Controllability of Autonomous Behaviors and Livšic Overdetermined Systems as 2D Behaviors with Pure Autonomy Degree One

G. M. Boquet and J. A. Ball

***Abstract*— In [8] discrete time Livšic systems are related to 2D behaviors with autonomy degree one. A necessary component for transitioning from a behavior to a Livšic system is a "controllability theory" for autonomous behaviors. It turns out that controllability for behaviors is a special case of the presented j-controllability. First we present a brisk overview of the ingredients that go into (algebraic) $\mathcal{D}_r$-controllability and (trajectory) j-controllability. We conclude with results demonstrating Livšic controllability implies 1-controllability and then $\mathcal{D}_r$-controllability by a series of reduction steps.**

I. Introduction

Behavioral systems theory contains a wealth of available abstraction from its connections to both algebraic geometry and commutative algebra. For instance, in [13] Oberst derives many results for discrete systems over coordinate rings via a series of simplifications. By focusing on autonomous behaviors with prime annihilators, the so-called induced duality in [13, pg. 46–47] can then be exploited. While this opens up some algebraic theory, it does not lend itself easily to a trajectory-based theory.

Outside of behavioral systems theory one encounters 2D discrete Livšic systems in the zoo of multidimensional input/state/output systems. Unlike Givonne-Rosser systems, Livšic systems have multiple *dependent* state evolution equations where the state-transition operators commute. Consistency of state evolution and the commutativity of the state-transition operators leads to restrictions on the input; that is, the input cannot be freely selected as one is accustomed. This means Livšic systems, from a behavioral point of view, are autonomous; however, since Livšic systems have their own form of controllability and transfer functions, this also indicates that the classical behavioral framework, where the operator ring is a polynomial ring, is insufficient.

In [8], one of the authors compiles a basic framework for allowing autonomous behaviors to have both controllable parts and transfer functions. The purpose of this paper is to present a bird's-eye-view of the mentioned approach. We first build up to the reduced behavior machinery and then move to its two forms of controllability: $\mathcal{D}_r$-controllability and j-controllability; these developments then lead us to a relationship between Livšic controllability and both 1-controllability and $\mathcal{D}_r$-controllability.

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Since the trajectory-based part of the theory is still rather new, it has some room for improvement; nevertheless, it already is able to allow behaviors to handle a broader class of problems both theoretically and computationally.

Notation. Let k be a field and $d \in \mathbb{N}$ be a positive integer. $\mathcal{V}(I)$ is the algebraic set of the ideal I. $\mathrm{NF}(-,-)$ denotes the normal form given a Gröbner basis. For more details on the normal form we refer the reader to [10].

II. Behavioral Systems

Note. The following survey of basic behavioral systems theory is meant to serve as a review rather than a detailed exposition. We heartily recommend [18], [13], [21], [14] for a more detailed treatment of single and multidimensional behavioral systems theory.

We begin with the discrete signal space

$$\mathcal{A} = \{w : \mathbb{N}^d \to k\}.$$

We are concerned with the shift operators $z_1, \ldots, z_d$ defined on $w \in \mathcal{A}$ by

$$z_i w(t_1, \ldots, t_i, \ldots, t_d) = w(t_1, \ldots, t_i + 1, \ldots, t_d).$$

We also consider $\mathcal{A}$ as a k-vector space via pointwise operations. Defining $\mathcal{D} = k[z_1, \ldots, z_d]$ immediately reveals that $\mathcal{A}$ is a $\mathcal{D}$-module.

We define the non-degenerate k-bilinear pairing

$$\langle -, - \rangle : \mathcal{A}^q \times \mathcal{D}^q \to \mathcal{A}$$

given by

$$\langle \{w\}_{i=1}^q, \{f\}_{i=1}^q \rangle = \sum_{i=1}^q f_i w_i \qquad w \in \mathcal{A}^q,\ f \in \mathcal{D}^q.$$

In this way, a polynomial matrix $R \in \mathcal{D}^{p \times q}$ induces the map $R : \mathcal{A}^q \to \mathcal{A}^p$.

We define a **behavior** as a linear shift-invariant $\mathcal{D}$-module that is the kernel of a polynomial matrix, i.e., a **kernel representation**; that is, for a behavior $\mathcal{B}$, there exists an $R \in \mathcal{D}^{p \times q}$ such that

$$\mathcal{B} = \ker_{\mathcal{A}}(R) = \{w \in \mathcal{A} : Rw = 0\} \subset \mathcal{A}^q.$$

We emphasize that R is in no way unique.

Consider a behavior $\mathcal{B} = \ker_A(R) \subset A^q$ where $R \in D^{p \times q}$. Let the **output index set**, $\mathcal{O} \subset \{1, \ldots, q\}$, be a subset of indices and the **input index set**, $\mathcal{I} = \{i \in \{1, \ldots, q\} : i \notin \mathcal{O}\}$ be its complement; note that, as consequence of the decomposition, we have $\mathcal{I} \cup \mathcal{O} = \{1, \ldots, q\}$ and $\mathcal{I} \cap \mathcal{O} = \emptyset$. We call the sets $(\mathcal{I}, \mathcal{O})$ an **i/o-structure** of $\mathcal{B}$.

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For a given i/o-structure of a behavior, decomposition of the kernel representation into an input/output form goes as follows. Let $R = [r_1, \ldots, r_q]$ be the columns of R; for an i/o-structure $(\mathcal{I}, \mathcal{O})$ we write

$$P = [r_i]_{i \in \mathcal{O}} \qquad\qquad Q = [-r_i]_{i \in \mathcal{I}}.$$

In this way, we have the equivalent ways of defining $\mathcal{B}$,

$$\mathcal{B} = \{w \in A^q : Rw = 0\} = \{(u, y) \in A^{\mathcal{I} \times \mathcal{O}} : Py = Qu\}.$$

For a behavior $\mathcal{B} = \ker_A(R) \subset A^q$, we define the **input dimension** as $n_\mathcal{I} = q - \operatorname{rank}(R)$. Similarly, we define the **output dimension** as $n_\mathcal{O} = \operatorname{rank}(R)$. Through this terminology, we introduce the following special type of i/o-structure that is commonly employed.

Definition 2.1: For a behavior $\mathcal{B}$ with input dimension $n_\mathcal{I}$ and output dimension $n_\mathcal{O}$ we say that an i/o-structure $(\mathcal{I}, \mathcal{O})$ is a **full i/o-structure** if $|\mathcal{I}| = n_\mathcal{I}$ and $|\mathcal{O}| = n_\mathcal{O}$.

A. The Cauchy Problem

In the presentation of j-controllability we implicitly employ specific details of the Cauchy problem. We begin with the concept of an initial condition set of a behavior given a free $\mathcal{D}$-module ordering and an i/o-structure.

Definition 2.2: For a behavior $\mathcal{B} = \ker_\mathcal{A} \begin{bmatrix} -Q & P \end{bmatrix}$ with given i/o-structure, $P \in \mathcal{D}^{p \times q}$, $\mathcal{D}$-module $U = \operatorname{im}_\mathcal{D}(P^T)$ and orders $\leq_l$ on $\mathbb{N}^d$ and $\leq_m$ on $\{1, \ldots, q\} \times \mathbb{N}^d$, we define its **leading subset** as

$$L^\mathbb{N}(U) = \{(i, \alpha) \in \{1, \ldots, q\} \times \mathbb{N}^d : z_1^{\alpha_1} \cdots z_d^{\alpha_d} e_i \in L(U)\},$$

where $L(U)$ is the leading submodule of U (see [10].) We also define the complementary set

$$G(U) = (\{1, \ldots, q\} \times \mathbb{N}^d) \setminus L^\mathbb{N}(U)$$

as the set of **initial conditions**. When a choice of orderings and i/o-structure is understood, we write $\partial\mathcal{B}$ to denote the set $G(U)$.

According to the following, the above is well-defined under any choice of kernel representation.

Lemma 2.1: [13] For a behavior $\mathcal{B} = \ker_\mathcal{A} \begin{bmatrix} -Q & P \end{bmatrix}$ with given i/o-structure, $\mathcal{D}$-module $U = \operatorname{im}_\mathcal{D}(P^T)$ and global orders $\leq_l$ on $\mathbb{N}^d$ and $\leq_m$ on $\{1, \ldots, q\} \times \mathbb{N}^d$, the sets $L^\mathbb{N}(U)$ and $G(U)$ depend on the i/o-structure used and the orderings, but not on the choice of P.

From the notation of Definition 2.2 we reach

Theorem 2.1: [13] Let $\mathcal{B} = \ker_\mathcal{A}(R)$, where $R \in \mathcal{D}^{p \times q}$, be a behavior with a given full i/o-structure through the decomposition $R = \begin{bmatrix} -Q & P \end{bmatrix}$ and initial condition set $\partial\mathcal{B}$ provided by some given global orderings. Then the **canonical Cauchy problem**

$$Py = Qu, \quad y|_G = x \quad u \in A^{n_\mathcal{I}},\ x \in k^{\partial\mathcal{B}},\ y \in \mathcal{A}^{n_\mathcal{O}}$$

is uniquely solvable.

On one hand, the Cauchy problem provides a way to solve for trajectories from input and initial conditions; on the other hand, it provides a way to understand the unique extension of solutions for when $n_\mathcal{I} = 0$. It is this latter situation that we now consider.

B. Autonomous Behaviors

Behaviors that satisfy polynomial relationships independent of the component form a special class of behaviors commonly referred to as autonomous behaviors. There are several ways of defining autonomous behaviors, however, all of these equivalent definitions are intrinsically tied to the Cauchy problem. We take the approach of defining autonomy via the annihilator of the behavior.

Definition 2.3: For the behavior $\mathcal{B} \subset \mathcal{A}^q$ we define the **annihilator ideal** as

$$\operatorname{Ann}(\mathcal{B}) = \{f \in \mathcal{D} : fw = 0 \text{ for all } w \in \mathcal{B}\}.$$

If $\mathcal{B}$ is such that $\operatorname{Ann}(\mathcal{B}) \neq 0$, then we say that $\mathcal{B}$ is an **autonomous behavior**.

Our focus is to allow autonomous behaviors to exhibit kernels (hence, zero annihilators over some ring.)

C. Extensions of Autonomy Degree

In [19] the notion of **autonomy degree** of a behavior is defined and tied to the dimension of the behavior's annihilator. This development can be seen as the module-theoretic extension of results that are well established in commutative/computational algebra. When working with ideals in the context of computational algebra, it is often beneficial to analyze their initial ideals (leading monomial ideals); such ideals not only contain a great deal of information, but, as seen in [13], directly relate behaviors to the Cauchy problem. Through the Hilbert function the dimension of the leading ideal determines the dimension of the ideal. In turn, this relationship allows us to use the "size" of the initial condition set as a property that classifies autonomous behaviors.

Rather than following the original treatment in [19], we diverge towards a somewhat broader approach; this additional generality proves to be useful when considering j-controllability in the following sections. Let us first start with dimension.

Definition 2.4: For a behavior $\mathcal{B}$, we define the **dimension** of $\mathcal{B}$ as $\dim(\mathcal{B}) = \dim(\mathcal{D}/\operatorname{Ann}(\mathcal{B}))$. Here, dimension of the quotient ring is given by the transcendence degree of the field extension from k. (See [2], [10].)

The next goal is to relate a behavior's dimension to the size of an initial condition set for the behavior. Recall briefly that a subset $G \subset \mathbb{N}^d$ is a **cw-ideal** if for any $p \in G$ we have $p + \mathbb{N}^d \subset G$. For its complement we have the following

Definition 2.5: We call a subset $G \subset \mathbb{N}^d$ a **staircase** if $\mathbb{N}^d \setminus G$ is a cw-ideal.

To handle arbitrary subsets, we have the following.

Definition 2.6: For a subset $G \subset \mathbb{N}^d$ we define the **staircase closure**[1] of G, denoted $\overline{G}$, as

$$\overline{G} = \mathbb{N}^d \setminus \left(\bigcup_{g \in \{p \in \mathbb{N}^d : (p + \mathbb{N}^d) \cap G = \emptyset\}} g + \mathbb{N}^d \right).$$

A straightfoward verification demonstrates that the staircase closure operation behaves as expected: for $G \subset \mathbb{N}^d$ we

[1] Compare the staircase closure to the erosion operation in binary morphology. See [9].

have $\overline{G}$ is a staircase, $G \subset \overline{G}$ and if G is staircase then $G = \overline{G}$.

Definition 2.7: Let $u \subset \{z_1, \ldots, z_d\}$ be a non-empty subset of the indeterminates of $\mathcal{D}$. For $u = \{z_{i_1}, \ldots, z_{i_{|u|}}\}$ we define the **sublattice**

$$\mathcal{L}[u] = \text{span}_{\mathbb{N}}\{e_{i_1}, \ldots, e_{i_{|u|}}\} \subset \mathbb{N}^d,$$

were $e_1, \ldots, e_d$ are the standard basis vectors of $\mathbb{N}^d$. In this case we say that $\mathcal{L}[u]$ is an $|u|$**-dimensional sublattice** of $\mathbb{N}^d$. For $u = \emptyset$ we define the 0-dimensional lattice as being the point $0 \in \mathbb{N}^d$.

By Dickson's Lemma (see [10]) the initial condition set $\partial\mathcal{B}$ for a behavior $\mathcal{B}$ has a complement in $\mathbb{N}^d$ that is a cw-ideal and hence is *always* a staircase.

Definition 2.8: For a staircase $G \subset \mathbb{N}^d$ we define the **dimension** of G to be the largest value ℓ for which there is an ℓ-dimensional sublattice contained in G. We denote the dimension of G by $\dim(G)$. For an arbitrary set $G' \subset \mathbb{N}^d$, we define $\dim(G') = \dim(\overline{G'})$ where $\overline{G'}$ is the staircase closure as in Definition 2.6. For $G'' = \cup_{i=1}^q \{i\} \times G''_i \subset (\mathbb{N}^d)^q$, we define $\dim(G'')$

$$\dim(G'') = \max_{1 \leq i \leq q} \dim(G''_i).$$

This leads us to the notion of the autonomy degree of a behavior.

Definition 2.9: [19] For a non-trivial d-dimensional behavior $\mathcal{B} \subset \mathcal{A}^q$ we define its **autonomy degree** as $d - \dim(\partial\mathcal{B})$. We denote the autonomy degree of $\mathcal{B}$ as $\text{adeg}(\mathcal{B})$.

D. Reduced Behaviors

The key process introduced here is that of **reduction** of a behavior by an ideal. Although the process is simple, under certain assumptions, it allows autonomous behaviors to be studied by the same tools that are commonly employed in the analysis of behaviors with free variables.

Definition 2.10: Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be an autonomous behavior and $I \subset \mathcal{D}$ be an ideal such that $I \subset \text{Ann}(\mathcal{B})$. Let $\mathcal{D}_r$ denote the **reduced** ring $\mathcal{D}_r = \mathcal{D}/I$. The signal space associated with $\mathcal{D}_r$ is

$$\mathcal{A}_r = \{w \in \mathcal{A} : pr = 0 \text{ for all } p \in I\}.$$

We define the behavior **reduced** by I as

$$\mathcal{B} \otimes \mathcal{D}_r = \{w \in \mathcal{A}_r : (R \otimes 1_{\mathcal{D}_r})w = 0\} = \ker_{\mathcal{A}_r}(R \otimes 1_{\mathcal{D}_r}).$$

If $\mathcal{B}$ is reduced by its annihilator, then we simply write $\mathcal{B}_r$. We reserve the symbols $\mathcal{B}_r$, $\mathcal{A}_r$, and $\mathcal{D}_r$ to denote, respectively, the reduced behavior, signal space, and ring of a behavior.

In [13, page 46-47] Oberst shows that $\mathcal{A}_r$, as stated above, is a large injective cogenerator over finitely generated $\mathcal{D}_r$-modules. We emphasize that our application of reduced behaviors is, for the most part, not new. Oberst's results in [13] set the stage or created many of the concepts we introduce; however, since the theory in [loc. cit.] is mainly of an algebraic nature, it is necessary to have consistent terminology when we link it to the trajectory-based concepts in the next section.

III. Controllability

The classical point-to-point sense of controllability is concerned with reaching desired state values from a finite string of inputs and arbitrary initial state. Since this concept requires the existence of a state space, it clearly is not suitable for the behavioral approach to systems. In [18] Willems presents a definition of controllability that is based on concatenation of trajectories rather than reaching a desired state; when a behavior is considered as an input/state/output system by constructing latent variables for the state, then these two definitions are equivalent.

In [14] Rocha presents the first multidimensional setting of controllability for the two-dimensional case. This can be seen as a natural extension of the original definitions presented by Willems. This development is then followed by the complete generalization to d-dimensional systems by Zerz and Wood [20]. From the equivalent definitions in [loc. cit.], it is clear that autonomous behaviors are not controllable. We now generalize controllability to accommodate certain annihilators.

A. Generalizations of Controllability

In the reduced behavior setting all of the algebraic aspects of controllability that exist in the classical theory continue to hold; note, however, that the change of ring/signal space is necessary for this to hold. The following theorem sums up results derived in both [13] and [8]; we refer the reader to either source for more elaboration on terminology not previously defined.

Theorem 3.1: Let $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ be a reduced behavior, where the associated ring $\mathcal{D}_r$ is an affine domain, and $\mathcal{M}_r = \text{coker}_{\mathcal{D}_r}(R^T)$ be its dual module. The following are equivalent.

1) $\mathcal{M}_r$ is torsion-free
2) $\mathcal{B}_r$ has an image representation.
3) R is generalized factor left prime over $\mathcal{D}_r$
4) $\mathcal{B}_r$ is minimal in its transfer class.
5) $\mathcal{B}_r$ is divisible.

Since these properties are all algebraic in nature, we have the following

Definition 3.1: Let $\mathcal{B}_r = \ker_{\mathcal{A}_r}(R)$ be a reduced behavior, where the associated ring $\mathcal{D}_r$ is an affine domain. We say that $\mathcal{B}_r$ is $\mathcal{D}_r$-**controllable** if it satisfies any of the equivalent properties in Theorem 3.1.

We now move onto a trajectory-based form of controllability for reduced behaviors.

Definition 3.2: Let $\mathcal{D}_r = \mathcal{D}/I$ be an affine domain, $R \in \mathcal{D}^{p \times q}$ and G be a reduced Gröbner basis of I under a provided global ordering. We define the **support** of R, denoted by $\text{supp}(R)$, as follows. Let $\text{NF}(R|G)$ be the matrix consisting of the normal form of the entries of R, i.e., for $R = \{r_{i,\ell}\}$ we have $\text{NF}(R|G)_{i,\ell} = \text{NF}(r_{i,\ell}|G)$. We may write

$$\text{NF}(R|G) = \sum R_i \mathbf{z}^i$$

where each R_i is a constant matrix. From this we define

$$\text{supp}(R) = \{i \in \mathbb{N}^d : R_i \neq 0\}.$$

We define the **closed support**, $\overline{\text{supp}}(R)$, to be the smallest ($d$-dimensional) interval containing $\text{supp}(R)$. We also define the **diameter** of $\text{supp}(R)$ as

$$\rho(\text{supp}(R)) = \max_{a,b \in \text{supp}(R)} \underbrace{\left(\sum_{i=1}^{d} |a_i - b_i| \right)}_{d(a,b)}.$$

Note that the choice of ordering provides a well-defined notion of support that would otherwise be absent when working over a quotient ring. This is crucial in the following

Definition 3.3: For $j \in \{1, \ldots, d\}$ and $\mathcal{B}$ a d-dimensional behavior we say that $\mathcal{B}$ is **j-controllable** if there exists a j-dimensional lattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$ and $|u| = j$, (the **motion sublattice**) such that $\mathcal{L}[u] \subset \partial\text{Ann}(\mathcal{B})$ and, for any $T_1 \subset \partial\text{Ann}(\mathcal{B})$ such that $\dim(T_1) < j$ and *finite* subset $J \subset \mathbb{N}^d$, there exists an integer (the **separation distance**) $\tau(T_1 + J) \geq 0$ such that, for any $w_1, w_2 \in \mathcal{B}$ and $b \in \mathcal{L}[u]$ with $d(T_1 + J, b + \mathbb{N}^d) > \tau(T_1 + J)$, there exists $w \in \mathcal{B}$ such that

$$w(t) = \begin{cases} w_1(t) & t \in T_1 + J \\ w_2(t - b) & t \in b + \mathbb{N}^d. \end{cases}$$

By extending the proof of Zerz and Wood [20] with some additional machinery we obtain the following.

Theorem 3.2: [8] Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ be a given behavior. Provided $\dim(\mathcal{B}) = j$, $\mathcal{B}$ is j-controllable, and $\text{Ann}(\mathcal{B})$ is a prime ideal, then $\mathcal{B}_r$ is $\mathcal{D}_r$-controllable.

A slight adjustment of the proof of Rocha in [14] yields the converse direction.

Theorem 3.3: [8] Let $\mathcal{B} \subset \mathcal{A}^q$ be a behavior. If $\text{Ann}(\mathcal{B})$ is a prime ideal, the reduced signal space $\mathcal{A}_r$ is j-controllable with motion sublattice $\mathcal{L}[u]$, $u \subset \{z_1, \ldots, z_d\}$, $|u| = j$, $\mathcal{L}[u] \subset \partial\text{Ann}(\mathcal{B})$, and $\mathcal{B}_r$ has image representation M over its reduced ring, then $\mathcal{B}$ is j-controllable with motion sublattice $\mathcal{L}[u]$.

There are a few interesting properties of j-controllable and $\mathcal{D}_r$-controllable behaviors. For the purposes of this paper, the most apparent is that $\mathcal{D}_r$-controllability is invariant under $\mathcal{D}$-automorphism.

Lemma 3.1: [8] Let $\mathcal{B} = \ker_{\mathcal{A}}(R)$ and $\mathcal{B}' = \ker_{\mathcal{A}}(R')$ be two given behaviors, $\phi : \mathcal{D} \to \mathcal{D}$ a k-algebra automorphism. If $\phi(R') = R$, $\text{Ann}(\mathcal{B})$ is a prime ideal, and $\mathcal{B}_r$ is $\mathcal{D}/\text{Ann}(\mathcal{B})$-controllable then $\mathcal{B}'_r$ is $\mathcal{D}/\text{Ann}(\mathcal{B}')$-controllable.

With behaviors over $\mathcal{D}$ the concept of a "change of variables" is not commonly employed. However, this attitude changes in the following section.

Remark. The definition of j-controllability differs from conventional behavioral controllability in two ways. One way is that the separation distance is not constant, but depends on the set $T_1 + J$. To understand why this is necessary, one can look at Lemma 4.1. It turns out that the separation distance should not really depend on $T_1 + J$, but the size of J. One can also see that the definition of d-controllability is not as accommodating as the classical definition of controllability as found in [20]. The reason is that the definition provided is the bare minimum required for proving the relation to having an image representation.

IV. Livšic Systems

Throughout this section we are concerned with discrete-time 2D input/state/output colligations of the form

$$\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \\ C & D \end{bmatrix} : \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{E} \end{bmatrix} \to \begin{bmatrix} \mathcal{X} \\ \mathcal{X} \\ \mathcal{E}_* \end{bmatrix},$$

which are given by

$$\begin{aligned} x(t + e_1) &= A_1 x(t) + B_1 u(t) \\ x(t + e_2) &= A_2 x(t) + B_2 u(t) \\ y(t) &= C x(t) + D u(t). \end{aligned} \tag{1}$$

In the above, $t = (t_1, t_2) \in \mathbb{N}^2$ and the coordinates e_1 and e_2 denote the standard basis vectors for the free $\mathbb{N}$-module $\mathbb{N}^2$. In this way, $t + e_1$ represents a backward shift in the horizontal direction and $t + e_2$ represents a backward shift in the vertical direction. The vector $x(t)$ takes values in the state space $\mathcal{X}$, $u(t)$ takes values in the input space $\mathcal{U}$, and $y(t)$ takes values in the output space $\mathcal{Y}$. The state, input, and output spaces are all finite dimensional complex vector spaces. References for Livšic systems include [5], [15], [6], [16], [17], [12], [1], [7], [11].

A. Preliminaries

The system given by (1) is **overdetermined** in the sense that there are two ways to compute $x(t + e_1 + e_2)$ from $x(t)$:

$$\begin{aligned} x(t) &\mapsto x(t + e_1) \mapsto x(t + e_1 + e_2) \\ x(t) &\mapsto x(t + e_2) \mapsto x(t + e_2 + e_1). \end{aligned}$$

This leads us to the discussion of **compatibility conditions** for the system to exhibit solutions.

(Zero Input Evolution). Let $u(t) = 0$ for all $t \in \mathbb{N}^2$ with an arbitrary initial state $x(0)$. By the state-evolution equations in (1), we reach the following.

$$\begin{aligned} x(e_1 + e_2) &= A_2 x(e_1) = A_2 A_1 x(0) \\ x(e_1 + e_2) &= A_1 x(e_1) = A_1 A_2 x(0). \end{aligned}$$

Because the above relation must hold for every choice of $x(0) \in \mathcal{X}$, it must be the case that A_1 and A_2 commute. This leads us to the first requirement.

$$A_1 A_2 = A_2 A_1. \tag{2}$$

(Input Signal Compatibility). If a nonzero input signal is present the two paths for state evolution lead us to the following equations.

$$\begin{aligned} x(t + e_1 + e_2) &= A_1 x(t + e_2) + B_1 u(t + e_2) \\ &= A_1 (A_2 x(t) + B_2 u(t)) + B_1 u(t + e_2) \\ x(t + e_1 + e_2) &= A_2 x(t + e_1) + B_2 u(t + e_1) \\ &= A_2 (A_1 x(t) + B_1 u(t)) + B_2 u(t + e_1). \end{aligned}$$

After subtracting the the top equation from the bottom equation and using the commutativity condition (2), we reach

$$B_2 u(t+e_1) - B_1 u(t+e_2) + (A_2 B_1 - A_1 B_2) u(t) = 0.$$

We now assume that we have factorizations of B_1, B_2, and $A_2B_1 - A_1B_2$ of the form

$$\begin{cases} B_1 = \widetilde{B}\sigma_1 \\ B_2 = \widetilde{B}\sigma_2 \\ A_2 B_1 - A_1 B_2 = A_2\widetilde{B}\sigma_1 - A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma \end{cases}$$

In the above, σ_1, σ_2, and γ are continuous linear mappings from the input space $\mathcal{U}$ into some auxiliary input space $\widetilde{\mathcal{U}}$; the continuous linear map $\widetilde{B} : \widetilde{\mathcal{U}} \to \mathcal{X}$ maps to the state space from the auxiliary input space. When the derived equalities are combined we reach

$$\widetilde{B}\left(\sigma_2 u(t+e_1) - \sigma_1 u(t+e_2) + \gamma u(t)\right) = 0.$$

One way for the above to hold is to assume that the input signal satisfies the following **input compatibility condition**:

$$\sigma_2 u(t+e_1) - \sigma_1 u(t+e_2) + \gamma u(t) = 0.$$

In a similar fashion the compatibility conditions for the output signals are derived. We refer the reader to [7], [4] for more details on this matter.

The stated factorizations and compatibility conditions lead us to the definition of a vessel.

Definition 4.1: We define a **vessel** to be a collection of operators

$$\mathfrak{V} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

where

$$\begin{aligned} A_1, A_2 \in \mathcal{L}(\mathcal{X}), \quad \widetilde{B} \in \mathcal{L}(\widetilde{\mathcal{U}}, \mathcal{X}), \quad C \in \mathcal{L}(\mathcal{X}, \mathcal{Y}), \\ D \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \quad \widetilde{D} \in \mathcal{L}(\widetilde{\mathcal{U}}, \widetilde{\mathcal{Y}}), \\ \sigma_1, \sigma_2, \gamma \in \mathcal{L}(\mathcal{U}, \widetilde{\mathcal{U}}), \quad \sigma_{1*}, \sigma_{2*}, \gamma_* \in \mathcal{L}(\mathcal{Y}, \widetilde{\mathcal{Y}}) \end{aligned}$$

are subject to conditions

$$\begin{cases} A_1 A_2 = A_2 A_1 \text{ (commutativity)} \\ A_2\widetilde{B}\sigma_1 - A_1\widetilde{B}\sigma_2 = \widetilde{B}\gamma \text{ (input vessel condition)} \\ \sigma_{1*} C A_2 - \sigma_{2*} C A_1 = \gamma_* C \text{ (output vessel condition)} \\ \sigma_{2*} C\widetilde{B}\sigma_1 - \sigma_{1*} C\widetilde{B}\sigma_2 + \gamma_* D = \widetilde{D}\gamma \\ \sigma_{1*} D = \widetilde{D}\sigma_1 \\ \sigma_{2*} D = \widetilde{D}\sigma_2 \text{ (linkage conditions)} \end{cases}$$

The derivation of the various compatibility conditions and factorizations leading up to the definition of a vessel naturally suggest the following properties of a vessel.

Proposition 4.1: [4] Let $\mathfrak{V}$ be a vessel associated with the input/state/output system

$$\begin{aligned} x(t+e_1) &= A_1 x(t) + B_1 u(t) \\ x(t+e_2) &= A_2 x(t) + B_2 u(t) \\ y(t) &= C x(t) + D u(t). \end{aligned}$$

The above system is consistent over $\mathbb{N}^2$ for any choice of initial state $x(0)$ as long as the input signal satisfies the **input compatibility condition**

$$\begin{aligned} U(z_1, z_2)u &= (\sigma_2 z_1 - \sigma_1 z_2 + \gamma)u \\ &= \sigma_2 u(t+e_1) - \sigma_1 u(t+e_2) + \gamma u(t) = 0. \end{aligned} \tag{3}$$

In this case, the output signal satisfies the **output compatibility condition**

$$\begin{aligned} U_*(z_1, z_2)y &= (\sigma_{2*} z_1 - \sigma_{1*} z_2 + \gamma_*)y \\ &= \sigma_{2*} y(t+e_1) - \sigma_{1*} y(t+e_2) + \gamma_* y(t) = 0. \end{aligned} \tag{4}$$

For the remainder of this paper we make the following assumptions.

ND-FD. The pencils $\xi_i\sigma_1 + \xi_2\sigma_2$ and $\xi_1\sigma_{1*} + \xi_2\sigma_{2*}$ are nonsingular.

Irreducibility/Maximality Assumption (IMA) *For the matrix pencils $U(z_1, z_2)$ and $U_*(z_1, z_2)$ we have $\det U(z_1, z_2) = f^r$ and $\det U_*(z_1, z_2) = f^{r_*}$ where f is an irreducible polynomial over $\mathbb{C}[z_1, z_2]$ and*

$$\dim \ker_{\mathbb{C}}(U(\boldsymbol{\lambda})) = r \text{ and } \dim \ker_{\mathbb{C}}(U_*(\boldsymbol{\lambda})) = r_*$$

for all $\boldsymbol{\lambda} \in \mathcal{V}(f) \setminus \mathbf{S}$, where $\mathbf{S}$ consists of the finite set of singularities of the curve $\mathcal{V}(f)$.

B. Vessels as 2D Behaviors with Degree One Autonomy

We now briefly present some material from [8], [3] that provides a behavioral interpretation of Livšic systems. Define the (tall) kernel representation

$$R^{i/s/o} = \begin{bmatrix} A_1 - z_1 I & \widetilde{B}\sigma_1 & 0 \\ A_2 - z_2 I & \widetilde{B}\sigma_2 & 0 \\ C & D & -I \\ 0 & U(z_1,z_2) & 0 \\ 0 & 0 & U_*(z_1,z_2) \end{bmatrix} \tag{5}$$

where U and U_* are as in (3) and (4). We then define the **augmented behavior associated with the vessel** as

$$\mathcal{B}^{i/s/o} = \left\{ w = \left[\begin{smallmatrix} x \\ u \\ y \end{smallmatrix}\right] \in (\mathcal{X} \oplus \mathcal{U} \oplus \mathcal{Y})^{\mathbb{N}^2} : R^{i/s/o} w = 0 \right\}.$$

It is a consequence of Proposition 4.1 that one can drop the last row of $R^{i/s/o}$ without changing the behavior but it is more convenient for our purposes to work with $R^{i/s/o}$ as it is in (5). The time domain is $\mathbb{N}^2$, the associated ring of operators is $\mathcal{D} = \mathbb{C}[z_1, z_2]$, the signal space is $\mathcal{A} = \mathbb{C}^{\mathbb{N}^2}$, $d = \dim(\mathcal{X}) + \dim(\mathcal{U}) + \dim(\mathcal{Y})$, and $\mathcal{B}^{i/s/o} \subset \mathcal{A}^d$.

After rewriting (5) in ARMA form we reach the following equations

$$\underbrace{\begin{bmatrix} \widetilde{B}\sigma_1 & 0 \\ \widetilde{B}\sigma_2 & 0 \\ D & -I \\ U(z_1,z_2) & 0 \\ 0 & U_*(z_1,z_2) \end{bmatrix}}_{P(z_1,z_2)} \begin{bmatrix} u \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} -(A_1 - z_1 I) \\ -(A_2 - z_2 I) \\ -C \\ 0 \\ 0 \end{bmatrix}}_{Q(z_1,z_2)} x. \tag{6}$$

Let $N(z_1, z_2)$ be the minimal left annihilator (MLA) of the Hautus pencil $Q(z_1, z_2)$. By definition, given (u, y) there exists x so that (6) holds if and only if

$$N(z_1, z_2) P(z_1, z_2) \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] = 0.$$

We thus define the **external behavior associated with the vessel** as

$$\mathcal{B}^{i/o} = \{[\begin{smallmatrix} u \\ y \end{smallmatrix}] \in (\mathcal{U} \oplus \mathcal{Y})^{\mathbb{N}^2} : N(z_1, z_2)P(z_1, z_2)[\begin{smallmatrix} u \\ y \end{smallmatrix}] = 0\}.$$

It is of interest to identify $N(z_1, z_2)$ more explicitly. Since we know that the MLA of the zero matrix is the identity matrix, we know that a minimal left annihilator of $N(z_1, z_2)$ is of the form

$$N(z_1, z_2) = \begin{bmatrix} 0 & 0 & 0 & I_{\mathcal{U}} & 0 \\ 0 & 0 & 0 & 0 & I_{\mathcal{Y}} \\ V_1 & V_2 & V_3 & 0 & 0 \end{bmatrix}, \quad (7)$$

for suitable polynomial matrices V_1, V_2 and V_3. In (7) $I_{\mathcal{U}}$ and $I_{\mathcal{Y}}$ are identity operators on the input space $\mathcal{U}$ and output space $\mathcal{Y}$ respectively. The kernel representation for the external behavior becomes

$$R^{i/o} = \begin{bmatrix} U & 0 \\ 0 & U_* \\ V_1\widetilde{B}\sigma_1 + V_2\widetilde{B}\sigma_2 + V_3 D & -V_3 \end{bmatrix}. \quad (8)$$

After a series of calculations presented in [8], [3] the following are obtained.

Theorem 4.1: [8], [3] For any vessel $\mathfrak{V}$ which satisfies the stated assumpsion we have $\mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/o}) = \langle f \rangle$ and thus the annihilator is a prime ideal.

Corollary 4.1: [8], [3] For any vessel $\mathfrak{V}$ which satisfies the stated assumptions the associated external behavior $\mathcal{B}_{\mathfrak{V}}^{i/o}$ has autonomy degree one.

C. Livšic Controllability

We now show that if a vessel $\mathfrak{V}$ is Livšic controllable (see Definition 4.3 below) then its associated augmented and external behaviors have image representations over the reduced ring. A first step is to establish this correspondence for a special class of vessels; we then use a change of variables to show that any vessel satisfying the nondegeneracy condition (ND) is isomorphic to a vessel in this special class. Before moving to the main results of this section we review some preliminary concepts from [4], [7].

Notation. For $A = (A_1, A_2)$ a pair of commuting operators on a finite-dimensional linear space and $\gamma = (\gamma_1, \gamma_2) \in \mathbb{N}^2$, we use the standard multivariable notation $A^\gamma = A_1^{\gamma_1} A_2^{\gamma_2}$.

Definition 4.2: Given a vessel $\mathfrak{V}$ as in Definition 4.1, we define the **controllability operator** $\mathcal{C}$ by

$$\mathcal{C} = \mathrm{row}_{\gamma \in \mathbb{N}^2}[A^\gamma \widetilde{B}]. \quad (9)$$

Definition 4.3: A vessel $\mathfrak{V}$ satisfying the nondegeneracy condition (ND) is said to be **Livšic controllable** if the controllability operator $\mathcal{C}$ (9) has image equal to the entire state space $\mathcal{X}$, i.e., for any point $T \in \mathbb{N}^2$ suitably far into the future and state $x_0 \in \mathcal{X}$, there exists an input trajectory $u(t)$ such that u drives the zero initial state to x_0 at time T.

We next analyze controllability properties for the behavior defined by a matrix pencil (3) with one of σ_1 and σ_2 invertible.

Lemma 4.1: [8] If σ_1 is invertible then for any $a \in \mathbb{N}$ and any two trajectories $u_1, u_2 \in \ker_{\mathcal{A}}(U) \subset \mathcal{A}^q$, there exists a unique trajectory $w \in \ker_{\mathcal{A}}(U)$ such that

$$w(t) = \begin{cases} u_1(t_1, t_2) & t_1 + t_2 < a \\ u_2(t_1 - a, t_2) & t_1 \geq a \end{cases}$$

If σ_2 is invertible then for any $a \in \mathbb{N}$ and any two trajectories $u_1, u_2 \in \ker_{\mathcal{A}}(U) \subset \mathcal{A}^q$, there exists a unique trajectory $w \in \ker_{\mathcal{A}}(U)$ such that

$$w(t) = \begin{cases} u_1(t_1, t_2) & t_1 + t_2 < a \\ u_2(t_1, t_2 - a) & t_2 \geq a \end{cases}$$

This concatenation of input trajectories can be used to relate Livšic controllability to 1-controllability.

Theorem 4.2: [8] If $\mathfrak{V}$ is Livšic controllable vessel and either σ_1 or σ_2 is invertible then

1) $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ is 1-controllable,
2) $\mathcal{B}_{\mathfrak{V}}^{i/o}$ is 1-controllable and
3) $\ker_{\mathcal{A}}(U)$ is 1-controllable.

If the input compatibility pencil has neither σ_1 or σ_2 invertible but does satisfy the nondegeneracy condition (ND), then a "change of coordinates" is used to bring it into a form where at least one of the matrices is invertible. We now produce such a change of coordinates.

Assume that $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$ is nondegenerate and hence there exists $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1\sigma_1 + \xi_2\sigma_2$ is invertible. Define $\mathcal{B} = \ker_{\mathcal{A}}(U)$. Without loss of generality, assume that $\xi_1^2 + \xi_2^2 = 1$. Define the ring $\mathcal{D}' = \mathbb{C}[w_1, w_2]$, associated signal space $\mathcal{A}' = (\mathbb{C}^{\mathbb{N}^2})^m$, the matrices

$$\sigma_1' = \xi_1\sigma_1 + \xi_2\sigma_2 \qquad \sigma_2' = \xi_1\sigma_2 - \xi_2\sigma_1,$$

the matrix pencil $U'(w_1, w_2) = \sigma_2' w_1 - \sigma_1' w_2 + \gamma$ and its associated behavior $\mathcal{B}' = \ker_{\mathcal{A}'}(U')$.

Let ψ be the $\mathbb{C}$-algebra isomorphism of $\mathcal{D}$ defined on the generators z_1 and z_2 by

$$\psi(z_1) = \xi_1 z_1 - \xi_2 z_2 \qquad \psi(z_2) = \xi_2 z_1 + \xi_1 z_2. \quad (10)$$

or, via a matrix presentation, by

$$\begin{bmatrix} \psi(z_1) \\ \psi(z_2) \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix} [\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}].$$

Then it is easily verified that ψ is invertible with inverse ϕ on generators given by

$$\begin{bmatrix} \phi(z_1) \\ \phi(z_2) \end{bmatrix} = \begin{bmatrix} \xi_1 & \xi_2 \\ -\xi_2 & \xi_1 \end{bmatrix} [\begin{smallmatrix} z_1 \\ z_2 \end{smallmatrix}]$$

or equivalently

$$\phi(z_1) = \xi_1 z_1 + \xi_2 z_2 \qquad \phi(z_2) = \xi_2 z_1 + \xi_1 z_2.$$

Thus we have

$$\psi \circ \phi = \mathrm{id}_{\mathcal{D}'} : \mathcal{D}' \to \mathcal{D}' \qquad \phi \circ \psi = \mathrm{id}_{\mathcal{D}} : \mathcal{D} \to \mathcal{D}.$$

Then ψ extends to polynomials in z_1 and z_2 (with scalar, vector, or matrix coefficients) via

$$\psi(U)(z_1, z_2) = U(\psi(z_1), \psi(z_2)) =: U'(z_1, z_2),$$

in addition to the relations

$$\begin{aligned} \phi(U'(z_1, z_2)) &:= U'(\xi_1 z_1 + \xi_2 z_2, \xi_1 z_2 - \xi_2 z_1) \\ &= U(z_1, z_2) \\ \psi(U(z_1, z_2)) &:= U(\xi_1 z_1 - \xi_2 z_2, \xi_2 z_1 + \xi_1 z_2) \\ &= U'(z_1, z_2). \end{aligned} \quad (11)$$

Combining these observations with Lemma 3.1 provides the following lemma.

Lemma 4.2: [8] Assume that the matrix pencil $U(z_1, z_2) = \sigma_2 z_1 - \sigma_1 z_2 + \gamma$ is nondegenerate and $\mathrm{Ann}(\ker_{\mathcal{A}}(U))$ is a prime ideal. Then $\ker_{\mathcal{A}}(U)$ has an image representation over its reduced ring.

The change-of-variable procedure also provides an isomorphism between its two associated vessels.

Theorem 4.3: [8] Let

$$\mathfrak{V} = (A_1, A_2, \widetilde{B}, C, D, \widetilde{D}, \sigma_1, \sigma_2, \gamma, \sigma_{1*}, \sigma_{2*}, \gamma_*)$$

be a vessel with $\xi_1, \xi_2 \in \mathbb{C}$ such that $\xi_1^2 + \xi_2^2 = 1$ and $\xi_1\sigma_1 + \xi_2\sigma_2$ an invertible matrix. Then the set of operators

$$\begin{aligned}\mathfrak{V}' = (&A_1' = \xi_1 A_1 + \xi_2 A_2, A_2' = \xi_1 A_2 - \xi_2 A_1, \widetilde{B}, C,\\ &D, \widetilde{D}, \sigma_1' = \xi_1\sigma_1 + \xi_2\sigma_2, \sigma_2' = \xi_1\sigma_2 - \xi_2\sigma_1,\\ &\gamma, \sigma_{1*}' = \xi_1\sigma_{1*} + \xi_2\sigma_{2*}, \sigma_{2*}' = \xi_1\sigma_{2*} - \xi_2\sigma_{1*}, \gamma_*)\end{aligned}$$

forms a vessel with σ_1' invertible. Furthermore, $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}_{\mathfrak{V}'}^{i/s/o}$ and $\mathcal{B}_{\mathfrak{V}}^{i/o}$ is $\mathcal{D}$-isomorphic to $\mathcal{B}_{\mathfrak{V}'}^{i/o}$. If $\mathfrak{V}$ is Livšic controllable, then $\mathfrak{V}'$ is also Livšic controllable.

When Theorem 4.2 is combined with Theorem 4.3 the desired relationship is thus obtained.

Theorem 4.4: Let $\mathfrak{V}$ and $\mathfrak{V}'$ be as stated in Theorem 4.3. If $\mathfrak{V}$ is Livšic controllable and $\mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/o})$ is a prime ideal, then $\mathcal{B}_{\mathfrak{V}}^{i/o}$ has an image representation over its reduced ring.

The above process shows how one may work with the trajectory form of j-controllability. Under a change of coordinates, there may be an isomorphic behavior whose reduced signal space is j-controllable. Theorems 3.3 and 3.1 then yield that at least $\mathcal{D}_r$-controllability is carried over through the isomorphism.

For completeness, we also have the other direction.

Lemma 4.3: [8] Let $\mathfrak{V}$ be a vessel and $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$ be its associated behavior. If $\mathrm{Ann}(\mathcal{B}_{\mathfrak{V}}^{i/s/o})$ is a prime ideal of height one and $\mathcal{B}_{\mathfrak{V}}^{i/s/o}$, when reduced, has an image representation, then $\mathfrak{V}$ is Livšic controllable.

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