch to be orthogonal to the columns of W, (i.e., it is not *observed* in specific directions) then we get

$$W^T V\dot{z} = w^T AVz + WBu, \quad \text{with} \quad y \approx y_r = CV. \quad (4)$$

There is however a problem with this approach. It can be shown that for single input single output (SISO) systems, a reduced model can be found having any *arbitrary* characteristic polynomial. In what sense can we then state that the POD-reduced model has an input-output behavior that is close to the full order model?

At this point balancing (in the sense of Moore) is invoked to provide a rationale for a good choice. It also turns out that the reduced order system $\hat{G}(s) = \hat{C}(sI - \hat{A})^{-1}\hat{B}$ has an H_∞ error norm bound, $\|G - \hat{G}\|_\infty \leq 2\sum_{i=k+1}^n \sigma_i$, where $\sigma_{k+1} \ldots, \sigma_n$ are the truncated singular values of the Hankel operator associated with the full order model. In fact the approximation corresponds to the optimal rank k approximation of the Hankel operator. The reason why one focuses on the Hankel operator and not on the system (input-output) operator $\mathcal{S} : L_2^m(-\infty, \infty) \rightarrow L_2^p(-\infty, \infty) : u \rightarrow y = \mathcal{S}[u]$ where

$$y(t) = \int_{-\infty}^t h(t - \tau)u(\tau)\,\mathrm{d}\tau$$

is that the latter is not a compact operator, and therefore many of the nice mathematical properties are absent.

Enter now SIB for the time-invariant system over an interval of length 1. It is now possible to associate with this the time-limited input output operator, which is compact: $\mathcal{S}_1 : L_2^m(0, 1) \rightarrow L_2^p(0, 1) : u \rightarrow y = \mathcal{S}_1[u]$ where now

$$y(t) = \int_0^t h(t - \tau)u(\tau)\,\mathrm{d}\tau, \quad 0 \leq t \leq 1.$$

The nice fact about a compact operator is that it comes close to having a finite dimensional range. [6, p. 381]. We note that a collection of finite time input-output operators as given above can also be defined for the general time varying case. The original motivation for introducing SIB was precisely to extend balancing to that case. Since $L_2(0, 1)$ is separable, a

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complete orthonormal set $\{\phi_k\}$ exists, and we can compute the *matrix elements* of the transformation $\mathcal{S}_1$:

$$S_{ij} = \langle \phi_i, \mathcal{S}_1 \phi_j \rangle.$$

For instance, in the LTI case, an obvious choice is the Fourier basis $\phi_k(t) = \exp(j2\pi kt)$. A simple computation for the case where A has no eigenvalues at integer multiples of $j2\pi$, leads to the matrix elements

$$S_{\ell k} = C(j2\pi kI - A)^{-1} B \delta_{\ell k} + \\ + C(A - j2\pi \ell I)^{-1}(\mathrm{e}^A - I)(A - j2\pi kI)^{-1} B$$

Note that in terms of the transfer function and the observability and reachability functions, respectively

$$\psi_k^o(C, A) = C(A - j2\pi kI)^{-1} \tag{5}$$
$$\psi_k^r(A, B) = (j2\pi kI - A)^{-1} B, \tag{6}$$

this gives

$$S_{\ell k} = G(j2\pi k)\delta_{\ell k} + \psi_\ell^o(C, A)(\mathrm{e}^A - I)\psi_k^r(A, B). \tag{7}$$

Thus the finite time system operator can be represented by the infinite matrix

$$\mathbf{S} = \Psi^o(\mathrm{e}^A - I)\Psi^r + \mathbf{G}.$$

where $\mathbf{G}$ is diagonal with in the k-th position the evaluation of the transfer function at $2\pi jk$. The other term has a factorization of at most rank n. We note that $\Psi^r = [\ldots, (j2\pi kI - A)^{-1}B, \ldots]$, which corresponds to the matrix V in the rational Krylov method [1]. Similarly, Ψ^o corresponds to W^*. This makes then a nice interpretation of that work.

We also note that $\mathbf{G}$ contains all the information about the response to periodic inputs (period 1) of the given system.

A. Reduction based on the finite time system operator

A rationale for model reduction based on the finite time system operator S_1 follows: Find a low rank approximation for $\mathbf{S}_0 = \Psi^o(e^A - I)\Psi^r$. This involves determining its singular values. Then truncate the diagonal matrix $\mathbf{G}$ accordingly. This gives an approximation $\widehat{\mathbf{S}}$ of $\mathbf{S}$. Alternatively, one could also directly compute the SVD of $\mathbf{S}$, which would be better to justify, if it were not for its added complexity.

Moreover nonuniqueness of the solution may be exploited to solve the interpolation problem

$$G(j2\pi k) = \hat{G}(j2\pi k).$$

but this conforms again with the truncation of $\mathbf{G}$.

The requisite SVD problem can be simplified by noting that the eigenvalues of $\mathbf{S}_0\mathbf{S}_0^*$ are also the eigenvalues of the matrix

$$(e^A - I)\Psi^r\Psi^{r*}(e^{A^T} - I)\Psi^{o*}\Psi^o,$$

thus involving the reachability 'Gramians', $\Psi^r\Psi^{r*}$ and its dual, $\Psi^{o*}\Psi^o$. Note that

$$\Psi^r\Psi^{r*} = (\mathrm{e}^A - I)\mathcal{R}^r(\mathrm{e}^{A^T} - I) \tag{8}$$

where we defined

$$\mathcal{R}^r = \sum_\ell (A - j2\pi\ell I)^{-1} BB^T (A - j2\pi\ell I)^{-*} \tag{9}$$

This may be compared to the frequency domain representation of the infinite interval reachability Gramian (in the sense of Moore):

$$\mathcal{R} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega I - A)^{-1} BB^T (j\omega I - A)^{-*}\,\mathrm{d}\omega. \tag{10}$$

Indeed, we note that $\mathcal{R}$ is of the form

$$\begin{aligned}
\mathcal{R} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} M(j\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_\ell \int_{-2\pi(\ell-\frac{1}{2})}^{2\pi(\ell+\frac{1}{2})} M(j\omega)\,\mathrm{d}\omega \\
&= \frac{1}{2\pi}\sum_\ell \int_{-\pi}^{\pi} M(j2\pi(\ell+\omega))\,\mathrm{d}\omega \\
&\approx \frac{1}{2\pi}\sum_\ell \int_{-\pi}^{\pi} \left(M(j2\pi\ell) + M'(j2\pi\ell)\omega + \cdots\right)\,\mathrm{d}\omega \\
&\approx \sum_\ell M(j2\pi\ell) \\
&= \mathcal{R}^r
\end{aligned}$$

The approximations stem from the truncation of the Taylor series in the integrand. In view of this approximate Gramian relation, it is expected that finite time Hankel and system operator approaches can be compared and/or related to each other.

III. Finite Time Hankel Operator

For the given system, the finite time Hankel operator $\mathcal{H}_1 : L_2^m(-1, 0) \to L_2^p(0, 1) : u \to y = \mathcal{H}_1[u]$ where now

$$y(t) = C\mathrm{e}^{At}\int_{-1}^{0} \mathrm{e}^{-A\tau} Bu(\tau)\,\mathrm{d}\tau, \quad 0 \le t \le 1.$$

We note

$$\|y\|^2 = \max \lambda(\mathcal{O}_1\mathcal{R}_1)\|u\|^2$$

where $\mathcal{O}_1$ and $\mathcal{R}_1$ are the finite time observability and reachability Gramians of the system (A, B, C). Thus the Hankel norm is obtained by the largest eigenvalue of $\mathcal{O}_1\mathcal{R}_1$. If the system is sliding interval balanced, $\mathcal{O}_1 = \mathcal{R}_1 = \Lambda_1$. We also note that for a stable time invariant system the finite time Gramian (interval of length δ), solving the Lyapunov equation

$$AP_\delta + P_\delta A' + bb' = \mathrm{e}^{A\delta} bb' \mathrm{e}^{A'\delta} \tag{11}$$

satisfies the functional equation

$$P_\infty = P_\delta + \mathrm{e}^{A\delta} P_\infty \mathrm{e}^{A'\delta}. \tag{12}$$

Hence

$$\|P_\infty - P_\delta\| \le \|\mathrm{e}^{A\delta}\|^2 \|P_\infty\|.$$

It follows that $P_\delta \approx P_\infty$ if $\mathrm{e}^{A\delta}$ is sufficiently large. We found in simulations that for δ approximately twice the largest characteristic time (time constant) in the system, the infinite Gramian is reasonably approximated by the finite time Gramian. The latter may be computed by direct integration. This is useful in nonlinear balancing, were in principle Gramians may be computed by integration [4].

A. State Space

It is a well known property, in fact probably the most important, that the Hankel operator factors into an input-to-state map and a state-to-output map. In fact this was the basis for the work on realization theory some decades earlier [7]. It is clear that the rank of the Hankel operator also coincides with the dimension of the minimal state space. So one naturally wonders if there is a state space construct for the finite time input-output operator $\mathcal{S}_1$. The answer is yes, and it is rather trivial. In terms of a complete orthonormal set $\{\phi_i\}$ for $L_2((0,1)$, we find (using Dirac's bra-ket notation for its intuitive power) and the resolution of the identity

$$\sum |\phi_k\rangle\langle\phi_k| = I$$

$$\begin{aligned} |y\rangle &= \mathcal{S}_1|u\rangle \\ |y\rangle &= \mathcal{S}_1 \sum_k |\phi_k\rangle\langle\phi_k||u\rangle \\ |y\rangle &= \sum_k \mathcal{S}_1|\phi_k\rangle\langle\phi_k|u\rangle \\ \sum_\ell |\phi_\ell\rangle\langle\phi_\ell|y\rangle &= \sum_{\ell,k} |\phi_\ell\rangle\langle\phi_\ell|\mathcal{S}_1|\phi_k\rangle\langle\phi_k|u\rangle \end{aligned}$$

We recognize the matrix elements $S_{\ell,k} = \langle\phi_\ell|\mathcal{S}_1|\phi_k\rangle$ and the components of the input and output function, respectively $u_k = \langle\phi_k|u\rangle$ and $y_\ell = \langle\phi_\ell|y\rangle$

A trivial factorization is $\mathcal{S}_1 = O_1 R_1$, where

$$R_1 : L_2(0,1) \to \ell_2 : u \to \mathbf{x} = \{\langle\phi_k|u\rangle\}_{k=-\infty}^{\infty} \quad (13)$$

$$O_1 : \ell_2 \to L_2(0,1) : \mathbf{x} = \{\langle\phi_k|u\rangle\}_{k=-\infty}^{\infty} \to \mathbf{y} \quad (14)$$

where

$$\mathbf{y} = \sum_\ell |\phi_l\rangle(\mathbf{S}\mathbf{x})_\ell. \quad (15)$$

The corresponding state space is ℓ_2. However we note that the factorization does not correspond to causal maps: The entit=re segment of the input u is needed before the state can be determined, and the past of the output can only be determined at $t = 1$. We also note that by Parseval's theorem (isometry property)

$$\frac{\|\mathcal{S}_1 u\|}{\|u\|} = \frac{\|\mathbf{S}\mathbf{u}\|}{\|\mathbf{u}\|}.$$

IV. BALANCING AND DECOUPLING

In this section we consider two systems that are weakly coupled, and show that the balanced realization of the combined system may be approximated by the direct sum of the balanced form of the decoupled systems. By weak coupling we mean that two systems (A, b, c) and $(\overline{A}, \overline{b}, \overline{c})$ have a weak interaction term in the state model. Let therefore the coupled equations be

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{\overline{x}} \end{bmatrix} &= \begin{bmatrix} A & \epsilon H \\ \epsilon\overline{H} & \overline{A} \end{bmatrix} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} + \begin{bmatrix} b & 0 \\ 0 & \overline{b} \end{bmatrix} \begin{bmatrix} u \\ \overline{u} \end{bmatrix} \\ \begin{bmatrix} y \\ \overline{y} \end{bmatrix} &= \begin{bmatrix} c & 0 \\ 0 & \overline{c} \end{bmatrix} \begin{bmatrix} x \\ \overline{x} \end{bmatrix} \end{aligned}$$

where ϵ is the small coupling parameter. The Lyapunov equation for the reachability Gramian is

$$\begin{aligned} &\begin{bmatrix} A & \epsilon H \\ \epsilon\overline{H} & \overline{A} \end{bmatrix} \begin{bmatrix} P & P_x \\ P_x' & \overline{P} \end{bmatrix} + \\ &+ \begin{bmatrix} P & P_x \\ P_x' & \overline{P} \end{bmatrix} \begin{bmatrix} A' & \epsilon\overline{H}' \\ \epsilon H' & \overline{A}'x \end{bmatrix} + \\ &+ \begin{bmatrix} bb' & 0 \\ 0 & \overline{bb}' \end{bmatrix} = 0 \end{aligned}$$

We get

$$AP + PA' + bb' + \epsilon(HP_x' + P_x H') = 0 \quad (16)$$

$$\overline{AP} + \overline{PA}' + \overline{bb}' + \epsilon(\overline{H}P_x + P_x'\overline{H'}) = 0. \quad (17)$$

and the cross term

$$AP_x + \epsilon H\overline{P} + \epsilon P\overline{H}' + P_x\overline{A}' = 0. \quad (18)$$

It follows from (18) that P_x will be of order $O(\epsilon)$. Similar conclusions are made for the observability Gramian, where

$$A'Q_x + \epsilon\overline{H}'\overline{Q} + \epsilon QH + Q_x\overline{A} = 0 \quad (19)$$

implies that Q_x is of order $O(\epsilon)$. Hence if ϵ is small and both realizations were in balanced form, then the combined system is balanced for $\epsilon = 0$. But then it follows from (16) and (17) that the matrix

$$\begin{bmatrix} P & P_x \\ P_x' & \overline{P} \end{bmatrix} = \begin{bmatrix} \Lambda & \\ & \overline{\Lambda} \end{bmatrix} + 0(\epsilon^2)$$

with an analogous conclusion for the observability Gramian. This implies that if two systems, respectively of dimension n_1 and n_2 are weakly coupled, then the full system of order $n_1 + n_2$ is approximately balanced by performing the balancing on the decoupled systems individually. This reduces the computational complexity from $(n_1 + n_2)^2$ to $n_1^2 + n_2^2$. In addition if the purpose of balancing is model order reduction, singular values of order ϵ may be lost in the decoupling approximation error.

V. CONCLUSIONS

In this short note we explored some properties of the finite time input-output operator. We have discovered some connections with the model reduction method by Krylov methods. A state space formalism was given for this operator, but unlike the factorization of the Hankel map, this state space does not correspond to a causal construction. We have also shown that approximations via sliding interval balancing may be used to approximate the infinite Gramians, provided that the interval is sufficiently large (in terms of $\|e^{A\delta}\|$.

REFERENCES

[1] A.C. Antoulas, *Approximation of large-scale dynamical systems*, SIAM 2008.

[2] B. Moore, "Principal component analysis in linear systems: controllability, observability and model reduction," *IEEE Transactions on Automatic Control*, Vol. 26, pp. 17-32, 1981.

[3] E.I. Verriest and T. Kailath, "On generalized balanced realizations," *IEEE Transactions on Automatic Control*, Vol. 28, No. 8, pp. 833-844.

[4] E.I. Verriest, "Time Variant and Nonlinear Balanced Realizations," in *Model Order Reduction*, Schilders, van der Vorst and Rommes, (Eds), Springer 2008.

[5] E.I. Verriest and W.S. Gray, "Nonlinear balanced realizations," *Proc. 43d IEEE Conference on Decision and Control* 2004, pp. 1164-1169, Paradise Island, Bahamas 2004.

[6] A.W. Naylor and G.R. Sell, *Linear Operator Theory in Engineering and Science*, Springer, 1982.

[7] R.E. Kalman, P.L. Falb and M.A. Arbib, *Topics in Mathematical System Theory*, McGraw-Hill 1969.