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# Tracking and Regulation in the Behavioral Framework

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***Abstract*— This paper considers the problem of tracking and regulation for the class of linear differential systems in the behavioral framework. Given a plant, together with an exosystem generating the disturbances and the reference signals, the problem of tracking and regulation is to find a controller such that the plant variable tracks the reference signal regardless of the disturbance acting on the system. A controller which achieves this design objective is called a regulator for the plant with respect to the exosystem. In this paper we formulate the tracking and regulation problem in the behavioral framework, with control as interconnection. We obtain necessary and sufficient conditions for the existence of a controller which acts like a regulator for the plant with respect to the exosystem. The problem formulation and its resolution are completely representation free, and specified only in terms of the plant and the exosystem dynamics.**

## I. INTRODUCTION

This paper deals with control in the behavioral context. We consider the problem of finding, for a given plant behavior with to-be-controlled variable $w$, reference signal $r$, and disturbances acting on the plant, a controller such that in the resulting system after interconnection of the plant and the controller, the plant variable $w$ follows (in some sense) the reference signal $r$, regardless of the disturbances acting on the plant. In other words, we consider the problem of *tracking and regulation* in the behavioral framework.

Of course, the problem of tracking and regulation has been studied before in the literature, in an input-output framework. See for instance [7], [8], [10], [11] and [9]. The theory has also been extended to nonlinear systems in [12]. Many results have been collected in the book [13] (see also [14], Chapter 9).

Our work can be seen as behavioral generalization of [8], [10], [11] and [9]. In the behavioral framework, controlling a plant means restricting its behavior to a desired subset of the behavior. This restriction is brought about by interconnecting the plant with a controller that we design. The restricted behavior is then called the controlled behavior, which is required to satisfy the design specifications. In terms of representations, control means that additional laws (e.g., in the form of differential equations representing the controller behavior) are put on some of the plant variables. Thus, the plant and controller are interconnected through some of their variables. In our context we do not distinguish between inputs and outputs, so the interconnection does not involve feedback. This idea was introduced by J. C. Willems in [2] in the context of stabilization and pole placement. In this paper we use these ideas to solve the problem of tracking and regulation. Necessary and sufficient conditions for the existence of controllers which solve the tracking and regulation problem are expressed in terms of the plant and the exosystem which generates the disturbances and the reference signal.

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### A. Notation and nomenclature

A few words about the notation and nomenclature used. We use standard symbols for the fields of real and complex numbers $\mathbb{R}$ and $\mathbb{C}$. $\mathbb{C}^-$, and $\bar{\mathbb{C}}_+$ will denote the open left half plane and closed right half plane, respectively. We use $\mathbb{R}^{\mathtt{n}}$, $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, etc., for the real linear spaces of vectors and matrices with components in $\mathbb{R}$.

$\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathbb{R}[\xi]$ denotes the ring of polynomials in the indeterminate $\xi$ with real coefficients. We use $\mathbb{R}^{\mathtt{n}}[\xi], \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, for the spaces of vectors and matrices with components in $\mathbb{R}[\xi]$. Elements of $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$ are called *real polynomial matrices*.

We use the notation $\det(A)$, to denote the determinant of a square matrix A. A square, nonsingular real polynomial matrix $R$ is called *Hurwitz* if all roots of $\det(R)$ lie in the open left half complex plane $\mathbb{C}^-$. It is called *anti-Hurwitz* if all roots of $\det(R)$ lie in the closed right half complex plane $\bar{\mathbb{C}}^+$. Given a real square matrix $A$, we use the notation $\sigma(A)$ for the spectrum of $A$. We use the notation $\mathrm{diag}(a_1, a_2, \ldots, a_n)$ to represent a $\mathtt{n}\times\mathtt{n}$ diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_n$. We abrivate greatest common divisor to $\gcd$. Finally, for $n \geq 1$, $\underline{n}$ is the set $\{1, \ldots, n\}$.

Finally, we use the notation $\mathrm{col}(w_1, w_2)$ to represent the column vector formed by stacking $w_1$ over $w_2$.

## II. LINEAR DIFFERENTIAL SYSTEMS AND POLYNOMIAL KERNEL REPRESENTATIONS

In this section we review the basic material on linear differential systems and their polynomial kernel representations that we need in this paper.

In the behavioral approach to linear systems, a dynamical system is given by a triple $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$, where $\mathbb{R}$ is the time axis, $\mathbb{R}^{\mathtt{w}}$ is the signal space, and the *behavior* $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ consisting of all solutions of a set of higher order, linear, constant coefficient differential equations. Such a triple is called a *linear differential system*. The set of all linear differential systems with $\mathtt{w}$ variables is denoted by $\mathfrak{L}^{\mathtt{w}}$.

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For any linear differential system $\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ there exists a polynomial matrix $R$ with $\mathtt{w}$ columns such that $\mathfrak{B}$ is equal to the space of solutions of

$$R(\tfrac{\mathrm{d}}{\mathrm{d}t})w = 0. \tag{1}$$

If a behavior $\mathfrak{B}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ (or: $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$, with $R(\xi)$ a real polynomial matrix, then we call this a *polynomial kernel representation* of $\mathfrak{B}$. If $R$ has $\mathtt{p}$ rows, then the polynomial kernel representation is said to be *minimal* if every polynomial kernel representation of $\mathfrak{B}$ has at least $\mathtt{p}$ rows. A given polynomial kernel representation, $\mathfrak{B} = \ker(R(\frac{\mathrm{d}}{\mathrm{d}t}))$, is minimal if and only if the polynomial matrix $R$ has full row rank (see [1], Theorem 3.6.4). In the remaining paper we omit the word polynomial from the polynomial kernel representation. The number of rows in any minimal kernel representation of $\mathfrak{B}$, denoted by $\mathtt{p}(\mathfrak{B})$, is called the *output cardinality* of $\mathfrak{B}$. This number corresponds to the number of outputs in any input/output representation of $\mathfrak{B}$. We speak of a system as the behavior $\mathfrak{B}$, one of whose representations is given by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$ or just $\mathfrak{B} = \ker(R)$. The '$\frac{\mathrm{d}}{\mathrm{d}t}$' is often suppressed to enhance readability.

The following Proposition from [1] relates two minimal kernel representations of a given behavior.

*Proposition 2.1:* Let $\mathfrak{B}_1 = \ker(R_1)$ and $\mathfrak{B}_2 = \ker(R_2)$ be minimal kernel representations. Then $\mathfrak{B}_1 = \mathfrak{B}_2$ if and only if there exists a unimodular matrix $U$ such that $R_1 = UR_2$.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $(w_1, w_2)$. Often we are interested only in the behavior of one of the components, say the variable $w_1$, obtained by projecting $\mathfrak{B}$ onto the first component $w_1$. This behavior $\mathfrak{B}_{w_1}$ is defined by $\mathfrak{B}_{w_1} := \{w_1 \mid \exists w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$. If $\mathfrak{B} = \ker(R_1\ R_2)$ is a kernel representation, then a kernel representation for $\mathfrak{B}_{w_1}$ is obtained as follows: choose a unimodular matrix $U$ such that $UR_2 = \begin{pmatrix} R_{12} \\ 0 \end{pmatrix}$, with $R_{12}$ full row rank, and conformably partition $UR_1 = \begin{pmatrix} R_{11} \\ R_{21} \end{pmatrix}$. Then $\mathfrak{B}_{w_1} = \ker(R_{21})$ (see [1], section 6.2.2).

*Definition 2.2:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $w$ partitioned as $w = (w_1, w_2)$. We will call $w_2$ *free in* $\mathfrak{B}$ if, for any $w_2 \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}_2})$, there exists $w_1$ such that $(w_1, w_2) \in \mathfrak{B}$.

The following result was shown in [1]:

*Proposition 2.3:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $(w_1, w_2)$. Let a minimal kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. Then $w_2$ is free in $\mathfrak{B}$ if and only if the polynomial matrix $R_1$ has full row rank.

*Definition 2.4:* A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called *autonomous* if it has no free variables, equivalently, $\mathtt{p}(\mathfrak{B}) = \mathtt{w}$. It is called *stable* if for all $w \in \mathfrak{B}$ we have $\lim_{t\to\infty} w(t) = 0$.

The following Proposition was shown in [1].

*Proposition 2.5:* If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is autonomous if and only if $R$ has full column rank and is stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. Note that a stable behavior is necessarily autonomous.

We denote the set of all linear autonomous differential systems with $\mathtt{w}$ variables by $\mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$.

*Definition 2.6:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then $\mathfrak{B}$ is called *anti-stable* if for all non-zero $w \in \mathfrak{B}$ we have either $\lim_{t\to\infty} w(t) \neq 0$ or $\lim_{t\to\infty} w(t)$ does not exist.

*Proposition 2.7:* If $\mathfrak{B} = \ker(R)$, then $\mathfrak{B}$ is anti-stable if and only if $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^-$.

*Definition 2.8:* A function of the form $H(t) = \sum_{i=1}^{N} \sum_{j=1}^{n_i} A_{ij} t^{j-1} e^{\lambda_i t}$ is called a *Bohl function*, i.e., a Bohl function is a finite sum of products of polynomials and exponentials. In the real case, a Bohl function is a finite sum of products of polynomials, real exponentials, sines, and cosines. A function $H(t)$ is called *stable Bohl* if it is Bohl and $\lim_{t\to\infty} H(t) = 0$. A function $H(t)$ is called *anti-stable Bohl* if it is Bohl and for non-zero $H(t)$ we have either $\lim_{t\to\infty} H(t) \neq 0$ or $\lim_{t\to\infty} H(t)$ does not exist.

Then we have the following Proposition.

*Proposition 2.9:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{aut}}$. Then

1) every $w \in \mathfrak{B}$ is a Bohl functions,
2) if $\mathfrak{B}$ is stable then every $w \in \mathfrak{B}$ is a stable Bohl function, and
3) if $\mathfrak{B}$ is anti-stable then every $w \in \mathfrak{B}$ is a anti-stable Bohl function.

We now recall from [1] the definitions of stabilizability and detectability.

*Definition 2.10:* A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is said to be *stabilizable*, if for every $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that $w'(t) = w(t)$ for $t \leq 0$, and $\lim_{t\to\infty} w'(t) = 0$.

The following result was shown in [1]:

*Proposition 2.11:* If $\mathfrak{B} = \ker(R)$ is a minimal kernel representation of $\mathfrak{B}$, then $\mathfrak{B}$ is stabilizable if and only if $R(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$.

*Definition 2.12:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with plant variable $w = (w_1, w_2)$. We say that $w_2$ is *observable* from $w_1$ in $\mathfrak{B}$ if, whenever $(w_1, w_2)$, $(w_1, w_2') \in \mathfrak{B}$, then $w_2 = w_2'$. We say that $w_2$ is *detectable* from $w_1$ in $\mathfrak{B}$ if, whenever $(w_1, w_2)$, $(w_1, w_2') \in \mathfrak{B}$, then $\lim_{t\to\infty}(w_2 - w_2')(t) = 0$.

The following result was shown in [1]:

*Proposition 2.13:* Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $(w_1, w_2)$. Let a minimal kernel representation of $\mathfrak{B}$ be given by $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2 = 0$. In $\mathfrak{B}$, $w_2$ is observable from $w_1$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. In $\mathfrak{B}$, $w_2$ is detectable from $w_1$ if and only if $R_2(\lambda)$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$.

*Definition 2.14:* Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\bullet}$, $C \in \mathbb{R}^{\bullet\times\mathtt{n}}$. We call the pair $(A, B)$ *stabilizable* if the behavior defined by $\ker\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A & -B \end{pmatrix}$ is stabilizable and we call the pair $(C, A)$ *detectable* if the behavior defined by $\ker\begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t}I - A \\ C \end{pmatrix}$ is stable.

## III. Review of stabilization by interconnection

In this section we will briefly recall the notion of stabilization by interconnection. We will first look at the full interconnection case, i.e. the case when all the plant variables are available for interconnection.

*Definition 3.1:* Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$ be a plant behavior. A controller for $\mathcal{P}$ is a system behavior $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$. The full interconnection of $\mathcal{P}$ and $\mathcal{C}$ is defined as the system with behavior

$\mathcal{P} \cap \mathcal{C}$. This behavior is called the *controlled behavior*, and is also an element of $\mathfrak{L}^{\mathtt{w}}$. The full interconnection is called *regular* if $\mathtt{p}(\mathcal{P} \cap \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. In that case we call $\mathcal{C}$ a *regular controller*.

In full interconnection, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the plant variable $w$ that are already present in the laws of $\mathcal{P}$ (see [2]).

A given plant is stabilizable if and only if we can stabilize it by interconnecting it with a suitable controller, called a stabilizing controller, which is defined as follows [4].

*Definition 3.2:* Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{w}}$ is said to be a *stabilizing controller* if the behavior $\mathcal{P} \cap \mathcal{C}$ is stable and the interconnection is regular.

Next we will look at the so called partial interconnection case, in which only a pre-specified subset of the plant variables is available for interconnection. Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ be a linear differential system, with system variable $(w, c)$, where $w$ takes its values in $\mathbb{R}^{\mathtt{w}}$ and $c$ in $\mathbb{R}^{\mathtt{c}}$. The variable $w$ should be interpreted as the variable to-be-controlled, the variable $c$ as the one through which we can interconnect the plant with a controller, called the control variable. Let $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ (to be interpreted as a controller behavior) with variable $c$.

*Definition 3.3:* The interconnection of $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$ and $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ through $c$ is defined as the system behavior $\mathcal{P} \wedge_c \mathcal{C} \in \mathfrak{L}^{\mathtt{w}+\mathtt{c}}$, given by $\mathcal{P} \wedge_c \mathcal{C} = \{(w, c) \mid (w, c) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}$. The behavior $\mathcal{P} \wedge_c \mathcal{C}$ is called *the full controlled behavior*. The behavior $(\mathcal{P} \wedge_c \mathcal{C})_w \in \mathfrak{L}^{\mathtt{w}}$ that is obtained by eliminating $c$ from $\mathcal{P} \wedge_c \mathcal{C}$ is called *the manifest controlled behavior*. The interconnection of $\mathcal{P}$ and $\mathcal{C}$ through $c$ is called *regular* if $\mathtt{p}(\mathcal{P} \wedge_c \mathcal{C}) = \mathtt{p}(\mathcal{P}) + \mathtt{p}(\mathcal{C})$. $\mathcal{C}$ is then called a *regular controller*.

In partial interconnection, the regularity condition is equivalent to: $\mathcal{C}$ does not re-impose restrictions on the control variable $c$ that are already present in the laws of $\mathcal{P}$ (see [5] and [3]).

If $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2}$ with system variable $(w_1, w_2)$, in this paper we use the notation $\mathcal{N}_{w_1}(\mathcal{P})$ to indicate the behavior obtained by putting $w_2 = 0$ and projecting onto the variable $w_1$ i.e., $\mathcal{N}_{w_1}(\mathcal{P}) = \{w_1 \mid (w_1, 0) \in \mathcal{P}\}$.

The following Proposition on polynomial matrices will be useful in the paper.

*Proposition 3.4:* Let $A \in \mathbb{R}^{\mathtt{p}\times\mathtt{p}}[\xi]$ be Hurwitz and $B \in \mathbb{R}^{\mathtt{q}\times\mathtt{q}}[\xi]$ be anti-Hurwitz. Then for any $C \in \mathbb{R}[\xi]^{\mathtt{p}\times\mathtt{q}}$ there exists a solution $(X, Y)$ of the equation $AX + YB = C$.

*Proof:* Let $U_1AU_2 = \Sigma_1 = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{\mathtt{p}})$ and $V_1BV_2 = \Sigma_2 = \text{diag}(\gamma_1, \gamma_2..., \gamma_{\mathtt{q}})$, where $U_1, U_2, V_1, V_2$ are unimodular matrices. As $A$ Hurwitz and $B$ anti-Hurwitz we have $\Sigma_1$ Hurwitz and $\Sigma_2$ anti-Hurwitz. Define $X' := U_2^{-1}XV_2$, $Y' := U_1YV_1^{-1}$ and $C' := U_1CV_2$. It is easy to see that the following statements are equivalent:

1) for any $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ there exists a solution $(X, Y)$ of the equation $AX + YB = C$.
2) for any $C' \in \mathbb{R}^{\mathtt{p}\times\mathtt{q}}[\xi]$ there exists a solution $(X', Y')$ of the equation $\Sigma_1X' + Y'\Sigma_2 = C'$.
3) for any $c'_{ij} \in \mathbb{R}[\xi]$ there exists a solution $(x'_{ij}, y'_{ij})$ of the equation $\lambda_i x'_{ij} + y'_{ij}\gamma_j = c'_{ij}$ where $\lambda_i$ and $\gamma_j$ are $i^{th}$ and $j^{th}$ diagonal elements of $\Sigma_1$ and $\Sigma_2$ respectively.
4) $\gcd(\lambda_i, \gamma_j) = 1$ for all $i \in \underline{\mathtt{p}}$ and $j \in \underline{\mathtt{q}}$.

As $\Sigma_1$ Hurwitz and $\Sigma_2$ anti-Hurwitz $\gcd(\lambda_i, \gamma_j) = 1$ for all $i \in \underline{\mathtt{p}}$ and $j \in \underline{\mathtt{q}}$. Hence if $A$ is Hurwitz and $B$ is anti-Hurwitz then the Equation $AX + YB = C$ is universally solvable for $(X, Y)$. □

In the next section we will formulate the tracking and regulation problem studied in this paper.

## IV. Tracking and regulation

An important synthesis problem in control is to design for a given plant behavior with its to-be-controlled variable $w$ and reference signal $r$, a controller such that in the resulting system after interconnecting the plant and the controller, the plant variable $w$ follows the reference signal $r$. This is called the *tracking problem*. The classical approach to this problem is to let the reference signal be generated by an autonomous system called the *exosystem*. One then incorporates the dynamics of the exosystem into the dynamics of the plant and defines a new variable $e$ as the difference between the reference signal $r$ and $w$. The tracking problem is then reformulated as: design a controller which drives the signal $e$ to zero if it is interconnected with the plant.

A second important synthesis problem is the problem of *regulation*. For a given plant with to-be-controlled variable $w$, and external disturbance acting on the plant (which is assumed to be free in the plant), the problem is here to design a controller such that in the resulting system after interconnection of the plant and the controller, disturbance remains free and the plant variable $w$ converges to zero as time tends to infinity, regardless of the disturbance acting on the plant. Similar to the tracking problem we approach this problem by assuming the disturbances to be generated by some linear time invariant autonomous system, again called the *exosystem*. Then one incorporates the dynamics of the exosystem into the dynamics of the plant, and requires the variable $w$ in this interconnected system to converge to zero as time tends infinity.

Combining these two synthesis problems we can formulate a new synthesis problem by requiring the design of a controller such that the interconnected system variable tracks a given reference signal, regardless of the disturbance. This is done by combing the two exosystems into a single one and requires regulation of the tracking error.

In addition to the requirements of tracking and regulation, a realistic design requires the system to go to rest in the absence of disturbances (if the disturbance signal is equal to zero).

In this section we will introduce the problem of tracking and regulation in the behavioral context, with control by general, regular, interconnection.

We start with a plant behavior $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variable $(w_1, w_2, c, v)$. Variables $w_2$, $c$, $v$ represent the to-be-regulated variable (like tracking error), the interconnection variable (like sensor measurements and actuator inputs), and external disturbances respectively. $w_1$ is an auxiliary variable of the plant which only needs to be driven to zero in the absence of disturbances (e.g. the state variable in the state space set-up). The interconnection variable $c$ is the system variable through which we are allowed to interconnect $\mathcal{P}$ with a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$. As the variable $v$ represents reference signal and external disturbances we assume it to be free in $\mathcal{P}$. In addition to the plant $\mathcal{P}$, let an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ which generates the disturbances and the reference signal be given. The interconnection of the plant $\mathcal{P}$ with a controller $\mathcal{C}$ is given by $\mathcal{P} \wedge_c \mathcal{C} := \{(w_1, w_2, c, v) \mid (w_1, w_2, c, v) \in \mathcal{P} \text{ and } c \in \mathcal{C}\}$. The interconnection of the plant $\mathcal{P}$, the exosystem $\mathcal{E}$ and a controller $\mathcal{C}$ is given by $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} := \{(w_1, w_2, c, v) \mid (w_1, w_2, c, v) \in \mathcal{P},\ v \in \mathcal{E} \text{ and } c \in \mathcal{C}\}$.

We have the following definition of a regulator.

*Definition 4.1:* A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ is called a *regulator* for $\mathcal{P}$ with respect to $\mathcal{E}$ if it satisfies the following conditions

1) the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular,
2) $v$ is free in $\mathcal{P} \wedge_c \mathcal{C}$,
3) for all $(w_1, w_2, c, 0) \in \mathcal{P} \wedge_c \mathcal{C}$ we have $\lim_{t\to\infty}(w_1(t), w_2(t), c(t)) = (0, 0, 0)$, i.e., $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable, and
4) for all $(w_1, w_2, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ we have $\lim_{t\to\infty} w_2(t) = 0$, i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ is stable.

Condition (2.) in the above definition asks the controller not to put any restrictions on the variable $v$ which represents the reference signal and external disturbances acting on the system. Condition (4.) asks the controller to achieve regulation of the tracking error, and condition (3.) asks the controller to drive the plant variables $w_1, w_2$ and $c$ to zero if $v = 0$, i.e., if the disturbance is absent. Condition (1.) about the regularity of the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ will make sure that $\mathcal{C}$ does not re-impose restrictions on the control variable $c$ that are already present in the laws of $\mathcal{P}$. A controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ satisfying conditions (1.), (2.) and (3.) is called *disturbance-free stabilizing* controller for $\mathcal{P}$.

We now formulate the main problem of this paper:

**Problem 1:** Given a plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variable $(w_1, w_2, c, v)$, with $v$ free in $\mathcal{P}$, and an exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable $v$, find necessary and sufficient conditions for the existence of a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$.

With out loss of generality, in this paper we make the following assumptions.

**Assumptions :**

A1. $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ is an anti-stable system, and

A2. $v$ is observable from $(w_2, c)$ in $(\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$, i.e., $\mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)}) = 0$.

Assumption A1 does not lose generality because, as can be easily seen, the asymptotically stable trajectories of the exosystem do not affect the regulation of the variable $w_2$ as long as the disturbance-free stability is achieved (as can be seen, for disturbance-free stability the exosystem is disregarded). As $v$ is free in the plant $\mathcal{P}$, if $\mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)}) \neq 0$, then it can be shown that the plant signals $(w_2, c)$ are already decoupled from all $v \in \mathcal{E} \cap \mathcal{N}_v((\mathcal{P})_{(w_2,c,v)})$. Hence, Assumption A2 is without loss of generality.

The following Theorem will be useful in solving Problem 1.

*Theorem 4.2:* Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}+\mathtt{v}}$ with system variable $(w, v)$. Assume $v$ is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ be an anti-stable system with system variable $v$. Then $(\mathcal{P} \wedge_v \mathcal{E})_w$ is stable if and only if the following conditions hold.

1) for all $(w, 0) \in \mathcal{P}$, $\lim_{t\to\infty} w(t) = 0$ i.e., $\mathcal{N}_w(\mathcal{P})$ is stable, and
2) for all $v \in \mathcal{E}$ we have $(0, v) \in \mathcal{P}$ i.e., $\mathcal{E} \subseteq \mathcal{N}_v(\mathcal{P})$.

*Proof: (if)* $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$ implies $(w, v) \in \mathcal{P}$ and $v \in \mathcal{E}$. As for all $v \in \mathcal{E}$, $(0, v) \in \mathcal{P}$, from linearity, we have $(w, v) - (0, v) \in \mathcal{P}$. Therefore $(w, 0) \in \mathcal{P}$. As $\lim_{t\to\infty} w(t) = 0$ holds for all $(w, 0) \in \mathcal{P}$ , we conclude that $\lim_{t\to\infty} w(t) = 0$ holds for all $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$ .

*(only if)*

We have $\{(w, 0) \mid (w, 0) \in \mathcal{P}\} \subseteq \mathcal{P} \wedge_v \mathcal{E}$. As for all $(w, v) \in \mathcal{P} \wedge_v \mathcal{E}$, $\lim_{t\to\infty} w(t) = 0$ we have for all $(w, 0) \in \mathcal{P}$, $\lim_{t\to\infty} w(t) = 0$. Let

$$\mathcal{P} = \{(w, v) \mid G_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w + G_2(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \tag{2}$$

be a minimal kernel representation. As $v$ is free in $\mathcal{P}$ for all $v \in \mathcal{E}$, there exists a $w$ such that

$$G_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w = -G_2(\tfrac{\mathrm{d}}{\mathrm{d}t})v. \tag{3}$$

As $(\mathcal{P} \wedge_v \mathcal{E})_w$ is stable, $w$ is a stable Bohl. Also, $v$ anti-stable Bohl. As in Equation (3) the LHS is stable Bohl and the RHS is unstable Bohl, we have $G_1 w = -G_2 v = 0$. From Equation (2) this implies that $(w, 0) \in \mathcal{P}$. From linearity we have $(w, v) - (w, 0) \in \mathcal{P}$, which implies that $(0, v) \in \mathcal{P}$. Therefore $v \in \mathcal{N}_v(\mathcal{P})$. □

The following Theorem provides a solution to the Problem 1.

*Theorem 4.3:* Let $\mathcal{P} \in \mathfrak{L}^{\mathtt{w}_1+\mathtt{w}_2+\mathtt{c}+\mathtt{v}}$ with system variable $(w_1, w_2, c, v)$. Assume $v$ is free in $\mathcal{P}$. Let $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}_{\text{aut}}$ with system variable $v$ satisfies the assumptions A1 and A2. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if the following conditions hold.

1) $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ is stabilizable,
2) $(w_1, w_2, v)$ is detectable from $c$ in $\mathcal{P} \wedge_v \mathcal{E}$, and
3) there exists polynomial matrices $L(\xi) \in \mathbb{R}[\xi]^{\mathtt{w}_1\times\mathtt{v}}$ and $M(\xi) \in \mathbb{R}[\xi]^{\mathtt{c}\times\mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(L(\tfrac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\tfrac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$.

*Proof:* Let $\mathcal{P}$ and $\mathcal{E}$ be given by minimal kernel representations

$$\mathcal{P} = \{(w_1, w_2, c, v) \mid R_1(\tfrac{\mathrm{d}}{\mathrm{d}t})w_1 + R_2(\tfrac{\mathrm{d}}{\mathrm{d}t})w_2 + R_3(\tfrac{\mathrm{d}}{\mathrm{d}t})c + R_4(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\}, \tag{4}$$

and

$$\mathcal{E} = \{v \mid V(\tfrac{\mathrm{d}}{\mathrm{d}t})v = 0\} \tag{5}$$

respectively.

*(necessity)*

There exists a unimodular matrix $U$ such that $U\begin{pmatrix} R_1 & R_2 & R_3 & R_4 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \end{pmatrix}$, where $R_{11}$ has full row rank. Using Prosposition 2.1 we have

$$\mathcal{P} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \end{pmatrix}. \quad (6)$$

From Equation (6) we have

$$\begin{aligned} \mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}) &= \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \end{pmatrix}, \quad (7)\\ (\mathcal{P})_{(w_2,c,v)} &= \ker \begin{pmatrix} R_{22} & R_{23} & R_{24} \end{pmatrix}, \quad (8)\\ \mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)}) &= \ker \begin{pmatrix} R_{22} & R_{23} \end{pmatrix}. \quad (9) \end{aligned}$$

From Equations (5) and (6) we have

$$\begin{aligned} \mathcal{P} \wedge_v \mathcal{E} &= \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \\ 0 & 0 & 0 & V \end{pmatrix}, \quad (10)\\ (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} &= \ker \begin{pmatrix} R_{22} & R_{23} & R_{24} \\ 0 & 0 & V \end{pmatrix}. \quad (11) \end{aligned}$$

For any $\mathcal{C} \in \mathfrak{L}^{\mathtt{c}}$ given by minimal kernel representation $\mathcal{C} = \ker(C)$ we have

$$\begin{aligned} \mathcal{P} \wedge_c \mathcal{C} &= \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \\ 0 & 0 & C & 0 \end{pmatrix}, \quad (12)\\ (\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C} &= \ker \begin{pmatrix} R_{22} & R_{23} & R_{24} \\ 0 & C & 0 \end{pmatrix}, \quad (13) \end{aligned}$$

and

$$\mathcal{N}_{(w_2,c)}((\mathcal{P})_{(w_2,c,v)} \wedge_c \mathcal{C}) = \ker \begin{pmatrix} R_{22} & R_{23} \\ 0 & C \end{pmatrix}. \quad (14)$$

*(necessity of condition 1))* Let $\mathcal{C} = \ker(C)$ be a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. From Equation (12) and Proposition 2.5 the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, $v$ is free in $\mathcal{P} \wedge_c \mathcal{C}$, and $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ stable implies that $\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & R_{22} & R_{23} \\ 0 & 0 & C \end{pmatrix}$ is square, nonsingular and Hurwitz. As a result $R_{11}$ and $\begin{pmatrix} R_{22} & R_{23} \\ 0 & C \end{pmatrix}$ are square, nonsingular and Hurwitz. Therefore $\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & R_{22}(\lambda) & R_{23}(\lambda) \end{pmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Equation (7) and Proposition 2.11 it is evident that $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ is stabilizable.

*(necessity of condition 2))* We have $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2} = ((\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2}$ stable. We have $\{(w_2,0,v) \mid (w_2,0,v) \in (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}\} \subseteq (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$. Therefore using Definition 4.1, for all $(w_2,0,v) \in (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$ we have $\lim_{t\to\infty} w_2(t) = 0$. Hence for all $(w_2,0,v) \in (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$, $w_2$ is a stable Bohl. As $v$ is observable from $(w_2,c)$ in $\mathcal{P} \wedge_v \mathcal{E}$, for all $(w_2,0,v) \in (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$ and $w_2$ stable Bohl we have $v$ stable Bohl. Therefore for all $(w_2,0,v) \in (\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$ we have $\lim_{t\to\infty}(w_2(t), v_2(t)) = 0$, in other words $(w_2,v)$ is detectable from $c$ in $(\mathcal{P})_{(w_2,c,v)} \wedge_v \mathcal{E}$. From Equation (11), $\begin{pmatrix} R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{14}(\lambda) \\ 0 & R_{22}(\lambda) & R_{24}(\lambda) \\ 0 & 0 & V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$ (use the fact that $R_{11}$ is Hurwitz). Using Equation (10) and Proposition 2.13 we conclude that $(w_1,w_2,v)$ is detectable from $c$ in $\mathcal{P} \wedge_v \mathcal{E}$.

*(necessity of condition 3))* There exists a unimodular matrix $U_2$ such that

$$U_2 \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & R_{22} & R_{23} & R_{24} \\ 0 & 0 & C & 0 \end{pmatrix} = \begin{pmatrix} 0 & \tilde{R}_{12} & 0 & \tilde{R}_{14} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} & \tilde{R}_{24} \end{pmatrix}, \quad (15)$$

where $\tilde{R}_{12}$ and $\begin{pmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{pmatrix}$ are square, non-singular and Hurwitz. From Equation (12), we have

$$\begin{aligned} \mathcal{P} \wedge_c \mathcal{C} &= \ker \begin{pmatrix} 0 & \tilde{R}_{12} & 0 & \tilde{R}_{14} \\ \tilde{R}_{21} & \tilde{R}_{22} & \tilde{R}_{23} & \tilde{R}_{24} \end{pmatrix}, \\ (\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)} &= \ker \begin{pmatrix} \tilde{R}_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & \tilde{R}_{14}(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix}, \\ \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)}) &= \ker(\tilde{R}_{14}(\tfrac{\mathrm{d}}{\mathrm{d}t})). \quad (16) \end{aligned}$$

From Theorem 4.2, $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_{w_2} = ((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)} \wedge_v \mathcal{E})_{w_2}$ stable implies that $\mathcal{E} \subseteq \mathcal{N}_v((\mathcal{P} \wedge_c \mathcal{C})_{(w_2,v)})$. From Equation (16) and (5), there exists a polynomial matrix $\tilde{N}$ such that

$$\tilde{R}_{14} = \tilde{N}V. \quad (17)$$

From Proposition 3.4, as $\begin{pmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{pmatrix}$ Hurwitz and $V$ anti-Hurwitz there exists a solution $\left(\begin{pmatrix} L \\ M \end{pmatrix}, \tilde{P}\right)$ for the Equation

$$\begin{pmatrix} \tilde{R}_{21} & \tilde{R}_{23} \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} + \tilde{R}_{24} = \tilde{P}V. \quad (18)$$

From Equations (17) and (18), we have

$$\begin{pmatrix} 0 & 0 \\ \tilde{R}_{21} & \tilde{R}_{23} \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} + \begin{pmatrix} \tilde{R}_{14} \\ \tilde{R}_{24} \end{pmatrix} = \begin{pmatrix} \tilde{N} \\ \tilde{P} \end{pmatrix} V. \quad (19)$$

Multiplying both side of Equation (19) with $U_2^{-1}$ we obtain

$$\begin{pmatrix} R_{11} & R_{13} \\ 0 & R_{22} \\ 0 & C \end{pmatrix} \begin{pmatrix} L \\ M \end{pmatrix} + \begin{pmatrix} R_{14} \\ R_{24} \\ 0 \end{pmatrix} = \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} V \quad (20)$$

where $\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} := U_2^{-1} \begin{pmatrix} \tilde{N} \\ \tilde{P} \end{pmatrix}$. Then we have

$$\begin{pmatrix} R_{11} \\ 0 \end{pmatrix} L + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} M + \begin{pmatrix} R_{14} \\ R_{24} \end{pmatrix} = NV. \quad (21)$$

Since $\mathcal{E} = \ker(V)$, for all $v \in \mathcal{E}$ we then have $\begin{pmatrix} R_{11} & R_{13} & R_{14} \\ 0 & R_{23} & R_{24} \end{pmatrix} \begin{pmatrix} L \\ M \\ I \end{pmatrix} (\frac{\mathrm{d}}{\mathrm{d}t}) v = 0$, i.e., $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$.

*(sufficiency)*

Let $\mathcal{P}$ be given by the Equation (4). There exists a unimodular matrix $U$ such that $U\begin{pmatrix} R_1 & R_2 & R_3 & R_4 \end{pmatrix} = \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & 0 & R_{23} & R_{24} \end{pmatrix}$, where $(R_{11} \ R_{12})$ has full row rank. Using Prosposition 2.1 we have

$$\mathcal{P} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & 0 & R_{23} & R_{24} \end{pmatrix}. \quad (22)$$

From Equation (22) we have

$$\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}) = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & 0 & R_{23} \end{pmatrix}, \quad (23)$$

$$(\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}))_c = \ker(R_{23}), \quad (24)$$

and

$$\mathcal{P} \wedge_v \mathcal{E} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & 0 & R_{23} & R_{24} \\ 0 & 0 & 0 & V \end{pmatrix}. \quad (25)$$

There exists polynomial matrices $L(\xi) \in \mathbb{R}[\xi]^{\mathtt{w}_1 \times \mathtt{v}}$ and $M(\xi) \in \mathbb{R}[\xi]^{\mathtt{c} \times \mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$. Hence $V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0$ implies $\begin{pmatrix} R_{11} \\ 0 \end{pmatrix} L(\frac{\mathrm{d}}{\mathrm{d}t})v + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} M(\frac{\mathrm{d}}{\mathrm{d}t})v + \begin{pmatrix} R_{14} \\ R_{24} \end{pmatrix} (\frac{\mathrm{d}}{\mathrm{d}t})v = 0$. Therefore there exists a polynomial matrix $Y = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix}$ such that

$$\begin{pmatrix} R_{11} \\ 0 \end{pmatrix} L + \begin{pmatrix} R_{13} \\ R_{23} \end{pmatrix} M + \begin{pmatrix} R_{14} \\ R_{24} \end{pmatrix} = \begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} V. \quad (26)$$

This implies

$$R_{23}M + R_{24} = Y_2 V. \quad (27)$$

From Equation (23), $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P})$ stabilizable implies that $\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{13}(\lambda) \\ 0 & 0 & R_{23}(\lambda) \end{pmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$, which in turn implies that $R_{23}(\lambda)$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+$. From Equation (24) and Proposition 2.11 we conclude that $(\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}))_c$ stabilizable. Factorize $R_{23} = DK$ where $D$ is Hurwitz and $K(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let $S$ be such that $\begin{pmatrix} K \\ S \end{pmatrix}$ is unimodular. Then for any arbitrary polynomial matrix $F$ and arbitrary Hurwitz polynomial matrix $H$ of suitable dimensions, it is easy to check that

$$C = FR_{23} + HS \quad (28)$$

acts as a stabilizing controller for $(\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P}))_c$.

We note that for all $C$ given by the Equation (28) we have $\begin{pmatrix} R_{23} \\ C \end{pmatrix}$ is Hurwitz.

From Equation (25), $(w_1, w_2, v)$ is detectable from $c$ in $\mathcal{P} \wedge_v \mathcal{E}$ implies that $\begin{pmatrix} R_{11}(\lambda) & R_{12}(\lambda) & R_{14}(\lambda) \\ 0 & 0 & R_{24}(\lambda) \\ 0 & 0 & V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. This implies that $(R_{11}\ R_{12})$ is square nonsingular and Hurwitz and $\begin{pmatrix} R_{24}(\lambda) \\ V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+$. As $V(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}^-$ (use the fact that $V$ is anti-Hurwitz) we conclude that $\begin{pmatrix} R_{24}(\lambda) \\ V(\lambda) \end{pmatrix}$ has full column rank for all $\lambda \in \mathbb{C}$. Hence there always exists a solution $(F, N)$ for the equation

$$FR_{24} + NV = HSM. \quad (29)$$

We now prove that any controller given by $\mathcal{C} = \ker(C)$ where $C = FR_{23} + HS$ with $F$ satisfying the Equation (29) acts as a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$. The following identities hold true.

$$\begin{aligned} CM &= FR_{23}M + HSM \\ &= FR_{23}M + FR_{24} + NV \text{ ( from Equation (29) )} \\ &= F(R_{23}M + R_{24}) + NV \\ &= FY_2V + NV \text{( from Equation (27))} \\ &= (FY_2 + N)V. \end{aligned}$$

We have

$$CM = WV, \quad (30)$$

where $W := FY_2 + N$. We have

$$\mathcal{P} \wedge_c \mathcal{C} = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} & R_{14} \\ 0 & 0 & R_{23} & R_{24} \\ 0 & 0 & C & 0 \end{pmatrix}, \quad (31)$$

$$\mathcal{N}_{(w,c)}(\mathcal{P} \wedge_c \mathcal{C}) = \ker \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & C \end{pmatrix}. \quad (32)$$

As $C$ is chosen such that $\begin{pmatrix} R_{23} \\ C \end{pmatrix}$ is Hurwitz, we have $\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & C \end{pmatrix}$ square, nonsingular and Hurwitz. Therefore from Equation (31), the interconnection $\mathcal{P} \wedge_c \mathcal{C}$ is regular, from Equation (32) and Proposition 2.5 $\mathcal{N}_{(w_1,w_2,c)}(\mathcal{P} \wedge_c \mathcal{C})$ is stable, and also from Proposition 2.3, $v$ is free in $\mathcal{P} \wedge_c \mathcal{C}$.

We have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w_1, w_2, c, v) \mid R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})w_1 + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})w_2 + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{14}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})c + R_{24}(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; C(\frac{\mathrm{d}}{\mathrm{d}t})c = 0; V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0\}$. Using Equation (26) we have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w_1, w_2, c, v) \mid R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})(w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v) + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})w_2 + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_1V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; C(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + CM(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0\}$. Using Equation (30) we have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{(w, c, v) \mid R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})(w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v) + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})w_2 + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_1V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + Y_2V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; C(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) + WV(\frac{\mathrm{d}}{\mathrm{d}t})v = 0; V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0\}$. Finally after eliminating $v$ from the first three equations we have $\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C} = \{R_{11}(\frac{\mathrm{d}}{\mathrm{d}t})(w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v) + R_{12}(\frac{\mathrm{d}}{\mathrm{d}t})w_2 + R_{13}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0; R_{23}(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0; C(\frac{\mathrm{d}}{\mathrm{d}t})(c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0; V(\frac{\mathrm{d}}{\mathrm{d}t})v = 0\}$.

From the above, for all $(w_1, w_2, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, $(w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v, w_2, c - M(\frac{\mathrm{d}}{\mathrm{d}t})v)$ satisfies the equation $\begin{pmatrix} R_{11}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{12}(\frac{\mathrm{d}}{\mathrm{d}t}) & R_{13}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & R_{23}(\frac{\mathrm{d}}{\mathrm{d}t}) \\ 0 & 0 & C(\frac{\mathrm{d}}{\mathrm{d}t}) \end{pmatrix} \begin{pmatrix} w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v \\ w_1 \\ c - M(\frac{\mathrm{d}}{\mathrm{d}t})v \end{pmatrix} = 0$. As $\begin{pmatrix} R_{11} & R_{12} & R_{13} \\ 0 & 0 & R_{23} \\ 0 & 0 & C \end{pmatrix}$ is Hurwitz, for all $(w_1, w_2, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$, we have $\lim_{t\to\infty}(w_1 - L(\frac{\mathrm{d}}{\mathrm{d}t})v, w_2, c - M(\frac{\mathrm{d}}{\mathrm{d}t})v) = 0$. Therefore we conclude that for all $(w_1, w_2, c, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C}$ we have $\lim_{t\to\infty} w(t) = 0$ i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_c \mathcal{C})_w$ is stable. □

*Remark 4.4:* We note that the conditions given in Theorem 4.3 are representation free and depends only on the plant and exosystem dynamics.

## V. Statespace case

So far we have obtained necessary and sufficient conditions for the existence of a regulator for a given plant with respect to a given exosystem which are representation free and depends only in terms of the plant and exosystem dynamics. In this section we obtain these conditions in terms of representations of the plant and the exosystem. For this we look at a special case when the plant and the exosystem are given by the first order linear state space equations.

In state space representation the plant $\mathcal{P} \in \mathfrak{L}^{\mathtt{x}+\mathtt{w}+(\mathtt{u}+\mathtt{y})+\mathtt{v}}$ and the exosystem $\mathcal{E} \in \mathfrak{L}^{\mathtt{v}}$ are described by

$$\mathcal{P} = \left\{ (x, w, (u, y), v) \mid \begin{array}{rcl} \dot{x} & = & A_3 v + A_2 x + B_2 u \\ y & = & C_1 v + C_2 x \\ w & = & D_1 v + D_2 x + Eu \end{array} \right\} \tag{33}$$

and

$$\mathcal{E} = \{ v \mid \dot{v} = A_1 v \} \tag{34}$$

respectively. Here, the variables $x$, $w$, $v$ in the plant represent the state variable of the plant, the to-be-regulated variable, and the external disturbances, respectively and the variable $(u, y)$ represents the control variable available for interconnection with the controller. Here the assumptions A1 and A2 given in section IV translates in to $\sigma(A_1) \subset \bar{\mathbb{C}}^+$ and $v$ is observable from $(w, u, y)$ in $\mathcal{P} \wedge_v \mathcal{E}$.

Rewriting the behaviors given in Equations (33) and (34) in kernel representations, we have

$$\mathcal{P} = \ker \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{x}} - A_2 & 0 & -B_2 & 0 & -A_3 \\ -C_2 & 0 & 0 & I & -C_1 \\ -D_2 & I & -E & 0 & -D_1 \end{pmatrix}, \tag{35}$$

and

$$\mathcal{E} = \ker(\tfrac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{v}} - A_1). \tag{36}$$

It is easy to see that the kernel representations given in Equations (35) and (36) are minimal. Then we have

$$\mathcal{P} \wedge_v \mathcal{E} = \ker \begin{pmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{x}} - A_2 & 0 & -B_2 & 0 & -A_3 \\ -C_2 & 0 & 0 & I & -C_1 \\ -D_2 & I & -E & 0 & -D_1 \\ 0 & 0 & 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} I_{\mathtt{v}} - A_1 \end{pmatrix} \tag{37}$$

The problem of regulation here is to design a controller $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ such that

1) the interconnection $\mathcal{P} \wedge_{(u,y)} \mathcal{C}$ is regular,
2) $v$ is free in $\mathcal{P} \wedge_{(u,y)} \mathcal{C}$,
3) for all $(x, w, u, y, v) \in \mathcal{P} \wedge_v \mathcal{E} \wedge_{(u,y)} \mathcal{C}$ we have $\lim_{t\to\infty} w(t) = 0$ i.e., $(\mathcal{P} \wedge_v \mathcal{E} \wedge_{(u,y)} \mathcal{C})_w$ is stable, and
4) for all $(x, w, u, y, 0) \in \mathcal{P} \wedge_{(u,y)} \mathcal{C}$ we have $\lim_{t\to\infty} \mathrm{col}(x(t), w(t), u(t), y(t)) = 0$ i.e., $\mathcal{N}_{(x,w,u,y)}(\mathcal{P} \wedge_{(u,y)} \mathcal{C})$ is stable.

Then we have the following Theorem.

*Theorem 5.1:* Let $\mathcal{P}$ and $\mathcal{E}$ are given by the Equations (33) and (34). Assume $\sigma(A_1) \subset \bar{\mathbb{C}}^+$ and $v$ is observable from $(w, u, y)$ in $\mathcal{P} \wedge_v \mathcal{E}$. Then there exists a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$ if the following conditions holds true.

$\mathfrak{C}_1$. pair $(A_2, B_2)$ is stabilizable,

$\mathfrak{C}_2$. pair $\left( \begin{pmatrix} C_2 & C_1 \end{pmatrix}, \begin{pmatrix} A_2 & A_3 \\ 0 & A_1 \end{pmatrix} \right)$ is detectable,

$\mathfrak{C}_3$. there exists $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ such that

$$\begin{aligned} SA_1 - A_2 S - B_2 T &= A_3 &(38)\\ D_1 + D_2 S + ET &= 0. &(39) \end{aligned}$$

Conditions $\mathfrak{C}_1$ and $\mathfrak{C}_2$ are also necessary for the existence of a regulator for $\mathcal{P}$ with respect to $\mathcal{E}$.

*Proof:* From Theorem 4.3, there exists a regulator $\mathcal{C} \in \mathfrak{L}^{\mathtt{u}+\mathtt{y}}$ for $\mathcal{P}$ with respect to $\mathcal{E}$ if and only if conditions given in Theorem 4.3 are satisfied.

1) From Equation (37), $(x, w, v)$ detectable from $c$ in $\mathcal{P} \wedge_v \mathcal{E} \Leftrightarrow \begin{pmatrix} \lambda I_{\mathtt{x}} - A_2 & 0 & -A_3 \\ -C_2 & 0 & -C_1 \\ -D_2 & I & -D_1 \\ 0 & 0 & \lambda I_{\mathtt{v}} - A_1 \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow \begin{pmatrix} \lambda I_{\mathtt{x}} - A_2 & -A_3 \\ 0 & \lambda I_{\mathtt{v}} - A_1 \\ -C_2 & -C_1 \end{pmatrix}$ has full column rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow$ pair $\left( \begin{pmatrix} C_2 & C_1 \end{pmatrix}, \begin{pmatrix} A_2 & A_3 \\ 0 & A_1 \end{pmatrix} \right)$ is detectable.
2) From Equation (35) $\mathcal{N}_{(x,w,u,y)}(\mathcal{P})$ stabilizable $\Leftrightarrow \begin{pmatrix} \lambda I_{\mathtt{x}} - A_2 & 0 & -B_2 & 0 \\ -C_2 & 0 & 0 & I \\ -D_2 & I & -E & 0 \end{pmatrix}$ has full row rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow \begin{pmatrix} \lambda I_{\mathtt{x}} - A_2 & -B_2 \end{pmatrix}$ full row rank for all $\lambda \in \bar{\mathbb{C}}^+ \Leftrightarrow$ pair $(A_2, B_2)$ stabilizable.
3) From Equation (35) there exists polynomial matrices $L(\xi) \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{v}}$ and $M(\xi) = \begin{pmatrix} M_1(\xi) \\ M_2(\xi) \end{pmatrix} \in \mathbb{R}[\xi]^{(\mathtt{u}+\mathtt{y})\times\mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M_1(\frac{\mathrm{d}}{\mathrm{d}t})v, M_2(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P} \Leftrightarrow$ there exists a polynomial matrix $\begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix}$ such that

$$\begin{aligned} (\xi I_{\mathtt{x}} - A_2)L - B_2 M_1 - A_3 - N_1(\xi I_{\mathtt{v}} - A_1) &= 0, &(40)\\ -C_2 L + M_2 - C_1 - N_2(\xi I_{\mathtt{v}} - A_1) &= 0, &(41)\\ -D_2 L - EM_1 - D_1 - N_3(\xi I_{\mathtt{v}} - A_1) &= 0. &(42) \end{aligned}$$

We have the following Lemma:

*Lemma 5.2:* There exists a polynomial matrices $L, M_1, M_2, N_1, N_2$ and $N_3$ such that Equations (40), (41) and (42) hold true if there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ satisfying

$$\begin{aligned} SA_1 - A_2 S - B_2 T &= A_3, &(43)\\ D_1 + D_2 S + ET &= 0. &(44) \end{aligned}$$

*Proof:*

Assume that there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$ such that Equations (43) and (44) hold. Choose $L = S$, $N_1 = S$, $N_2$ any arbitrary polynomial matrix (with appropriate dimensions), $N_3 = 0$, $M_1 = T$ and $M_2 = (\xi I_{\mathtt{v}} - A_1)N_2 + C_1 + C_2 S$. Then it is easy to verify that these $L, M_1, M_2, N_1, N_2$ and $N_3$ satisfy the Equations (40), (41) and (42). □

From above we have shown that there exists polynomial matrices $L(\xi) \in \mathbb{R}[\xi]^{\mathtt{x}\times\mathtt{v}}$ and $M(\xi) \in \mathbb{R}[\xi]^{(\mathtt{u}+\mathtt{y})\times\mathtt{v}}$ such that for all $v \in \mathcal{E}$ we have $(L(\frac{\mathrm{d}}{\mathrm{d}t})v, 0, M(\frac{\mathrm{d}}{\mathrm{d}t})v, v) \in \mathcal{P}$ if there exist $S \in \mathbb{R}^{\mathtt{x}\times\mathtt{v}}$ and $T \in \mathbb{R}^{\mathtt{u}\times\mathtt{v}}$

$$\begin{aligned} SA_1 - A_2 S - B_2 T &= A_3, &(45)\\ D_1 + D_2 S + ET &= 0. &(46) \end{aligned}$$

completes the proof of the Theorem 5.1. □

*Remark 5.3:* The conditions in Theorem 5.1 coincide with classical results on state space systems. For example see Theorem 9.2 of [14] and references therein.

## VI. Conclusion

In this paper, we discussed the problem tracking and regulation in the behavioral framework. We have formulated and resolved the problem of tracking and regulation in a completely representation free manner. Given the plant and the exosystem, we have established necessary and sufficient conditions for the existence of a regulator only in terms of the plant and the exosystem dynamics.

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