Balanced truncation for linear interconnected systems: the state feedback case

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***Abstract*— Model order reduction is an important tool in control systems theory. In particular, it is useful for controller design since the dimension of the controller becomes very high when we use advanced control theory. Balanced truncation is one of the most useful model order reduction methods. In general, however, the stability of the feedback system is not maintained when the order of the controller is reduced by balanced truncation. This paper proposes a novel method of state feedback controller reduction by which we can preserve the stability of the resulting reduced order state feedback system. A numerical example demonstrates the effectiveness of the proposed method.**

I. Introduction

Recently, the development of CAD softwares enable one to execute controller design and stability analysis for large scale systems. Accordingly, the importance of the model order reduction increases. So far, many techniques were proposed. See e.g. [9], [1]. Among them, balanced realization and its application to model order reduction is known as one of the most effective model order reduction methods [8]. Balanced realization is a state-space realization whose states are balanced in a sense that the relationship between the input and the output, and the relationship among the coordinate axes are balanced. On these coordinates, one can easily eliminate less important states and consequently obtain a reduced order model. This model order reduction method is called balanced truncation. There are many extensions of this result. Its nonlinear version was proposed in [12], [3]. Balanced realization along periodic orbits are proposed in [14]. Lall et al. investigates an identification method [5]. Laub et al. [6] and Ionescu et al. [4] study a special class of balanced realizations using so called cross Gramians.

In this paper, we consider a state feedback controller reduction problem, that is, a problem to reduce a subsystem of an interconnected system via state feedbacks. Since conventional balanced truncation is only applicable to an open-loop system, it was difficult to reduce the dimension of the controller and maintain the stability of the whole closed-loop system. This problem is called controller reduction and investigated within the framework of weighted balanced realization [2], [13], [1]. Although the existing results are enough effective, it is difficult to ensure the stability of the resulting feedback system with the original plant and the reduced order controller. Recently, Li [7] and Sandberg et al. [10], [11] proposed a variation of balanced realization applicable to a feedback system. Although these method can solve a class of controller reduction problems, there are some restrictions on the system to be applied.

This paper considers a feedback system of two subsystems interconnected by state feedbacks. The problem to be solved is to reduce the dimension of a subsystem, which typically is a state feedback controller, in such a way that the resulting feedback system with the reduced order subsystem maintains its stability and that the input-output behavior of the feedback system is also preserved. To this end, we propose so-called partial Gramians and partial Hankel singular values for the subsystem to be reduced in a similar way to that adopted in [10], [11]. It is proved that the partial Hankel singular values thus defined are invariant under a special class of state feedback transformations. By using the freedom in choosing those state feedbacks, we can obtain a order reduction method for the subsystem which guarantees the stability of the resulting feedback system. Furthermore, a numerical example illustrates how the proposed method works for a controller reduction problem.

II. Preliminaries

This section refers to conventional model order reduction method based on balanced truncation [8]. Consider the following linear system.

$$\begin{cases} \dot{x} &= Ax + Bu, \qquad x(0) = 0 \\ y &= Cx \end{cases}$$

Here $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, and $y(t) \in \mathbb{R}^r$. The system matrix A is Hurwitz. The controllability and observability Gramians P and Q are the symmetric solutions to the following Lyapunov equations.

$$AP + PA^{\mathrm{T}} + BB^{\mathrm{T}} = 0, \quad A^{\mathrm{T}}Q + QA + C^{\mathrm{T}}C = 0 \quad (1)$$

The positive definiteness of the Gramians P and Q is equivalent to the controllability and observability of the system. In addition, they are quantitative indices of the controllability and observability of the system. A state-space realization on which the following equation holds is called a balanced realization.

$$P = Q = \mathrm{diag}(\sigma_1, \cdots, \sigma_n), \ \sigma_1 \geq \cdots \geq \sigma_n \qquad (2)$$

Here the relation between the input and the output, and the relation among the coordinate axes are balanced. The scalars σ_i's indicating the importance of the states are called Hankel singular values.

Remark 1: Instead of using the Lyapunov equations (1), we can use the following Lyapunov inequalities (LMIs).

$$A\check{P} + \check{P}A^{\mathrm{T}} + BB^{\mathrm{T}} \prec 0, \ A^{\mathrm{T}}\check{Q} + \check{Q}A + C^{\mathrm{T}}C \prec 0 \quad (3)$$

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(Non-unique) symmetric solutions to these inequalities are called pseudo controllability Gramian and pseudo observability Gramian, respectively.

Furthermore, the Gramians P and Q define the controllability and observability functions L_c and L_o as follows [12].

$$L_c(x) = \frac{1}{2}x^{\mathrm{T}}P^{-1}x \quad (4)$$

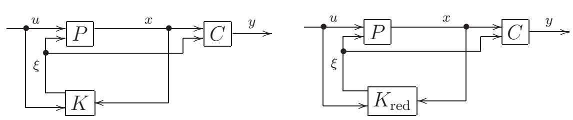
$$L_o(x) = \frac{1}{2}x^{\mathrm{T}}Q\, x \quad (5)$$

The procedure of model order reduction based on balanced realization (balanced truncation) is as follows. Suppose that the Hankel singular values defined in Equation (2) satisfy $\sigma_k \gg \sigma_{k+1}$, $1 \le k \le n-1$. This means that the set of the states $x_1, \cdots, x_k$ have a bigger effect to the input-output map than the rest $x_{k+1}, \cdots, x_n$. Therefore we can obtain a k-dimensional reduced order model by substituting 0 for the unimportant states $x_{k+1}, \cdots, x_n$.

III. Model order reduction for interconnected systems via state feedbacks

A. Problem setting

This section focuses on balanced realization for interconnected systems via state feedbacks and its application to model oder reduction. The biggest advantage of the proposed method is that the reduced order interconnected system preserves the stability of the original one. Fig. 1 shows the configuration of the proposed model order reduction problem. Here P and K denote state-space realizations with the state x and that ξ, respectively. C denotes a matrix determining the output signal y from x and ξ.



(i) Original system (ii) Reduced system

Fig. 1. Outline

Suppose that the feedback system with P and K as in Fig.1 (i) is stable. The problem considered here is to obtain a reduced order controller K_{red} in such a way that the feedback system with P and K_{red} as in Fig.1 (ii) maintains its stability and furthermore the input output map u-y is also preserved.

B. Subspace balanced realization

Let us consider the following linear system depicted as in Fig.1 (i).

$$\begin{cases} \dot{x} = A_{11}x + A_{12}\xi + B_1 u \\ \dot{\xi} = A_{21}x + A_{22}\xi + B_2 u \\ y = C_1 x + C_2 \xi \end{cases} \quad (6)$$

Here $x(t) \in \mathbb{R}^l$ is the state of the plant (the state to be preserved), $\xi(t) \in \mathbb{R}^n$ is the state of the controller (the state to be reduced), and $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^r$ are the input and the output, respectively.

In order to reduce the order of the controller K, let us replace its state ξ by a lower order vector $\hat{\xi}_r \in \mathbb{R}^k, k < n$ and obtain a reduced order controller K_{red} as

$$\begin{cases} \dot{x} = A_{11}x + A_{12}\xi + B_1 u \\ \dot{\hat{\xi}}_r = \hat{A}_{21_r}x + \hat{A}_{22_r}\hat{\xi}_r + \hat{B}_{2_r}u \\ \xi = E_1 x + E_2 \hat{\xi}_r \\ y = C_1 x + C_2 \xi \end{cases}. \quad (7)$$

Here we employ the following assumption for this system.

Assumption 1: The system (6) is asymptotically stable, controllable and observable.

Let us denote the controllability and observability Gramians $P \in \mathbb{R}^{(l+n)\times(l+n)}$ and $Q \in \mathbb{R}^{(l+n)\times(l+n)}$ in the following form where the division of the matrices corresponds to the division of the states x and ξ.

$$P^{-1} = \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix}, \quad Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix} \quad (8)$$

Then the controllability and observability functions L_c and L_o of the system (6) are obtained as follows.

$$L_c(x,\xi) = \frac{1}{2}x^{\mathrm{T}}R_{11}x + x^{\mathrm{T}}R_{12}\xi + \frac{1}{2}\xi^{\mathrm{T}}R_{22}\xi \quad (9)$$

$$L_o(x,\xi) = \frac{1}{2}x^{\mathrm{T}}Q_{11}x + x^{\mathrm{T}}Q_{12}\xi + \frac{1}{2}\xi^{\mathrm{T}}Q_{22}\xi \quad (10)$$

Since we are interested in the effect of the partial state ξ to the input-output behavior of the whole system, it is natural to define Gramians describing the properties of the partial state ξ. To this end, let us define the partial controllability Gramian P_{sub} and the partial observability Gramian Q_{sub} respectively as follows.

Definition 1: The matrices P_{sub} and Q_{sub} defined by

$$P_{\text{sub}} := R_{22}^{-1}, \qquad Q_{\text{sub}} := Q_{22} \quad (11)$$

are called *partial controllability Gramian* and *partial observability Gramian* with respect to the partial state ξ, respectively.

Note that if $P, Q \succ 0$ then $P_{\text{sub}}, Q_{\text{sub}} \succ 0$. We are interested in the Hankel singular values with respect to the partial Gramians P_{sub} and Q_{sub}. That is, we should obtain the Hankel singular value $\sigma \in \mathbb{R}$ and the corresponding coordinate axis $\zeta \in \mathbb{R}^n$ by solving the following equation for singular value analysis.

$$P_{\text{sub}}Q_{\text{sub}}\zeta = \sigma^2\zeta, \quad \sigma \ge 0 \in \mathbb{R}, \quad \zeta \in \mathbb{R}^n. \quad (12)$$

Then the partial Gramians are balanced (diagonalized) on the new coordinates ζ's. Consequently we can obtain a reduced order controller K_{red} of K by balanced truncation. We call σ's satisfying the above equation partial Hankel singular values as in the following definition.

Definition 2: The scalars σ's satisfying Equation (12) are called *partial Hankel singular values* with respect to the partial state ξ.

A question arises now: Is the feedback system (7) with the reduced order controller K_{red} stable? The answer is generally no. Here we use a trick to ensure the stability of the feedback system (7) by employing a freedom in choosing the partial state ξ itself. For this purpose, the following property is useful.

Theorem 1: Consider the system (6). For any linear coordinate transformation in the following form, the partial Hankel singular values with respect to $\hat{\xi}$ are equivalent to those with respect to ξ.

$$(x,\hat{\xi}) = \theta(x,\xi) := (x, Mx + N\xi), \qquad (13)$$
$$M \in \mathbb{R}^{n\times l},\ N \in \mathbb{R}^{l\times l},\ \det N \neq 0$$

Proof: Equation (13) can be described by

$$\begin{pmatrix} x \\ \hat{\xi} \end{pmatrix} = \Theta \begin{pmatrix} x \\ \xi \end{pmatrix} = \begin{bmatrix} I_l & 0_{ln} \\ M & N \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix}. \qquad (14)$$

Since N is nonsingular, this is a coordinate transformation. Let us denote the partial controllability and observability Gramians defined in (6) by $\hat{P}_{\text{sub}}$ and $\hat{Q}_{\text{sub}}$. Then we have

$$\begin{aligned} \hat{P}^{-1} &= (\Theta P \Theta^{\mathrm{T}})^{-1} \\ &= \begin{bmatrix} I_l & -M^{\mathrm{T}}N^{-\mathrm{T}} \\ 0_{nl} & N^{-\mathrm{T}} \end{bmatrix} \begin{bmatrix} R_{11} & R_{12} \\ R_{12}^{\mathrm{T}} & R_{22} \end{bmatrix} \begin{bmatrix} I_n & 0_{ln} \\ -N^{-1}M & N^{-1} \end{bmatrix} \\ &\therefore\ \hat{P}_{\text{sub}}^{-1} = N^{-\mathrm{T}} R_{22} N^{-1}, \qquad (15) \\ \hat{Q} &= \Theta^{-\mathrm{T}} Q \Theta^{-1} \\ &= \begin{bmatrix} I_l & -M^{\mathrm{T}}N^{-\mathrm{T}} \\ 0_{nl} & N^{-\mathrm{T}} \end{bmatrix} \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_{22} \end{bmatrix} \begin{bmatrix} I_n & 0_{ln} \\ -N^{-1}M & N^{-1} \end{bmatrix} \\ &\therefore\ \hat{Q}_{\text{sub}} = N^{-\mathrm{T}} Q_{22} N^{-1}. \qquad (16) \end{aligned}$$

This implies

$$\begin{aligned} &\hat{P}_{\text{sub}}\hat{Q}_{\text{sub}} \\ &= (N^{-\mathrm{T}}R_{22}N^{-1})^{-1}(N^{-\mathrm{T}}Q_{22}N^{-1}) \\ &= NR_{22}^{-1}Q_{22}N^{-1} \\ &= NP_{\text{sub}}Q_{\text{sub}}N^{-1} \qquad (17) \end{aligned}$$

that is, $\hat{P}_{\text{sub}}\hat{Q}_{\text{sub}}$ is similar to $P_{\text{sub}}Q_{\text{sub}}$. Therefore, the eigenvalues of $\hat{P}_{\text{sub}}\hat{Q}_{\text{sub}}$ which are equivalent to the partial Hankel singular values are invariant under the coordinate transformation (13). This completes the proof. ■

This theorem suggests that the freedom in choosing the coordinate transformation $\theta(\cdot)$ in Equation (13) can be used for the partial balanced realization and truncation. Please also note that the coordinate transformation $\theta(\cdot)$ can be regarded as a *state feedback transformation* for the subsystem with respect to ξ as explained later in Fig. fig:sfc.

C. Partial balanced truncation with stability

Let us consider a candidate Lyapunov function V

$$V(x,\xi) := (1-\alpha)L_c(x,\xi) + \alpha L_o(x,\xi), \qquad (18)$$
$$0 \leq \alpha \leq 1$$

Here L_c and L_o are the controllability and observability functions given in Equations (9) and (10). Assumption 1 implies $V \succ 0$.

Lemma 1: Suppose that Assumption 1 holds. Consider the function V in Equation (18) and the coordinate (state feedback) transformation θ in Equation (13). Then the function V satisfies

$$V(\theta^{-1}(x,\hat{\xi})) = x^{\mathrm{T}}V_x x + \hat{\xi}^{\mathrm{T}}V_\xi \hat{\xi} \qquad (19)$$

with positive definite matrices $V_x \in \mathbb{R}^{l\times l}$ and $V_\xi \in \mathbb{R}^{n\times n}$ if and only if

$$M = N\left((1-\alpha)R_{22} + \alpha Q_{22}\right)^{-1}\left((1-\alpha)R_{12} + \alpha Q_{12}\right)^{\mathrm{T}} \qquad (20)$$

Proof: Substituting the coordinate transformation (13) for Equation (18), we obtain

$$\begin{aligned} &V(\theta^{-1}(x,\hat{\xi})) \\ &= (1-\alpha)\Big(\frac{1}{2}x^{\mathrm{T}}R_{11}x + x^{\mathrm{T}}R_{12}N^{-1}(\hat{\xi}-Mx) \\ &\quad + \frac{1}{2}(\hat{\xi}-Mx)^{\mathrm{T}}N^{-\mathrm{T}}R_{22}N^{-1}(\hat{\xi}-Mx)\Big) \\ &\quad + \alpha\Big(\frac{1}{2}x^{\mathrm{T}}Q_{11}x + x^{\mathrm{T}}Q_{12}N^{-1}(\hat{\xi}-Mx) \\ &\quad + \frac{1}{2}(\hat{\xi}-Mx)^{\mathrm{T}}N^{-\mathrm{T}}Q_{22}N^{-1}(\hat{\xi}-Mx)\Big). \qquad (21) \end{aligned}$$

Hence the cross term with respect to x and $\hat{\xi}$ vanishes if and only if

$$(1-\alpha)(R_{12}-(N^{-1}M)^{\mathrm{T}}R_{22}) + \alpha(Q_{12}-(N^{-1}M)^{\mathrm{T}}Q_{22}) = 0 \qquad (22)$$

which is equivalent to (20). This completes the proof. ■

The procedure of model order reduction is as follows. First of all, apply a coordinate transformation θ in such a way that the candidate Lyapunov function V satisfy the condition (19) in Lemma 1. Then apply the partial balanced realization. Let us describe the partially balanced realization by the following equation.

$$\begin{cases} \dot{x} = A_{11}x + \bar{A}_{12}\zeta_a + \bar{A}_{13}\zeta_b + B_1 u \\ \dot{\zeta}_a = \bar{A}_{21}x + \bar{A}_{22}\zeta_a + \bar{A}_{23}\zeta_b + \bar{B}_2 u \\ \dot{\zeta}_b = \bar{A}_{31}x + \bar{A}_{32}\zeta_a + \bar{A}_{33}\zeta_b + \bar{B}_3 u \\ y = C_1 x + \bar{C}_2\zeta_a + \bar{C}_3\zeta_b \end{cases} \qquad (23)$$

Here the division of the state is $\zeta = (\zeta_a^{\mathrm{T}}, \zeta_b^{\mathrm{T}})^{\mathrm{T}}, \zeta_a = (\zeta_1, \cdots, \zeta_k)^{\mathrm{T}}, \zeta_b = (\zeta_{k+1}, \cdots, \zeta_n)^{\mathrm{T}}, k < n$. Suppose that the partial states ζ_a and ζ_b have bigger and smaller effect to the input-output behavior, respectively. The system matrices $(\bar{A}, \bar{B}, \bar{C})$, and the Gramians $\bar{P}$ and $\bar{Q}$ are divided according to the division of the state.

$$\bar{P}^{-1} = \begin{bmatrix} R_{11}^{\zeta} & R_{12}^{\zeta} & R_{13}^{\zeta} \\ R_{12}^{\zeta\mathrm{T}} & \Sigma_a^{-1} & 0 \\ R_{13}^{\zeta\mathrm{T}} & 0 & \Sigma_b^{-1} \end{bmatrix}, \bar{Q} = \begin{bmatrix} Q_{11}^{\zeta} & Q_{12}^{\zeta} & Q_{13}^{\zeta} \\ Q_{12}^{\zeta\mathrm{T}} & \Sigma_a & 0 \\ Q_{13}^{\zeta\mathrm{T}} & 0 & \Sigma_b \end{bmatrix} \qquad (24)$$

Here R_{ij}^{ζ}'s and Q_{ij}^{ζ}'s are matrices with the appropriate dimensions. $\Sigma_a = \mathrm{diag}(\sigma_1, \cdots, \sigma_k)$, and $\Sigma_b = \mathrm{diag}(\sigma_{k+1}, \cdots, \sigma_n)$ hold. The reduced order model can be obtained by substituting $\zeta_b \equiv 0$ for Equation (23) as follows.

$$\begin{cases} \dot{x} = A_{11}x + \bar{A}_{12}\zeta_a + B_1 u \\ \dot{\zeta}_a = \bar{A}_{21}x + \bar{A}_{22}\zeta_a + \bar{B}_2 u\ . \\ y = C_1 x + \bar{C}_2\zeta_a \end{cases} \qquad (25)$$

Stability of the corresponding feedback system is proven in the following theorem.

Theorem 2: Suppose that Assumption 1 holds. The reduced order systems obtained by substituting $\zeta_b \equiv 0$ for

Equation (23) is stable. Furthermore, if the pseudo Gramians satisfying (3) are used instead of the conventional Gramians, then the resultant reduced order system is asymptotically stable.

Proof: Let us define a matrix $W := (1-\alpha)\bar{P}^{-1} + \alpha\bar{Q}$. Then Equations (19) and (24) imply

$$W\bar{A} + \bar{A}^{\mathrm{T}}W + \bar{D}^{\mathrm{T}}\bar{D} = 0 \tag{26}$$

$$W = \begin{bmatrix} (1-\alpha)R_{11}^{\zeta} + \alpha Q_{11}^{\zeta} & 0 & 0 \\ 0 & \Sigma_a & 0 \\ 0 & 0 & \Sigma_b \end{bmatrix} \succ 0,$$

$$\bar{D} = \begin{bmatrix} \sqrt{\alpha}\bar{C} \\ \sqrt{1-\alpha}\bar{B}^{\mathrm{T}}\bar{P}^{-\mathrm{T}} \end{bmatrix}$$

where $\bar{A}$ is defined by

$$\bar{A} := \begin{pmatrix} A_{11} & \bar{A}_{12} & \bar{A}_{13} \\ \bar{A}_{21} & \bar{A}_{22} & \bar{A}_{23} \\ \bar{A}_{31} & \bar{A}_{32} & \bar{A}_{33} \end{pmatrix}$$

the system matrix of (23). The matrix W can be decomposed as

$$W = \begin{bmatrix} W_{11} & 0 \\ 0 & \Sigma_b \end{bmatrix},\ W_{11} := \begin{bmatrix} (1-\alpha)R_{11}^{\zeta} + \alpha Q_{11}^{\zeta} & 0 \\ 0 & \Sigma_a \end{bmatrix}. \tag{27}$$

According to this decomposition, decompose the matrices $\bar{A}$ and $\bar{D}$ in a similar way as

$$\bar{A} = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix},\ \ \bar{D} = \begin{bmatrix} \tilde{D}_1 & \tilde{D}_2 \end{bmatrix}. \tag{28}$$

Then Equation (26) reduces to

$$W_{11}\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}W_{11} + \tilde{D}_1^{\mathrm{T}}\tilde{D}_1 = 0, \tag{29}$$
$$\Sigma_b\tilde{A}_{22} + \tilde{A}_{22}^{\mathrm{T}}\Sigma_b + \tilde{D}_2^{\mathrm{T}}\tilde{D}_2 = 0,$$
$$W_{11}\tilde{A}_{12} + \tilde{A}_{21}^{\mathrm{T}}\Sigma_b + \tilde{D}_1^{\mathrm{T}}\tilde{D}_2 = 0.$$

Equation (29) is a Lyapunov function for the reduced order system 25. Since $\tilde{D}_1^{\mathrm{T}}\tilde{D}_1 \succeq 0$ and $W_{11} \succ 0$, the eigenvalues of the matrix $\tilde{A}_{11}$ are in the closed left half plane, that is, the reduced order system (25) is stable in the Lyapunov sense. Furthermore, if the pseudo Gramians satisfying (3) are used instead of the conventional Gramians, the reduced order system (25) becomes asymptotically stable, since (29) becomes

$$W_{11}\tilde{A}_{11} + \tilde{A}_{11}^{\mathrm{T}}W_{11} + \tilde{D}_1^{\mathrm{T}}\tilde{D}_1 \prec 0. \tag{30}$$

This completes the proof. ■

Finally, let us summarize the proposed model order reduction procedure as follows.

Procedure of the partial model order reduction

1) Apply the coordinate (state feedback) transformation θ in Equation (13) with the matrix M given in Equation (20). This procedure corresponds to the conversion from Fig.2 (i) to (ii).
2) Compute a partial balanced realization using the partial Gramians P_{sub} and Q_{sub} with respect to the partial state $\hat{\xi}$.
3) Truncating the less important state ζ^b, we can obtain the reduced order system. This procedure corresponds to the conversion from Fig.2 (ii) to (iii).

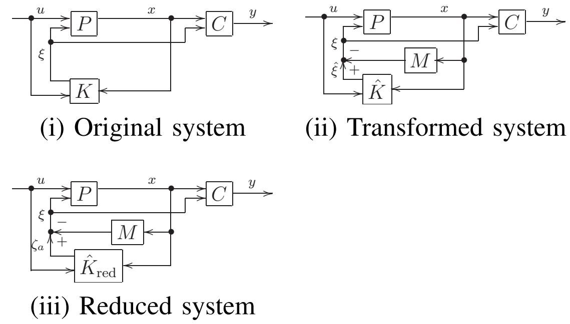


Fig. 2. Partial model reduction for state feedback systems

IV. Numerical Example

This section gives a numerical example illustrating how the proposed method works for a concrete problem.

A. Modelling

Let us consider a cart-pendulum system depicted in Fig.3.

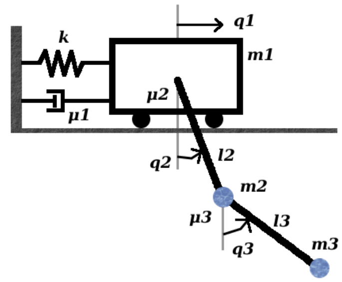


Fig. 3. The cart-pendulum

The dynamics of the system can be calculated by the Euler-Lagrange equation with the following Lagrangian function

$$\mathcal{L}_1 = \mathcal{T}_1 - \mathcal{V}_1 \tag{31}$$
$$\mathcal{T} = \frac{1}{2}m_1\dot{q_1}^2 + \frac{1}{2}m_2\dot{q_2}^2 + \frac{1}{2}m_3\dot{q_3}^2 \tag{32}$$
$$\mathcal{V} = \frac{1}{2}k_1q_1^2 + m_2gl_2(1-\cos q_2) + m_3g(l_2(1-\cos q_2) + l_3(1-\cos q_3)) \tag{33}$$

where $\mathcal{T}$ and $\mathcal{V}$ denote the kinetic energy, and the potential energy, respectively. The physical parameters are summarized in Table I

The input is the force applied to the mass in the q_1 direction. Then the Euler-Lagrange equation of this system can be obtained as follows.

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q_1}} - \left(\frac{\partial\mathcal{L}}{\partial q_1}\right) = u \tag{34}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{q_i}} - \left(\frac{\partial\mathcal{L}}{\partial q_i}\right) = 0, \quad i \in \{2,3\} \tag{35}$$

TABLE I

PARAMETERS OF THE CART-PENDULUM

parameter	description	value
q_1	displacement of the cart	variable
q_2	displacement of Link1	variable
q_3	displacement of Link2	variable
u	input force to the cart	variable
m_1	mass of the cart	1.0
m_2	mass of Link1	1.0
m_3	mass of Link2	1.0
k	elastic constant for cart	1.0
l_2	length of Link1	1.0
l_3	length of Link2	1.0
μ_1	viscous resistance constant for cart	1.0
μ_2	viscous resistance constant for Link1	1.0
μ_3	viscous resistance constant for Link2	1.0
g	constant of gravitation	9.8

Let us define the state as $(x^{\mathrm T}, \xi^{\mathrm T})^{\mathrm T} \in \mathbb{R}^6$, $x = (x_1, x_2)^{\mathrm T} := (q_1, \dot q_1)^{\mathrm T}$, $\xi = (\xi_1, \xi_2, \xi_3, \xi_4)^{\mathrm T} := (q_2, \dot q_2, q_3, \dot q_3)^{\mathrm T}$. Select the output function as $y = (q_1, \dot q_1)^{\mathrm T}$. Then we obtain the state-space realization of the system as follows.

$$\left\{\begin{array}{l} \begin{pmatrix} \dot x_1 \\ \dot x_2 \\ \dot\xi_1 \\ \dot\xi_2 \\ \dot\xi_3 \\ \dot\xi_4 \end{pmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 98/5 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & -196/5 & -2 & 49/5 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 98/5 & 1 & -98/5 & -2 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} \end{array}\right. \tag{36}$$

B. Partial Gramians and Hankel singular values

The controllability and observability Gramians P and Q of the system (36) can be calculated as follows.

$$\begin{aligned} P^{-1} &= \left[\begin{array}{cc|cccc} 2.31 & -0.353 & -8.96 & -0.353 & -0.252 & 0.909 \\ * & 6.44 & 0.637 & 4.13 & -0.605 & 0.637 \\ \hline * & * & 5.29\times 10^2 & -13.1 & -2.00\times 10^2 & -53.5 \\ * & * & * & 11.8 & 20.7 & 3.06 \\ * & * & * & * & 2.15\times 10^2 & 21.9 \\ * & * & * & * & * & 13.9 \end{array}\right] \\ &= \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline * & R_{22}(= P_{\mathrm{sub}}^{-1}) \end{array}\right] \end{aligned} \tag{37}$$

$$\begin{aligned} Q &= \left[\begin{array}{cc|cccc} 2.48 & 1.48 & 1.25 & 0.982 & 0.624 & 0.492 \\ * & 6.12 & 0.334 & 4.14 & 0.195 & 2.10 \\ \hline * & * & 2.58 & 0.132 & 0.984 & -7.02\times 10^{-2} \\ * & * & * & 0.982 & 0.221 & 1.48 \\ * & * & * & * & 1.08 & 0.110 \\ * & * & * & * & * & 0.796 \end{array}\right] \\ &= \left[\begin{array}{c|c} Q_{11} & Q_{12} \\ \hline * & Q_{22}(= Q_{\mathrm{sub}}) \end{array}\right] \end{aligned} \tag{38}$$

The corresponding Hankel singular values are

$$\sigma = \begin{pmatrix} 1.26 \\ 0.834 \\ 0.123 \\ 0.115 \\ 4.06\times 10^{-2} \\ 3.10\times 10^{-2} \end{pmatrix}. \tag{39}$$

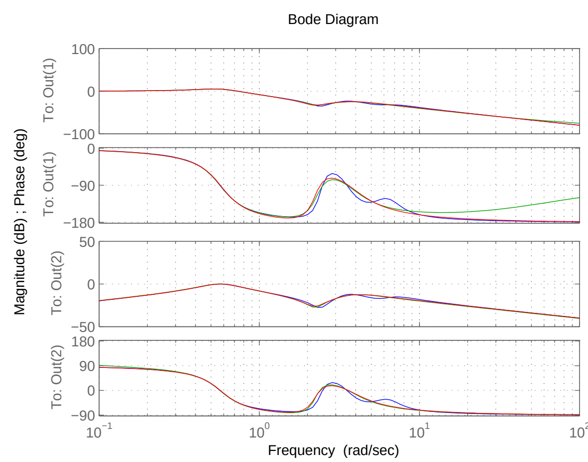
The corresponding partial Hankel singular values defined in Definition 1 can be computed by the partial controllability and observability Gramians P_{sub} and Q_{sub} as follows

$$\sigma_{\mathrm{sub}} = \begin{pmatrix} 0.580 \\ 0.147 \\ 4.81\times 10^{-2} \\ 3.38\times 10^{-2} \end{pmatrix} \tag{40}$$

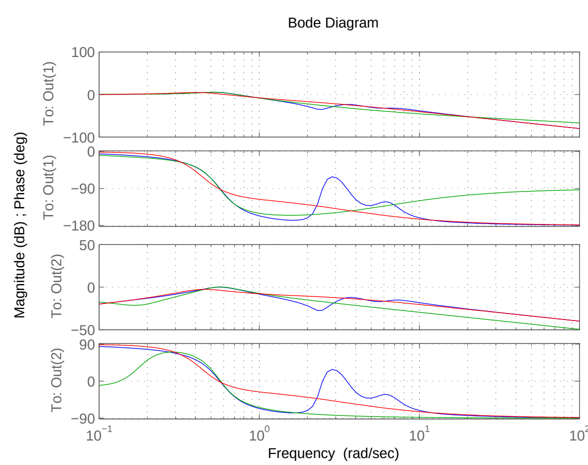
by which we can determine the dimension of the reduced order model. In this case, the dimension of the reduced order subsystem with respect to ξ should be 2 (or 1).

C. Simulations

This section shows simulation results. Recall that the original system is of 6 dimension. We compute two lower order systems. One is of 4 dimension (the order of the ξ subsystem is reduced from 4 to 2) and the other is of 3 dimension (the order of the ξ subsystem is reduced from 4 to 1).



(i) 4 dimensional system



(ii) 3 dimensional system

Fig. 4. Bode diagrams

The bode diagrams of the original system and the two reduced order ones are plotted in Fig. 4. The parameter α is selected as $\alpha = 1/2$. In the figures, the blue line depicts the bode plots of the original system, the green one depicts those of the reduce order system derived by the proposed method, and the red one depicts those of the reduced order system obtained by the conventional method. The conventional case is computed by just applying the balanced truncation method to the ξ subsystem directly.

First of all, Figs. 4 (i) and (ii) show that the approximation error of the 4th order system (i) is much smaller than that of the 3rd order system (ii). This result is consistent with the order of the partial Hankel singular values given in Equation (40) which suggests the dimension of the reduced order model to be 2. Fig. 4 (i) shows that both the proposed and the conventional methods derive reasonably good reduced order models. Although the conventional one is slightly better in the high frequency range, this does not affect the total approximation error much since the gain is small in this region. Fig. 4 (ii) shows that the proposed method gives better approximation than the conventional one. In particular the phase in the lower frequency of the proposed one is better than the conventional result. It should also be recalled that the stability of the resulting feedback system is guaranteed by the proposed method, whereas it is not guaranteed by the conventional one.

The transfer functions of the three systems are given as follows.

- The original system:

$$G(s) = \frac{1}{(s+1.6+j6.63)(s+1.6-j6.63)(s+0.743-j3.48)} \times \frac{1}{(s+0.743+j3.48)(s+0.157-j0.548)(s+0.157+j0.548)} \times \left[\frac{0.888(s+1.13)}{(s+1.28-j5.63)(s+1.28+j5.63)(s+0.213-j2.39)(s+0.213+j2.39)}\right]$$

- The conventional method (4 dimension):

$$G(s) = \begin{bmatrix} \frac{1.46\times 10^{-2}(s+61.1)(s+0.315-j2.27)(s+0.315+j2.27)}{(s+1.41-j3.46)(s+1.41+j3.46)(s+0.161-j0.551)(s+0.161+j0.551)} \\ \frac{0.946(s+0.0138)(s+0.229-j2.23)(s+0.229+j2.23)}{(s+1.41-j3.46)(s+1.41+j3.46)(s+0.161-j0.551)(s+0.161+j0.551)} \end{bmatrix}$$

- The proposed method (4 dimension):

$$G(s) = \begin{bmatrix} \frac{-0.489(s-2.05)}{(s+1.53-j3.50)(s+1.53+j3.50)(s+0.162-j0.557)(s+0.162+j0.557)} \\ \frac{-4.59\times 10^{-15}(s+0.27-j2.21)(s+0.27+j2.21)}{(s+1.53-j3.50)(s+1.53+j3.50)(s+0.162-j0.557)(s+0.162+j0.557)} \end{bmatrix}$$

- The conventional method (3 dimension):

$$G(s) = \begin{bmatrix} \frac{0.064(s+6.03)(s+0.209)}{(s+0.169)(s+0.143-j0.541)(s+0.143+j0.541)} \\ \frac{0.333(s+0.0516-j0.17)(s+0.0516+j0.17)}{(s+0.169)(s+0.143-j0.541)(s+0.143+j0.541)} \end{bmatrix}$$

- The proposed method (3 dimension):

$$G(s) = \begin{bmatrix} \frac{0.222(s+4.5)}{(s+3.08)(s+0.156-j0.408)(s+0.156+j0.408)} \\ \frac{(s+0.558)}{(s+3.08)(s+0.156-j0.408)(s+0.156+j0.408)} \end{bmatrix}$$

The simulation results show that the proposed reduction method maintains the stability of the resulting feedback system and that it preserves the input-output map of the feedback system as well.

V. Conclusion

This paper focuses on stability preserving balanced truncation for linear interconnected systems with state feedback. The main feature of the proposed method is to preserve asymptotic stability of the reduced order closed loop system without any restrictive assumptions. The effectiveness of the proposed method is demonstrated by a numerical example on a controller reduction problem.

An important future work following the present paper is to extend Theorem 2 to the output feedback case which will be discusses in the future work.

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