

Structured Noncommutative Multidimensional Linear Systems and Scale-Recursive Modeling

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Abstract—Recently, the multiscale signal and image processing community has recognized that a suitable model for multiresolution processes is a model with time-like variable indexed by the nodes on a homogeneous tree with different depths in the tree corresponding to different spatial scales associated with the signal or image. It turns out that these system models are close relatives of the Structured Noncommutative Multidimensional Linear Systems (SNMLSs) introduced by Ball-Groenewald-Malakorn [5], but with system operators dependent on the node of the tree at which the state update occurs. This provides engineering motivation for the introduction of a “parameter-varying” version of SNMLS.

I. INTRODUCTION

Recent development of the so-called *Multidimensional Linear System Theory* has been investigated extensively and drawn considerable attention in the control literature and applications for over decades. The prototypical 2D systems independently proposed by a group of researchers, such as Attasi [3]–[4], Fornasini and Marchesini [25]–[27], Givone and Roesser [28]–[29], and Roesser [34], have been generalized to the multidimensional (d D) linear models with $d \geq 2$. One application emerges from the robust control literature. It turns out that the state-space representation for linear time-invariant systems with structured uncertainty modeled via an upper linear fractional transformation (LFT) is an input/output map from an input sequence to an output sequence. Lu-Zhou-Doyle studied uncertain systems using the LFT as a tool for modeling systems with structured perturbations on a nominal model in [30] (see the monograph [22] for an overview). They considered each perturbation δ_j as an arbitrary time-varying operator on the square summable sequence space ℓ^2 or as a complex-valued parameter uncertainty—the former can be viewed as noncommuting indeterminates z_j ; while the latter can be regarded as the standard complex variables. By replacing δ_j with z_j , the input/output map of the original system can be considered as a noncommutative d -variable transfer function of a linear system, which can be expressed as a formal power series in several noncommuting indeterminates. The role of formal power series in analyzing linear time-invariant systems having time-varying structured uncertainties originates in [10]–[13].

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The author in [31] presented a connection between the robust control theory using the LFT framework and multidimensional linear system theory more explicitly by introducing an input-state-output (i/s/o) linear system where the “time axis” is a homogeneous tree of order d with root generated by a free semigroup. He also showed that the corresponding transfer function of such a system is represented by a formal power series in noncommuting indeterminates. He formulated two linear models, namely the noncommutative Fornasini-Marchesini (NCFM) model and the noncommutative Givone-Roesser (NCGR) model, introduced various concepts of stability, reachability, controllability, observability and similarity of such systems, and established the realization theory for noncommutative systems by adapting the noncommutative shift realization of Schützenberger and Fliess originally developed for the class of the “recognizable series” (see [23]–[24],[36]–[38], for more details).

Ball-Groenewald-Malakorn [5] introduced a new class of linear systems to incorporate NCFM and NCGR models into a general structured model using the notion of an *admissible graph*. This class of systems is called a *Structured Noncommutative Multidimensional Linear System (SNMLS)* which is a linear system having evolution along a free semigroup. They also defined the formal noncommutative Z -transform of a sequence to be a formal power series in several noncommuting indeterminates. Application of the formal Z -transform to the SNMLS yields the transfer function of an SNMLS which is a formal power series in noncommuting indeterminates rather than an analytic function of several complex variables. Standard system-theoretic properties such as cascade/parallel connection and inversion, reachability, controllability, observability, Kalman decomposition, state space similarity theorem, minimal state space realizations, Hankel operators, and realization theory have been developed. Sequels to the paper of [5] are the papers [6]–[7]. In [6], the authors studied the conservative properties of SNMLS and provided the converse realization theorem—any formal power series satisfying a noncommutative von Neumann inequality can be realized as the transfer function of a conservative SNMLS. A standard Bounded Real Lemma and a strict Bounded Real Lemma for SNMLS, closely related to results of Paganini [33], were established in [7]. The authors in [7] also developed the structured singular value introduced by Doyle (see [21], [35]) along with the related concepts of robust stability and robust performance for the LFT model for structured uncertainty in both time-invariant and time-varying cases.

It should be noted that the system parameters in the standard SNMLS are constant, i.e., A , B , C , and D are independent of “time”. In some engineering applications, this condition is too restrictive and should be removed. Recently, multiresolution signal and image analysis methods have been intensively studied under a variety of names including multi-rate filters, subband coding, Laplacian pyramids, multiscale analysis, and “scale-space” image processing; all of which can be modeled as a process with time-like variable indexed by nodes on a homogeneous tree in which different depths in the tree correspond to different spatial scales in representing a signal or image. This is an engineering motivation to consider a new class of systems called a *Parameter-Varying Structured Non-commutative Multidimensional Linear System (PV-SNMLS)*, which is an extension of the SNMLS where the system parameters are allowed to be “time”-dependent. Any readers interested in more details on the Multiscale theory and some applications may wish to consult [9], [14]–[15], [18]–[19], [39] and references therein.

The remainder of the paper is organized as follows: Section II provides a brief overview on the multiscale theory with some connections to the SNMLSs defined in [5]. The notion of a parameter-varying structured noncommutative multidimensional linear system is presented in Section III along with some properties. We then draw some conclusions in Section IV.

II. BRIEF OVERVIEW ON MULTISCALE SYSTEM MODELS

In this section, we offer a brief overview on Multiscale system models in connection with the structured noncommutative multidimensional linear systems (SNMLSs). It turns out that the scale-recursive dynamic models proposed by Benveniste-et-al in [18]–[19] have a hybrid connection to the system equations associated with SNMLSs as developed in [5]: the coarse-to-fine model is a “parameter varying” version of forward recognizable systems; while the fine-to-coarse model is a “parameter varying” version of backward noncommutative Fornasini-Marchesini systems.

A. The settings

To make comparisons between structured noncommutative multidimensional linear systems (SNMLSs) introduced in [5] versus the multiscale system equations of Benveniste-et-al [9], [14]–[15], [18]–[19], it suffices to restrict SNMLSs to the noncommutative Fornasini-Marchesini (NCFM) and recognizable cases, as these are the most relevant for comparison with the multiscale systems of Benveniste-et-al. Also, for purposes of comparison, we may as well take $d = 2$ and take an alphabet—a set of letters—to be $\{\alpha, \beta\}$ rather than $\{1, 2\}$, and the generic element (or word) in \mathcal{F}_d we write as t rather than w . In [5] the authors consider only two homogeneous trees with roots, namely the *future* $\mathcal{T}_F = \mathcal{F}_d$ and the *past* $\mathcal{T}_P = \mathcal{F}_d \setminus \{\emptyset\}$. To make the comparisons more compelling we take the feedthrough operator D to be zero. Also it is only a matter of notation to consider words to read from left to right rather than right to left. With these modifications, the NCFM system equations (Σ^{NCFM}) and

the recognizable system equations (Σ^{Rec}) with evolution on the future time can be expressed in the following forms:

$$\Sigma_F^{NCFM} : \begin{cases} x_{t\alpha} &= A_\alpha x_t + B_\alpha u_t \\ x_{t\beta} &= A_\beta x_t + B_\beta u_t \\ y_t &= Cx_t, \end{cases} \quad (1)$$

and

$$\Sigma_F^{Rec} : \begin{cases} x_{t\alpha} &= A_\alpha x_t + Bu_{t\alpha} \\ x_{t\beta} &= A_\beta x_t + Bu_{t\beta} \\ y_t &= Cx_t. \end{cases} \quad (2)$$

B. Multiscale theory

The authors in [14]–[15] consider the system as acting on a homogeneous (of order 2 or dyadic) tree with no root, \mathcal{T} . They designate $-\infty$ as an arbitrary boundary point, which is an equivalence class of infinite paths on the tree, where two paths are equivalent if they differ by a finite number of nodes, i.e., have a common tail. Any fixed infinite path originating from $-\infty$ is called a *skeleton* of an associated translation operator along the tree. Each point t on the skeleton is the immediate successor of a unique previous point t' on the skeleton; to go from t' to t one must go either left (α) or right (β). Here α and β are two shift operators on tree—the former moving one step away from $-\infty$ toward the left; while the latter moving one step toward the right. They define a signal as a family $z(t)$ of scalars or vectors indexed by the nodes of \mathcal{T} . Then the induced (dual) operators on signals is obtained by considering the associated composition operators:

$$(\alpha(z))(t) = z(t\alpha) := z_{t\alpha}, \text{ and } (\beta(z))(t) = z(t\beta) = z_{t\beta}.$$

If one restricts to signals of finite energy (i.e., with values which are norm square-summable over the tree), the operators α, β form a row unitary operator (i.e., the operator-block row matrix $[\alpha \ \beta]$ is an isometry), or equivalently, give rise to a representation of the Cuntz algebra; this representation can be identified as a particular instance of those studied in [8], [17], [20]. With these operators, Benveniste-et-al introduce the system equations in the state-space form as follows (see (27) on page 9 of [15]):

$$\Sigma^{multi} : \begin{cases} x_{t\alpha} &= A_\alpha x_t + \alpha \vec{H} u_t \\ x_{t\beta} &= A_\beta x_t + \beta \vec{H} u_t \\ y_t &= Cx_t. \end{cases} \quad (3)$$

If one is acting on a homogeneous tree with root, the operator-block row matrix $[\alpha \ \beta]$ is only a row isometry rather than a row unitary and generates a representation of the Cuntz-Toeplitz algebra. In this case the transfer function \vec{H} can be collapsed to a constant (corresponding to a power series expansion $\vec{H} = H(z)$ consisting of only the constant term). Then we identify \vec{H} with $\vec{H} := B$ and (3) becomes

$$\Sigma^{multi'} : \begin{cases} x_{t\alpha} &= A_\alpha x_t + Bu_{t\alpha} \\ x_{t\beta} &= A_\beta x_t + Bu_{t\beta} \\ y_t &= Cx_t. \end{cases} \quad (4)$$

This is in fact the system equations describing a recognizable linear system as in (2) (see also in Section 12 of [5]). It

should be noted that the Benveniste-et-al transfer function in [15]

$$H = C(I - \bar{\alpha}A_\alpha - \bar{\beta}A_\beta)^{-1}\bar{H}$$

reduces to the recognizable form (with no feedthrough term)

$$H = C(I - \bar{\alpha}A_\alpha - \bar{\beta}A_\beta)^{-1}B,$$

where $\bar{\alpha}$ and $\bar{\beta}$ here play the role of indeterminants. Note also that the realization theorem in [14]–[15] is derived as an adaptation of the Fliess realization theorem for recognizable formal power series (see [16], [23]).

We note that Alpay-Volok [1]–[2] have followed up on the multiscale system theory of Benveniste-et-al in a different way by developing various parallels with the theory of classical time-varying systems.

C. Scale recursive dynamic models

In [9], [18]–[19], the authors use the notation $t = (m, n)$ for any node t on \mathcal{T} with m and n integers, where $m := m(t)$ is the scale-of-resolution index at node t and n is its translation offset. Note that the scale-of-resolution corresponds to the horocycle level defined in [14]–[15]. If the tree we are considering is a dyadic tree (as in [9], [18]–[19]), it is natural to introduce three shift operators on \mathcal{T} : the unique backward shift $\bar{\gamma}$ and two forward shifts α and β . Here α and β are defined without reference to a skeleton, and the backward shift $\bar{\gamma}$ amounts to $\bar{\alpha} + \bar{\beta}$ where $\bar{\alpha}$ and $\bar{\beta}$ are the backward shifts (viewed as acting on the tree nodes) in [14]–[15]. Thus, if we let $t = (m, n)$, then $t\alpha = (m + 1, 2n)$, $t\beta = (m + 1, 2n + 1)$ and $t\bar{\gamma} = (m - 1, \lceil n/2 \rceil)$ where $\lceil \cdot \rceil : \mathbb{R} \rightarrow \mathbb{Z}$ defined by $\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}$ for any $x \in \mathbb{R}$ (i.e., the smallest integer not less than x). Here for convenience we write m rather than $m(t)$ when a node t under consideration is clear from the context.

The authors in [19] introduced two classes of scale-recursive linear dynamic models with evolution along a dyadic tree, namely the *coarse-to-fine* and the *fine-to-coarse* state space models. The coarse-to-fine system model is given by (see (2.1) and (2.2) in [19])

$$\Sigma^{CF} : \begin{cases} x(t) &= A(t)x(t\bar{\gamma}) + B(t)w(t) \\ y(t) &= C(t)x(t) + v(t). \end{cases} \quad (5)$$

By using the notation $t = (m, n)$, (5) can be rewritten in the form

$$\begin{aligned} x(m + 1, 2n) &= A(m + 1, 2n)x(m, n) \\ &\quad + B(m + 1, 2n)w(m + 1, 2n) \\ x(m + 1, 2n + 1) &= A(m + 1, 2n + 1)x(m, n) \\ &\quad + B(m + 1, 2n + 1)w(m + 1, 2n + 1) \\ y(m, n) &= C(m, n)x(m, n) + v(m, n) \end{aligned}$$

which in the notation of [14]–[15] amounts to

$$\Sigma_1^{CF} : \begin{cases} x(t\alpha) &= A(t\alpha)x(t) + B(t\alpha)w(t\alpha) \\ x(t\beta) &= A(t\beta)x(t) + B(t\beta)w(t\beta) \\ y(t) &= C(t)x(t) + v(t). \end{cases} \quad (6)$$

If we assume that the system operator $A(\cdot)$ is t -independent but dependent only on the shift operators, i.e.,

$$A(t\alpha) = A(\alpha) := A_\alpha, \quad A(t\beta) = A(\beta) := A_\beta,$$

the input operator $B(\cdot)$ and the output operator $C(\cdot)$ are constant for all nodes t on \mathcal{T} , i.e.,

$$B(t\alpha) = B = B(t\beta), \quad C(t) = C,$$

then (6) collapses to the *recognizable* system equations (cf. (2)). Thus, the coarse-to-fine system models in (5) (or (6)) amount to a “parameter-varying” version of the recognizable system equations discussed in [5].

The second type of scale-recursive linear dynamic models introduced in [19] is the fine-to-coarse recursive models given by

$$\Sigma^{FC} : \begin{cases} x(t) &= F(t\alpha)x(t\alpha) + F(t\beta)x(t\beta) \\ &\quad + G(t\alpha)x(t\alpha) + G(t\beta)x(t\beta) \\ y(t) &= C(t)x(t) + v(t). \end{cases} \quad (7)$$

As in the previous case, we make the assumptions that $F(\cdot)$ and $G(\cdot)$ are t -independent but dependent only on the shift operators, i.e.,

$$F(t\alpha) = F(\alpha) := F_\alpha, \quad F(t\beta) = F(\beta) := F_\beta,$$

and

$$G(t\alpha) = G(\alpha) := G_\alpha, \quad G(t\beta) = G(\beta) := G_\beta,$$

then (7) amounts to the NCFM backward system equations having evolution in \mathcal{T}_P (see (2.5) in [5]). Thus, the fine-to-coarse models can be viewed as a “parameter-varying” version of the NCFM backward system equations discussed in [5]. Thus from the point of view of [5], Benveniste-et-al have a hybrid collection of system equations: the coarse-to-fine model is a parameter-varying recognizable system with evolution in the future, while the fine-to-coarse model is a parameter-varying NCFM system with evolution in the past.

To end this Subsection, let us consider a special case of the system equations given in (6) and (7) when the system parameters are constant at each scale—we shall write $A(t)$ as $A(m)$, etc. Thus, the scale-varying version of the coarse-to-fine model (6) and the fine-to-coarse model (7) are of the forms

$$\Sigma_1^{SV-CF} : \begin{cases} x(t\alpha) &= A(m + 1)x(t) + B(m + 1)w(t\alpha) \\ x(t\beta) &= A(m + 1)x(t) + B(m + 1)w(t\beta) \\ y(t) &= C(m)x(t) + v(t), \end{cases}$$

and

$$\Sigma^{SV-FC} : \begin{cases} x(t) &= F(m + 1)[x(t\alpha) + x(t\beta)] \\ &\quad + G(m + 1)[x(t\alpha) + x(t\beta)] \\ y(t) &= C(m)x(t) + v(t). \end{cases}$$

D. Multiresolution Stochastic Processes

The main emphasis in [18]–[19] is the implementation of Kalman filtering by means of a Lyapunov equation on a tree. By assuming that the input is a white noise, this implies that the system trajectories are random variables and one uses the expectation of the variance of an error signal as the quantity to be minimized—just as in the standard Kalman filter. The Riccati equations derived for this in [18]–[19] are tree-versions of the usual Riccati equations for the Kalman filter.

To be more specific, let us consider the coarse-to-fine system equations (5). Suppose now that $w(t)$ and $v(t)$ are independent, zero-mean white noise processes with covariances I and $R(t)$, respectively, and let $P_x(t) = \mathbb{E}[x(t)x^T(t)]$ be the covariance. Then a Lyapunov equation on the tree is given by (see (2.3) in [18])

$$P_x(t) = A(t)P_x(t\bar{\gamma})A^T(t) + B(t)B^T(t). \quad (8)$$

For any nodes s, t on \mathcal{T} , there must exist the shortest paths from s to $-\infty$ and from t to $-\infty$. This implies immediately that both paths must meet at some node, say $s \wedge t$ on \mathcal{T} . Then the autocovariance $K_{xx}(t, s) := \mathbb{E}[x(t)x^T(s)]$ is given by

$$K_{xx}(t, s) = \Phi(t, s \wedge t)P_x(s \wedge t)\Phi^T(s, s \wedge t), \quad (9)$$

where $\Phi(t_1, t_2)$ is the state transition matrix on the tree

$$\Phi(t_1, t_2) = \begin{cases} I & \text{if } t_1 = t_2, \\ A(t_1)\Phi(t_1\bar{\gamma}, t_2) & m(t_1) > m(t_2). \end{cases} \quad (10)$$

III. PARAMETER-VARYING STRUCTURED NONCOMMUTATIVE MULTIDIMENSIONAL LINEAR SYSTEMS (PV-SNMLSs)

We present here the notion of the parameter-varying structured noncommutative multidimensional linear systems, which is an extension of the system model introduced in [5].

A. Admissible Graph

As in the standard graph theory, a finite graph G consists of a finite set of vertices $V = V(G)$ and edges $E = E(G)$ connecting vertices. A graph G is called an *admissible graph* if G is a bipartite graph such that each connected component is a complete bipartite graph. This means simply that:

- 1) the set of vertices V has a disjoint partitioning $V = S \dot{\cup} R$ into the set of *source* vertices S and *range* vertices R ,
- 2) S and R in turn have disjoint partitionings $S = \dot{\cup}_{k=1}^K S_k$ and $R = \dot{\cup}_{k=1}^K R_k$ into nonempty subsets S_1, \dots, S_K and R_1, \dots, R_K such that, for each $s_k \in S_k$ and $r_k \in R_k$ (with the same value of k) there is a unique edge $e = e_{s_k, r_k}$ connecting s_k to r_k ; i.e., $s(e) = s_k, r(e) = r_k$, and
- 3) every edge of G is of this form.

For any vertex v of G (so either $v \in S$ or $v \in R$), let $[v]$ denote the path-connected component p (i.e., the complete bipartite graph $p = G_k$ with set of source vertices equal to S_k and set of range vertices equal to R_k for some $k = 1, \dots, K$)

containing v . Thus, given two distinct vertices $v_1, v_2 \in S \cup R$, there is a path of G connecting v_1 to v_2 if and only if $[v_1] = [v_2]$ and this path has length 2 if both v_1 and v_2 are either in S or in R and has length 1 otherwise. In case $s \in S$ and $r \in R$ are such that $[s] = [r]$, let $e_{s,r}$ be the unique edge having s as source vertex and r as range vertex:

$$e_{s,r} \in E \text{ determined by } s(e_{s,r}) = s, r(e_{s,r}) = r. \quad (11)$$

It should be noted that $e_{s,r}$ is well-defined only for $s \in S$ and $r \in R$ with $[s] = [r]$.

B. Definition of PV-SNMLS

Now we define a notion of the *parameter-varying structured noncommutative multidimensional linear systems* (PV-SNMLS) to be a collection $\Sigma^{PV} = (G, \mathcal{H}, U^{PV})$ where G is an admissible graph, where $\mathcal{H} = \{\mathcal{H}_p : p \in P\}$ is a collection of (separable) Hilbert spaces (the *component state spaces*) indexed by the path-connected components $P = P(G)$ of the graph G , and where U^{PV} is a *connection matrix* or *colligation* of the form

$$U^{PV} = \begin{bmatrix} A(\cdot) & B(\cdot) \\ C(\cdot) & D(\cdot) \end{bmatrix} = \begin{bmatrix} [A_{r,s}(\cdot)] & [B_r(\cdot)] \\ [C_s(\cdot)] & D(\cdot) \end{bmatrix} : \begin{bmatrix} \oplus_{s \in S} \mathcal{H}_{[s]} \\ \mathcal{U} \end{bmatrix} \rightarrow \begin{bmatrix} \oplus_{r \in R} \mathcal{H}_{[r]} \\ \mathcal{Y} \end{bmatrix} \quad (12)$$

where \mathcal{U} and \mathcal{Y} are additional (separable) Hilbert spaces (to be interpreted as the *input space* and the *output space*, respectively) with all system operators— $A(\cdot), B(\cdot), C(\cdot)$ and $D(\cdot)$ —are well-defined.

C. System Equations

With any PV-SNMLS we associate an input/state/output linear system with evolution along a free semigroup as follows. We denote by \mathcal{F}_E the free semigroup generated by the edge set E . An element of \mathcal{F}_E is then a word w of the form $w = e_N e_{N-1} \dots e_1$ where each e_k is an edge of G for $k = 1, \dots, N$. We also denote the empty word (consisting of no letters) by \emptyset . The semigroup operation is concatenation: if $w = e_N \dots e_1$ and $w' = e'_{N'} \dots e'_1$, then ww' is defined to be

$$ww' = e_N \dots e_1 e'_{N'} \dots e'_1.$$

Note that the empty word \emptyset acts as the identity element for this semigroup.

If $\Sigma^{PV} = (G, \mathcal{H}, U^{PV})$ is a PV-SNMLS, we associate the system equations (with evolution along \mathcal{F}_E)

$$\Sigma^{PV} : \begin{cases} x_{s(e)}(ew) &= \sum_{s \in S} A_{r(e),s}(w)x_s(w) \\ &\quad + B_{r(e)}(w)u(w), \\ x_{s'}(ew) &= 0 \text{ if } s' \neq s(e), \\ y(w) &= \sum_{s \in S} C_s(w)x_s(w) \\ &\quad + D(w)u(w). \end{cases} \quad (13)$$

Here the *state vector* $x(w)$ at position w (for $w \in \mathcal{F}_E$) has the form of a column vector

$$x(w) = \text{col}_{s \in S} x_s(w),$$

with column entries indexed by the source vertices $s \in S$ and with column entry $x_s(w)$ taking values in the auxiliary state space $\mathcal{H}_{[s]}$ (and thus $x(w)$ takes values in the *state space* $\oplus_{s \in S} \mathcal{H}_{[s]}$), while $u(w) \in \mathcal{U}$ and $y(w) \in \mathcal{Y}$ denote the *input* and the *output*, respectively, at position w .

It should be noted that the system equations (13) can be written more compactly in operator-theoretic form as

$$\Sigma^{PV} : \begin{cases} x(ew) &= I_{\Sigma;e} A(w)x(w) + I_{\Sigma;e} B(w)u(w) \\ y(w) &= C(w)x(w) + D(w)u(w), \end{cases} \quad (14)$$

where $I_{\Sigma;e} : \oplus_{r \in R} \mathcal{H}_{[r]} \rightarrow \oplus_{s \in S} \mathcal{H}_{[s]}$ with matrix entries $[I_{\Sigma;e}]_{s \in S, r \in R} = I_{\mathcal{H}_{[s(e)]}} = I_{\mathcal{H}_{[r(e)]}}$ if $s = s(e)$ and $r = r(e)$, and we set $[I_{\Sigma;e}]_{s \in S, r \in R} = 0$ otherwise.

D. System solution

Now let i_s be the natural injection $h \mapsto \text{col}_{s' \in S} [\delta_{s',s} h]$ of $\mathcal{H}_{[s]}$ into $\oplus_{s \in S} \mathcal{H}_{[s]}$, and set

$$\Delta_e(\cdot) : \oplus_{s \in S} \mathcal{H}_{[s]} \rightarrow \oplus_{s \in S} \mathcal{H}_{[s]},$$

where $\Delta_{e_k}(e_{k-1} \dots e_1) = i_{s(e_k)} A_{\mathbf{r}(e_k)}(e_{k-1} \dots e_1)$ with $e_k \in E$, and $e_0 = \emptyset$.

For any $w \in \mathcal{F}_E$, w_0 is called a *suffix* of w if there exists $v \in \mathcal{F}_E$ such that $w = vw_0$. Suppose now that $w \in \mathcal{F}_E$ is of the form $w = e_N e_{N-1} \dots e_r \dots e_1$ with $e_k \in E$ and let w_0 be any suffix of w , say $w_0 = e_r \dots e_1$. Then from the noncommutative functional calculus, we write

$$\begin{aligned} \Delta_{w_0}^w &:= \Delta_{e_N}(e_{N-1} \dots e_{r+1} w_0) \Delta_{e_{N-1}}(e_{N-2} \dots e_{r+1} w_0) \\ &\quad \dots \Delta_{e_{r+2}}(e_{r+1} w_0) \Delta_{e_{r+1}}(w_0), \end{aligned}$$

where $\Delta_{e_0}^w = I_{\mathcal{H}}$ and $e_0 = \emptyset$. It is easy to verify that for any suffix w_0 of $w \in \mathcal{F}_E$,

$$\Delta_{\emptyset}^w = \Delta_{w_0}^w \Delta_{\emptyset}^{w_0}. \quad (15)$$

We need another piece of notation here. For a word $w \in \mathcal{F}_E$ and an edge $e \in E$, the notation $e^{-1}w$ means

$$e^{-1}w = \begin{cases} w' & \text{if } w = ew', \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Let $w \in \mathcal{F}_E$ be of the form $w = e_N e_{N-1} \dots e_1$, then by convention w^{-1} means $w^{-1} = e_1^{-1} \dots e_{N-1}^{-1} e_N^{-1}$.

Given $x(w_0)$ for some $w_0 \in \mathcal{F}_E$ and $\{u(w)\}_{w \in \mathcal{F}_E}$, the state vector $x(w)$ and the output vector $y(w)$, where $w = vw_0$ for some $v \in \mathcal{F}_E$, can be computed recursively via the equations (14) as follows:

$$\begin{aligned} x(w) &= \Delta_{w_0}^w x(w_0) \\ &+ \sum_{\substack{w', w'' \in \mathcal{F}_E \\ e \in E: w' ew'' = w}} \Delta_{(w')^{-1} w}^{w'} B_{\mathbf{r}(e)}(w'' w_0) u(w'' w_0), \end{aligned} \quad (16)$$

and

$$\begin{aligned} y(w) &= C(w) \Delta_{w_0}^w x(w_0) + D(w) u(w) \\ &+ \sum_{\substack{w', w'' \in \mathcal{F}_E \\ e \in E: w' ew'' = w}} C(w) \Delta_{(w')^{-1} w}^{w'} B_{\mathbf{r}(e)}(w'' w_0) u(w'' w_0). \end{aligned} \quad (17)$$

Here we set $x_{s'}(ew) = 0$ unless $s' = s(e)$.

E. Backward System Equations

Suppose we are given $x(w)$ for some $w \in \mathcal{F}_E \setminus \{\emptyset\}$ and a sequence of input string $\{u(w)\}_{w \in \mathcal{F}_E \setminus \{\emptyset\}}$, it is of interest to compute $x(\emptyset)$. This involves the system evolution in reverse direction of time. Thus, we need a notion of “time” rather than just a free semigroup \mathcal{F}_E . Up to this point we have been considering the system evolution only on the “future time” $\mathcal{T}_F := \mathcal{F}_E$. We now define the “past time” \mathcal{T}_P to be a second copy of \mathcal{F}_E but with the empty word deleted: $\mathcal{T}_P := \mathcal{F}_E \setminus \{\emptyset\}$. It is worth emphasizing that \mathcal{T}_F and \mathcal{T}_P are to be considered as disjoint sets; given a nonempty word w in \mathcal{F}_E , we will specify in the particular context whether it is to be considered as an element of \mathcal{T}_F or \mathcal{T}_P .

We now introduce the PV-SNMLS with evolution on the past, which is given by

$$\Sigma_P^{PV} : \begin{cases} x_s(w) &= \sum_{e: s(e)=s} \sum_{s' \in S} A_{s'}(we) x_{s'}(we) \\ &+ \sum_{e: s(e)=s} B(we) u(we) \\ y(w) &= \sum_{s \in S} C_s(w) x_s(w) + D(w) u(w), \end{cases} \quad (18)$$

or, in aggregate form,

$$\Sigma_P^{PV} = \begin{cases} x(w) &= \sum_{e \in E} I_{\Sigma;e} A(we) x(we) \\ &+ \sum_{e \in E} I_{\Sigma;e} B(we) u(we) \\ y(w) &= C(w) x(w) + D(w) u(w). \end{cases} \quad (19)$$

If the system operators are constant in the sense that

$$\begin{aligned} A_{s'}(we) &= A_{\mathbf{r}(e), s'}, & B(we) &= B_{\mathbf{r}(e)}, \\ C_s(w) &= C_s, & D(w) &= D, \end{aligned} \quad (20)$$

then (18) collapses to the standard SNMLS introduced in [5]; i.e.,

$$\Sigma_P : \begin{cases} x_s(w) &= \sum_{e: s(e)=s} \sum_{s' \in S} A_{\mathbf{r}(e), s'} x_{s'}(we) \\ &+ \sum_{e: s(e)=s} B_{\mathbf{r}(e)} u(we) \\ y(w) &= \sum_{s \in S} C_s x_s(w) + D u(w). \end{cases}$$

F. Examples

In case of parameter-varying noncommutative Fornasini-Marchesini (PV-NCFM) systems, the admissible graph G is a complete bipartite graph having only one source vertex and d range vertices. We then take $S = \{1\}$, and $R = E = \{1, \dots, d\}$ with $s(i) = 1$, $r(i) = i$. For any $i \in E$, we have

$$I_{\Sigma^{PV-NCFM}; i} = [0 \quad \dots \quad 0 \quad I_{\mathcal{H}} \quad 0 \quad \dots \quad 0],$$

where $I_{\mathcal{H}}$ occurs in the i -th column. Thus,

$$I_{\Sigma^{PV-NCFM}; i} A(\cdot) = A_i(\cdot), \quad I_{\Sigma^{PV-NCFM}; i} B(\cdot) = B_i(\cdot),$$

and therefore the associated PV-NCFM system is given by

$$\Sigma^{PV-NCFM} : \begin{cases} x(1w) &= A_1(w)x(w) + B_1(w)u(w) \\ &\vdots \\ x(dw) &= A_d(w)x(w) + B_d(w)u(w) \\ y(w) &= C(w)x(w) + D(w)u(w). \end{cases} \quad (21)$$

In addition, the backward PV-NCFM system with evolution on the past time is

$$\Sigma_P^{PV-NCFM} : \begin{cases} x(w) &= \sum_{i=1}^d A_i(w_i)x(w_i) \\ &+ \sum_{i=1}^d B_i(w_i)u(w_i) \\ y(w) &= C(w)x(w) + D(w)u(w). \end{cases} \quad (22)$$

If now we take $d = 2$ with constant system operators, (21) and (22) collapse to the standard NCFM as follows:

$$\Sigma^{NCFM} : \begin{cases} x(1w) &= A_1x(w) + B_1u(w) \\ x(2w) &= A_2x(w) + B_2u(w) \\ y(w) &= Cx(w) + Du(w), \end{cases} \quad (23)$$

and

$$\Sigma_P^{NCFM} : \begin{cases} x(w) &= A_1x(w1) + A_2x(w2) \\ &+ B_1x(w1) + B_2x(w2) \\ y(w) &= Cx(w) + Du(w). \end{cases} \quad (24)$$

Before ending this Section, let us consider the NCFM system equations (23) for a moment. Suppose we introduce a new state vector

$$\xi(w) = \begin{bmatrix} x(w) \\ u(w) \end{bmatrix},$$

it is straightforward calculation to show that (23) is equivalent to

$$\Sigma^{NCFM'} : \begin{cases} \xi(jw) &= \tilde{A}_j\xi(w) + \tilde{B}u(jw) \\ y(w) &= \tilde{C}\xi(w), \end{cases} \quad (25)$$

where

$$\tilde{A}_j = \begin{bmatrix} A_j & B_j \\ 0 & 0 \end{bmatrix}, \tilde{B} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \tilde{C} = [C \quad D].$$

Obviously, this is identical to the recognizable linear system as described in (12.1) of [5]. Now we let system operators be “word: w ”-dependent and thus we arrive at a parameter-varying recognizable linear system which is an extended version of (25); i.e.,

$$\Sigma^{PV-Rec} : \begin{cases} \xi(jw) &= \tilde{A}(jw)\xi(w) + \tilde{B}(jw)u(jw) \\ y(w) &= \tilde{C}(w)\xi(w). \end{cases} \quad (26)$$

G. Tree-version of Lyapunov Equations

Consider the unforced PV-SNMLS with noise:

$$\Sigma : \begin{cases} x_{\mathbf{s}(e)}(ew) &= \sum_{s \in S} A_{\mathbf{r}(e),s}(w)x_s(w) \\ &+ B_{\mathbf{r}(e)}(w)\nu(w), \\ x_{s'}(ew) &= 0 \quad \text{if } s' \neq \mathbf{s}(e), \\ y(w) &= \sum_{s \in S} C_s(w)x_s(w). \end{cases} \quad (27)$$

Here $\nu(w)$ is assumed to be independent, zero-mean white noise processes with covariance $Q(w)$, and $x(w)$ is a zero-mean stochastic process. The system Σ in (27) can be written in more compact form as

$$\Sigma : \begin{cases} x(ew) &= I_{\Sigma;e}A(w)x(w) + I_{\Sigma;e}B(w)\nu(w) \\ y(w) &= C(w)x(w) \end{cases} \quad (28)$$

As in [18]–[19], we let P denote the covariance operator defined by $P(w) = \mathbb{E}[x(w)x^T(w)]$. Thus, we have the following:

$$\begin{aligned} P(ew) &= \mathbb{E}[x(ew)x^T(ew)] \\ &= (I_{\Sigma;e}A(w))P(w)(I_{\Sigma;e}A(w))^T \\ &\quad + (I_{\Sigma;e}B(w))Q(w)(I_{\Sigma;e}B(w))^T. \end{aligned} \quad (29)$$

The above expression is called a *Lyapunov equation on the tree*.

Given words $ew, \tilde{e}\tilde{w} \in \mathcal{F}_E$, the autocovariance operator $K(ew, \tilde{e}\tilde{w})$ is given by

$$\begin{aligned} K(ew, \tilde{e}\tilde{w}) &= \mathbb{E}[x(ew)x^T(\tilde{e}\tilde{w})] \\ &= (I_{\Sigma;e}A(w))\mathbb{E}[x(w)x^T(\tilde{w})](I_{\Sigma;\tilde{e}}A(\tilde{w}))^T \\ &\quad + (I_{\Sigma;e}B(w))\mathbb{E}[\nu(w)\nu^T(\tilde{w})](I_{\Sigma;\tilde{e}}B(\tilde{w}))^T. \end{aligned}$$

Since $\nu(w)$ is white noise processes, $\mathbb{E}[\nu(w)\nu^T(\tilde{w})] = 0$ unless $w = \tilde{w}$. In addition, for any words $w, \tilde{w} \in \mathcal{F}_E$, there always exists a suffix $w_0 \in \mathcal{F}_E$ of w and \tilde{w} so that $w = vw_0$ and $\tilde{w} = \tilde{v}w_0$ for some $v, \tilde{v} \in \mathcal{F}_E$. Thus,

$$\begin{aligned} x(w) &= \Delta_{w_0}^w x(w_0) + \text{noise terms}, \\ x(\tilde{w}) &= \Delta_{w_0}^{\tilde{w}} x(w_0) + \text{noise terms}. \end{aligned}$$

Thus,

$$\mathbb{E}[x(w)x^T(\tilde{w})] = (\Delta_{w_0}^w)\mathbb{E}[x(w_0)x^T(w_0)](\Delta_{w_0}^{\tilde{w}})^T,$$

and hence, the autocovariance operator can be given by the following expression:

$$\begin{aligned} K(ew, \tilde{e}\tilde{w}) &= (I_{\Sigma;e}A(w))\mathbb{E}[x(w)x^T(\tilde{w})](I_{\Sigma;\tilde{e}}A(\tilde{w}))^T \\ &= (I_{\Sigma;e}A(w)\Delta_{w_0}^w)P(w_0)(I_{\Sigma;\tilde{e}}A(\tilde{w})\Delta_{w_0}^{\tilde{w}})^T \end{aligned}$$

In the forthcoming paper [32] we plan to apply the notion of covariance $P(ew)$ and the autocovariance $K(ew, \tilde{e}\tilde{w})$ derived here to obtain the tree-versions of the Riccati equations for the Kalman filter in the general SNMLS setting.

IV. CONCLUSIONS

The multiscale systems of Benveniste-et-al in [15], [18]–[19] have original motivation from the theory of wavelets and the desire to develop Kalman filters for this setting. On the other hand, the SNMLSs of [5]–[6] had motivation coming from robust control of standard 1-D systems in the presence of structured time-varying uncertainty. The main contribution of this paper is to make explicit the close connection between these two formalisms. It turns out that the coarse-to-fine system equations of Benveniste-et-al [19] amounts to a parameter-varying version of a forward-time recognizable system as defined in [5], while the fine-to-coarse model in [19] is a parameter-varying version of the backward noncommutative Fornasini-Marchesini system from [5]. Here we propose a new class of systems, called a parameter-varying structured noncommutative multidimensional linear system (PV-SNMLS), as a general model containing both the models of [19] and of [5] as particular cases. We have also derived a Lyapunov equation and a recursive definition of an autocovariance in this more general setting.

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