Dilatability of Linear Cellular Automata

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***Abstract*— We introduce a notion of dilatability between two LCAs and relate it with the notion of (power) dilatability between the corresponding global transition functions. We prove that a partial isometric LCA can be dilated to a quantum LCA which is reversible. In particular, any isometric LCA $\mathcal{A}$ can be dilated to a quantum LCA $\mathcal{B}$ such that the global rule of $\mathcal{B}$ extends the global rule of $\mathcal{A}$.**

I. PRELIMINARIES

A. Linear Cellular Automata

Cellular automata (CAs) were introduced by S. Ulam [16] and J. von Neumann [1], [5] in order to provide an appropriate mathematical model for the biological self-reproduction. A classical CA is represented as a discrete and homogeneous lattice of cells. The state of each cell is updated according to the states, at the previous time instant, of the cells in its neighborhood. This updating procedure is done synchronously by a set of local rules. Nowadays various CA based models are encountered in physics, biology, chemistry, cryptography, etc. ([17]).

In the recent years it was often more and more difficult to describe complex real life systems using only discrete state space CAs. Many applications based on CAs with continuous state space were considered. In this context we proposed in [7] a general CA model in which the state space is a Hilbert space and the local transition function is a bounded linear map. More precisely, a *linear cellular automaton* (LCA) is a triple $\mathcal{A} = (\mathcal{H}, N, \delta)$ where $\mathcal{H}$ (the *state space*) is a complex Hilbert space, $N = N_1 = \{-1, 0, 1\}$ (the *neighborhood*) and $\delta : \mathcal{H}^3 \to \mathcal{H}$ (the *local rule*) is a linear and bounded map. The *configuration space* is, in this case, the Hilbert space $\mathcal{C}_\mathcal{A} := \ell^2_\mathbb{Z}(\mathcal{H})$ of all square summable sequences $(h_n)_{n\in\mathbb{Z}}$ of vectors in $\mathcal{H}$. The *global transition function*, which describes the CA evolution, is defined as

$$\mathcal{C}_\mathcal{A} \ni c = (h_n)_{n\in\mathbb{Z}} \mapsto F_\mathcal{A}(c) := (\delta(h_{n+N}))_{n\in\mathbb{Z}} \in \mathcal{C}_\mathcal{A}.$$

Remark 1: (a) Since δ is linear and bounded there exist bounded linear operators δ_{-1}, δ_0 and δ_1 (acting on $\mathcal{H}$) such that

$$\delta(h_{-1}, h_0, h_1) = \delta_{-1}(h_{-1}) + \delta_0(h_0) + \delta_1(h_1), \quad h_{-1}, h_0, h_1 \in \mathcal{H}. \tag{1}$$

More precisely,

$$\begin{aligned} \delta_{-1}(h) &= \delta(h, 0, 0) \\ \delta_0(h) &= \delta(0, h, 0) \\ \delta_1(h) &= \delta(0, 0, h), \quad h \in \mathcal{H}. \end{aligned}$$

In addition, $\|\delta_i\| \le \|\delta\|,\ i \in N$. Conversely, if defined by (1) then δ is linear and bounded with $\|\delta\| \le \left(\sum_{i\in N} \|\delta_i\|^2\right)^{1/2}$.

(b) The notion of linear cellular automaton can be also extended for neighborhoods of arbitrary (but finite) radius r: $N = N_r = \{-r, \ldots, -1, 0, 1, \ldots, r\}$. Our main results, included in the following two sections, hold true for LCAs having neighborhoods of any (finite) radius. They will be presented, for simplicity, only for the case $r = 1$. ■

The *adjoint* of a linear cellular automaton $\mathcal{A} = (\mathcal{H}, N, \delta)$ is the LCA $\mathcal{A}^* = (\mathcal{H}, N, \delta_*)$, where the local rule δ_* is defined as

$$\delta_*(h_{-1}, h_0, h_1) := \delta_1^*(h_{-1}) + \delta_0^*(h_0) + \delta_{-1}^*(h_1), \quad h_{-1}, h_0, h_1 \in \mathcal{H},$$

the operators δ_{-1}, δ_0 and δ_1 being defined by the previous remark.

B. Reversible Linear Cellular Automata

A cellular automaton $\mathcal{A}$ is said to be *reversible* if there exists a cellular automaton $\mathcal{B}$ such that their global transition functions are inverses of each other. In other words, if we pass from one configuration c to another c' by iterating $\mathcal{A}$ on a number n of steps we can start with c' to recover c by iterating $\mathcal{B}$ for exactly n steps. G.A. Hedlund [2] characterized CA by a topological point of view: a map $F : S^\mathbb{Z} \to S^\mathbb{Z}$ which commutes with the shift operators is continuous if and only if F is the global transition function of a CA $\mathcal{A} = (S, N, \delta)$ (cf. also [8]). D. Richardson used this nice result in [11] to prove that a CA $\mathcal{A}$ is reversible if and only if its global transition function $F_\mathcal{A}$ is bijective.

The notion of reversibility for LCAs is introduced similarly. We only require that the inverse $\mathcal{B}$ (of a given reversible LCA $\mathcal{A}$) to be a LCA. In these settings the equivalence between the reversibility of $\mathcal{A}$ and the bijectivity of $F_\mathcal{A}$ is, in general, not true.

A LCA $\mathcal{A}$ is said to be *isometric* if its global transition function is an isometric map, i.e.,

$$\|F_\mathcal{A}((h_n)_{n\in\mathbb{Z}})\|_{\mathcal{C}_\mathcal{A}} = \|(h_n)_{n\in\mathbb{Z}}\|_{\mathcal{C}_\mathcal{A}}$$

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for every $(h_n)_{n\in\mathbb{Z}} \in \mathcal{C}_{\mathcal{A}}$. It was proved in [7] that a LCA $\mathcal{A} = (\mathcal{H}, N, \delta)$ is isometric if and only if

$$\delta_{-1}^*\delta_1 = \delta_0^*\delta_1 + \delta_{-1}^*\delta_0 = 0$$

and

$$\delta_{-1}^*\delta_{-1} + \delta_0^*\delta_0 + \delta_1^*\delta_1 = 1_{\mathcal{H}}.$$

If the global transition function $F_{\mathcal{A}}$ of $\mathcal{A}$ is a partial isometry (i.e., $F_{\mathcal{A}}^* F_{\mathcal{A}}$ or, equivalently, $F_{\mathcal{A}} F_{\mathcal{A}}^*$ is an orthogonal projection) then $\mathcal{A}$ is said to be a *partial isometric* LCA.

If the global transition function of a LCA $\mathcal{A} = (\mathcal{H}, N, \delta)$ is a unitary operator then $\mathcal{A}$ is called a *quantum* LCA. Equivalently, $\mathcal{A}$ is a quantum LCA if and only if both $\mathcal{A}$ and $\mathcal{A}^*$ are isometric; or, in terms of the local rule, if and only if

$$\delta_{-1}^*\delta_1 = \delta_1\delta_{-1}^* = \delta_0^*\delta_1 + \delta_{-1}^*\delta_0 = \delta_1\delta_0^* + \delta_0\delta_{-1}^* = 0$$

and

$$\delta_{-1}^*\delta_{-1} + \delta_0^*\delta_0 + \delta_1^*\delta_1 = \delta_{-1}\delta_{-1}^* + \delta_0\delta_0^* + \delta_1\delta_1^* = 1_{\mathcal{H}}.$$

Obviously, any quantum LCA is both isometric and reversible. Maybe the simplest example of a quantum LCA is the *bilateral shift* LCA $\mathcal{S} = (\mathcal{H}, N, \delta)$, where the local transition function is given (according to (1)) by the operators $\delta_{-1} = 1_{\mathcal{H}}$ and $\delta_0 = \delta_1 = 0_{\mathcal{H}}$.

C. The Contents of the Paper

We introduce a notion of dilatability between two LCAs and relate it with the notion of (power) dilatability between the corresponding global transition functions. We prove that a partial isometric LCA can be dilated to a quantum LCA which is reversible. In particular, any isometric LCA $\mathcal{A}$ can be dilated to a quantum LCA $\mathcal{B}$ such that the global rule of $\mathcal{B}$ extends the global rule of $\mathcal{A}$.

D. The Motivation

One of the most important characteristics of microscopic mechanisms in physics is reversibility. This means that all the information is preserved and, consequently, the physical system follows a deterministic rule. As noted before, CAs provide an excellent modeling environment for physical systems. They can capture reversibility without sacrificing other essential properties as computational universality, locality of the interactions or simultaneity of the motion. T. Toffoli and N. Margolus [13], [14], [15] proposed some dedicated cellular automata machines which were capable to efficiently support the simulation of reversible CAs. These machines were able to handle a large amount of numerical processing without any loss of information.

In the recent studies on quantum mechanical models of computing a special class of CAs, namely the class of quantum CAs, played an essential role ([4]). Quantum CAs are reversible CAs because their state evolution is determined by a unitary transformation.

We want to also emphasize the role of reversible CAs for the implementation of a robust environment in public-key cryptography [3], [10].

In this context it is important to find larger classes of cellular automata with a behaviour at least comparable, as concerning the application point of view, with the behaviour of reversible CAs. The class of CAs which are dilatable (or extensible) to a reversible CA represent a good example in this direction. We can perform computations and recover the past information (when needed) using only the dilation CA and, finally, compress the results into the configuration space of the original CA.

We choosed to use continuous state CAs because, in real life systems, one can rarely find situations described by only a finite set of states. In order to obtain better results provided by some powerfull tools of functional analysis we decided to add aditional structure on the state space (a Hilbert space) and on the local rule (a linear and bounded map).

II. DILATABILITY OF LOCAL RULES

Definition 2: Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ and $\mathcal{B} = (\mathcal{K}, N, \varepsilon)$ be two LCAs.

- $\mathcal{B}$ is a *dilation of* $\mathcal{A}$ if $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and

$$\delta_{i_1}\delta_{i_2}\ldots\delta_{i_n}h = P_{\mathcal{H}}^{\mathcal{K}}\varepsilon_{i_1}\varepsilon_{i_2}\ldots\varepsilon_{i_n}h,$$

  for every $n \in \mathbb{N}^*,\ i_1, i_2, \ldots, i_n \in N$ and $h \in \mathcal{H}$ ($P_{\mathcal{H}}^{\mathcal{K}}$ denotes the orthogonal projection of $\mathcal{K}$ onto $\mathcal{H}$);
- $\mathcal{B}$ is an *extension of* $\mathcal{A}$ if $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and ε is an extension of δ.

If the local rule of a given LCA $\mathcal{A}$ is a row contraction then $\mathcal{A}$ can be dilated to a LCA having a local rule with isometric components. More precisely, it holds:

Theorem 3: Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a LCA. If

$$\delta_{-1}\delta_{-1}^* + \delta_0\delta_0^* + \delta_1\delta_1^* \le 1_{\mathcal{H}}$$

then there exists a LCA $\mathcal{B} = (\mathcal{K}, N, \varepsilon)$ with the following properties:

(a) *$\mathcal{B}^*$ is an extension of $\mathcal{A}^*$ (hence, $\mathcal{B}$ is a dilation of $\mathcal{A}$).*
(b) $\varepsilon_i^*\varepsilon_j = \delta_{ij}, \quad i, j \in N.$
(c) $\frac{1}{\sqrt{3}}\mathcal{B} := (\mathcal{K}, N, \frac{1}{\sqrt{3}}\varepsilon)$ *is an isometric LCA.*

Proof: Conditions (a) and (b) are reformulations, for our context, of a dilation theorem of G. Popescu [6].

(c) follows by the remark that $F_{\mathcal{B}}^* F_{\mathcal{B}}$ has the matrix representation $(a_{ij})_{i,j\in\mathbb{Z}}$, where

$$\begin{aligned}
a_{ii} &= \varepsilon_{-1}^*\varepsilon_{-1} + \varepsilon_0^*\varepsilon_0 + \varepsilon_1^*\varepsilon_1 = 3 \cdot 1_{\mathcal{H}},\\
a_{i,i-1} &= \varepsilon_1^*\varepsilon_0 + \varepsilon_0^*\varepsilon_{-1} = 0,\\
a_{i,i+1} &= \varepsilon_0^*\varepsilon_1 + \varepsilon_{-1}^*\varepsilon_0 = 0,\\
a_{i,i-2} &= \varepsilon_1^*\varepsilon_{-1} = 0,\\
a_{i,i+2} &= \varepsilon_{-1}^*\varepsilon_1 = 0 \quad (i \in \mathbb{Z})
\end{aligned}$$

and

$$a_{ij} = 0 \ \text{ for the rest of the cases.}$$

■

III. DILATABILITY OF GLOBAL RULES

Definition 4: Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_s, \varepsilon)$ be two LCAs such that $r \leq s$. $\mathcal{B}$ is said to be a *power dilation* of $\mathcal{A}$ if

(a) $\mathcal{H}$ is a closed subspace of $\mathcal{K}$;
(b) $F^m_{\mathcal{A}}(c) = P^{\mathcal{C}_{\mathcal{B}}}_{\mathcal{C}_{\mathcal{A}}} F^m_{\mathcal{B}}(c)$

for every $m \geq 0$ and $c \in \mathcal{C}_{\mathcal{A}}$.

Let us observe that, for a given configuration $c = (h_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathcal{H})$, its orbit described by the evolution of the LCA $\mathcal{A} = (\mathcal{H}, N_1, \delta)$ can be computed according to the formulas:

$$F^m_{\mathcal{A}}(c) = c' = (h'_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathcal{H}),$$

where

$$h'_n = \sum_{l=-m}^{m} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_1}} \delta_{i_1}\delta_{i_2}\ldots\delta_{i_m}(h_{l+n}), \qquad m \geq 1,\ n \in \mathbb{Z}. \quad (2)$$

We proceed inductively. Let us firstly note that, for $m = 1$,

$$h'_n = \delta_{-1}(h_{n-1}) + \delta_0(h_n) + \delta_1(h_{n+1}), \quad n \in \mathbb{Z}.$$

We suppose now that (2) holds true for a given $m \geq 1$. Then

$$F^{m+1}_{\mathcal{A}}(c) = F_{\mathcal{A}}(c') = c'' = (h''_n)_{n\in\mathbb{Z}} \in \ell^2_{\mathbb{Z}}(\mathcal{H}),$$

where

$$\begin{aligned} h''_n &= \delta_{-1}(h'_{n-1}) + \delta_0(h'_n) + \delta_1(h'_{n+1}) \\ &= \delta_{-1} \sum_{l=-m}^{m} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_1}} \delta_{i_1}\ldots\delta_{i_m}(h_{l+n-1}) \\ &\quad + \delta_0 \sum_{l=-m}^{m} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_1}} \delta_{i_1}\ldots\delta_{i_m}(h_{l+n}) \\ &\quad + \delta_1 \sum_{l=-m}^{m} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_1}} \delta_{i_1}\ldots\delta_{i_m}(h_{l+n+1}) \\ &= \sum_{l=-m-1}^{m-1} \sum_{\substack{i_1+\cdots+i_m=l+1 \\ i_1,\ldots,i_m\in N_1}} \delta_{-1}\delta_{i_1}\ldots\delta_{i_m}(h_{l+n}) \\ &\quad + \sum_{l=-m}^{m} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_1}} \delta_0\delta_{i_1}\ldots\delta_{i_m}(h_{l+n}) \\ &\quad + \sum_{l=-m+1}^{m+1} \sum_{\substack{i_1+\cdots+i_m=l-1 \\ i_1,\ldots,i_m\in N_1}} \delta_1\delta_{i_1}\ldots\delta_{i_m}(h_{l+n}), \\ &\qquad n \in \mathbb{Z}. \end{aligned}$$

If $i_1, i_2, \ldots, i_m \in N_1$ then

$$-m \leq i_1 + i_2 + \cdots + i_m \leq m.$$

Consequently equalities of the form $i_1 + i_2 + \cdots + i_m = k$, with $k \in \{-m-2, -m-1, m+1, m+2\}$ are not possible. In other words,

$$h''_n = \sum_{l=-m-1}^{m+1} \sum_{\substack{i_1+\cdots+i_m+i=l \\ i_1,\ldots,i_m,i\in N_1}} \delta_i\delta_{i_1}\ldots\delta_{i_m}(h_{l+n}), \quad n \in \mathbb{Z},$$

as required.

In general, for a neighborhood of radius r, formula (2) becomes

$$h'_n = \sum_{l=-rm}^{rm} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_r}} \delta_{i_1}\ldots\delta_{i_m}(h_{l+n}), \qquad m \geq 1,\ n \in \mathbb{Z}. \quad (3)$$

We are now in position to characterize the notion of power dilatability using only local rules:

Proposition 5: Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_s, \varepsilon)$ be two LCAs such that $\mathcal{H}$ is a closed subspace of $\mathcal{K}$ and $r \leq s$. Then $\mathcal{B}$ is a power dilation of $\mathcal{A}$ if and only if

$$\sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_r}} \delta_{i_1}\ldots\delta_{i_m}(h) = P^{\mathcal{K}}_{\mathcal{H}} \sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_s}} \varepsilon_{i_1}\ldots\varepsilon_{i_m}(h), \qquad m \geq 1,\ l \in N_{rm},\ h \in \mathcal{H}$$

and

$$\sum_{\substack{i_1+\cdots+i_m=l \\ i_1,\ldots,i_m\in N_s}} \varepsilon_{i_1}\ldots\varepsilon_{i_m}(h) \perp \mathcal{H}, \qquad m \geq 1,\ l \in N_{sm} \setminus N_{rm},\ h \in \mathcal{H}.$$

The connection between the notions of dilatability and power dilatability follows as a consequence:

Corollary 6: Let $\mathcal{A} = (\mathcal{H}, N_r, \delta)$ and $\mathcal{B} = (\mathcal{K}, N_r, \varepsilon)$ be two LCAs. If $\mathcal{B}$ is a dilation of $\mathcal{A}$ then it is also a power dilation of $\mathcal{A}$.

We combine Theorem 3 with Corollary 6 to obtain that:

Corollary 7: Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a LCA such that δ is a row contraction. Then $\mathcal{A}$ has a power dilation $\mathcal{B}$ such that $\frac{1}{\sqrt{3}}F_{\mathcal{B}}$ is an isometric operator (if we consider neighborhoods of radius r then $\frac{1}{\sqrt{3}}F_{\mathcal{B}}$ must be replaced by $\frac{1}{\sqrt{2r+1}}F_{\mathcal{B}}$).

Our main theorems are (power) dilatability results for partial isometric LCAs:

Theorem 8: Let $\mathcal{A} = (\mathcal{H}, N, \delta)$ be a partial isometric LCA. Then there exists an isometric LCA $\mathcal{B} = (\mathcal{K}, N_2, \varepsilon)$ which power dilates $\mathcal{A}$.

Proof: The proof uses some arguments proposed by J.J. Schäffer [12] for the matrix construction of the unitary dilation for a given contraction.

Let us firstly observe that the map

$$\ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}}) \ni (c_n)_{n\geq 0} \overset{F}{\mapsto} (F_{\mathcal{A}}(c_0), c_0 - F^*_{\mathcal{A}}F_{\mathcal{A}}(c_0), c_1, c_2, \ldots) \in \ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}})$$

is isometric:

$$\begin{aligned}\|F_{\mathcal{A}}(c)\|^2 + \|c - F^*_{\mathcal{A}}F_{\mathcal{A}}(c)\|^2 &\\ = & \ \|F_{\mathcal{A}}(c)\|^2 + \langle c - F^*_{\mathcal{A}}F_{\mathcal{A}}(c), c\rangle \\ = & \ \|F_{\mathcal{A}}(c)\|^2 + \|c\|^2 - \|F_{\mathcal{A}}(c)\|^2 \\ = & \ \|c\|^2\end{aligned}$$

for any $c \in \mathcal{C}_{\mathcal{A}}$. In addition, for any $m \geq 1$, it holds

$$F^m((c_n)_{n\geq 0}) = (F^m_{\mathcal{A}}(c_0), (1 - F^*_{\mathcal{A}}F_{\mathcal{A}})F^{m-1}_{\mathcal{A}}(c_0), \ldots, \\ (1 - F^*_{\mathcal{A}}F_{\mathcal{A}})(c_0), c_1, c_2, \ldots), \quad (c_n)_{n\geq 0} \in \ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}}).$$

We identify $\mathcal{C}_{\mathcal{A}}$ with a (closed) subspace of $\ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}})$ via the embedding

$$\mathcal{C}_{\mathcal{A}} \ni c \mapsto (c, 0, 0, \ldots) \in \ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}})$$

and deduce that

$$F^m(c) = P^{\ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}})}_{\mathcal{C}_{\mathcal{A}}} F^m_{\mathcal{A}}(c), \quad m \geq 0, \ c \in \mathcal{C}_{\mathcal{A}}.$$

Any element $(c_n)_{n\geq 0} \in \ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}})$ of the form $c_n = (h_{mn})_{m\in\mathbb{Z}}$, $n \geq 0$ can be written as $(k_m)_{m\in\mathbb{Z}}$, where $k_m = (h_{mn})_{n\geq 0}$, $m \geq 0$. Therefore $\ell^2_{\mathbb{Z}_+}(\mathcal{C}_{\mathcal{A}}) = \ell^2_{\mathbb{Z}_+}(\mathcal{K})$, where $\mathcal{K} = \ell^2_{\mathbb{Z}_+}(\mathcal{H})$ and, under this identification, the elements of $\mathcal{K}$ correspond to the position of k_0, while the elements of $\mathcal{H}$ to the position of h_{00}.

F is the global transition function of a LCA $\mathcal{B} = (\mathcal{K}, N_2, \varepsilon)$, where $\varepsilon_i = (\varepsilon^i_{mn})_{m,n\geq 0}$, $i \in N_2$ are diagonal matrices,

$$\varepsilon^{-2}_{00} = \begin{pmatrix} 0_{\mathcal{H}} \\ -\delta^*_1\delta_{-1}\end{pmatrix}, \quad \varepsilon^{-2}_{mm} = 0_{\mathcal{H}} \ (m \geq 1),$$

$$\varepsilon^{-1}_{00} = \begin{pmatrix} \delta_{-1} \\ -\delta^*_1\delta_0 - \delta^*_0\delta_{-1}\end{pmatrix}, \quad \varepsilon^{-1}_{mm} = 0_{\mathcal{H}} \ (m \geq 1),$$

$$\varepsilon^{0}_{00} = \begin{pmatrix} \delta_{0} \\ 1_{\mathcal{H}} - \delta^*_{-1}\delta_{-1} - \delta^*_0\delta_0 - \delta^*_1\delta_1\end{pmatrix}, \\ \varepsilon^{0}_{mm} = 1_{\mathcal{H}} \ (m \geq 1),$$

$$\varepsilon^{1}_{00} = \begin{pmatrix} \delta_{1} \\ -\delta^*_0\delta_1 - \delta^*_{-1}\delta_0\end{pmatrix}, \quad \varepsilon^{1}_{mm} = 0_{\mathcal{H}} \ (m \geq 1)$$

and

$$\varepsilon^{2}_{00} = \begin{pmatrix} 0_{\mathcal{H}} \\ -\delta^*_{-1}\delta_1\end{pmatrix}, \quad \varepsilon^{2}_{mm} = 0_{\mathcal{H}} \ (m \geq 1).$$

■

Let us now suppose that the LCA $\mathcal{A}^*$ is isometric. Then $\mathcal{A}$ is partial isometric so, according to Theorem 8, can be power dilated to an isometric LCA $\mathcal{B}$. If we restrict $F_{\mathcal{B}}$ to the (closed) subspace $\mathcal{C}_{\mathcal{A}} \oplus \ell^2_{\mathbb{Z}^*_+}(\ker F_{\mathcal{A}})$ we obtain a unitary transformation which is the global transition function of a quantum LCA $\mathcal{B}'$. It follows that $\mathcal{A}$ is power dilated to $\mathcal{B}'^*$. Since a unitary dilation of an isometric operator is actually an extension we obtain:

Corollary 9: Any isometric LCA $\mathcal{A}$ can be power dilated to a quantum LCA $\mathcal{B}$ such that F_A is extended by $F_{\mathcal{B}}$.

Theorem 8 and Corollary 9 show that:

Theorem 10: Any partial isometric LCA can be power dilated to a quantum LCA.

IV. Conclusion and Future Work

The paper continues our previous investigations on linear cellular automata [7], [9]. We introduced two notions of dilatability between LCAs. The first one involves the local rules. We prove, in this situation, that any LCA $\mathcal{A} = (\mathcal{H}, N, \delta)$ such that δ is a row contraction can be dilated to a LCA $\mathcal{B} = (\mathcal{K}, N, \varepsilon)$ such that ε is a row contraction of isometric operators having pairwise orthogonal ranges. The second notion of dilatability, called here power dilatability, relates the global rules and it is weaker than the first one. But, by our point of view, it is more important. Our reasons are that any partial isometric LCA can be dilated to a quantum LCA, hence a reversible LCA. In particular, any isometric LCA can be dilated to a quantum LCA. The transitions are made, in this case, only with the dilation LCA, which is information preserving. We can always go back to the original LCA by compressing.

It is our aim, in the near future, to find other classes or even to characterize the class of LCAs that can be dilated to quantum or at least to reversible LCAs. The multidimensional case also remains open. In this context we also wish to enlarge the applicability area.

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