Proceedings of the 19th International Symposium on Mathematical Theory of Networks and Systems – MTNS 2010 • 5–9 July, 2010 • Budapest, Hungary

A polynomial approach to the realization of J-lossless behaviours

Shodhan Rao, Paolo Rapisarda and Lewis Moody

***Abstract*— In this paper, a class of behaviours known as J-lossless behaviours is introduced, where J is a symmetric two-variable polynomial matrix. For a certain J, it is shown that the resulting set of J-lossless behaviours are SISO behaviours such that for each of such behaviours, there exists a quadratic differential form which is positive for nonzero trajectories of the behaviour and whose derivative is equal to the product of the input variable and the derivative of the output variable. Earlier, Van der Schaft and Oeloff had considered a specific form of realization for such behaviours that plays an important role in their model reduction procedure. In our paper, we give a method of computation of a state space realization from a transfer function of such a behaviour in the same form as considered by Van der Schaft and Oeloff, using polynomial algebraic methods. Apart from being useful in enlarging the scope of the model reduction procedure of Van der Schaft and Oeloff, we show that our method of realization also has application in the synthesis of lossless mechanical systems with given transfer functions using springs and masses.**

I. Introduction

This paper deals with the realization of linear SISO lossless systems with external control u in the form:

$$\begin{aligned}\begin{pmatrix}\dot q\\ \dot p\end{pmatrix} &= \begin{pmatrix}0 & P\\ -Q & 0\end{pmatrix}\begin{pmatrix}q\\ p\end{pmatrix} + \begin{pmatrix}0\\ B\end{pmatrix}u \\ y &= B^\top q\end{aligned} \quad (1)$$

where $P = P^T > 0$, $Q = Q^T > 0$. The equations (1) arise naturally when considering conservative mechanical systems, in which case q is the vector of positions, and p that of momenta. In this case, the matrices P and Q define the total energy of the system as in $p^TPp + q^TQq = E(p,q)$, which is conserved in the sense that $\frac{d}{dt}E(p,q) = u(\frac{dy}{dt})$ for all trajectories (p,q,u,y) satisfying (1) where the functional $u(\frac{dy}{dt})$ appearing on the right hand side is the mechanical power.

Van der Schaft [2] has showed that a state-space representation of the form (1) exists for time-reversible Hamiltonian systems whose transfer function G is such that

$$G(s) = G(-s)^\top = G(-s) \quad (2)$$

Shodhan Rao is with Control Engineering group, Faculty of Electrical Engineering, Mathematics and Computer Science, University of Twente, 7500 AE Enschede, The Netherlands. Email of corresponding author: s.rao@ewi.utwente.nl

Paolo Rapisarda and Lewis Moody are with Information: Signals, Images, Systems (ISIS) Research group, School of Electronics and Computer Science (ECS), University of Southampton, United Kingdom. Email: pr3@ecs.soton.ac.uk, ljm405@ecs.soton.ac.uk

In [1], equation (1) represents a *conservative time-reversible Hamiltonian system* if P and Q are positive definite. Representations (1) play an important role in the model reduction procedure of [1] which preserves their structure after reduction. This paper, by concentrating on the computation of a state space representation of the form (1) from a transfer function, enlarges the scope of application of the model reduction procedure of [1] to those situations when the system under consideration is a conservative time-reversible Hamiltonian system.

In this paper, we use the behavioural framework and the calculus of quadratic differential forms; the reader is referred to [4] and [8] for a thorough exposition. For a given nonzero finite-dimensional symmetric two-variable polynomial square matrix J, we first define a class of behaviours known as J-lossless behaviours. We then show that for a certain J that is associated with conservative SISO mechanical systems a realization of a J-lossless behaviour of the form (1) can be obtained from its transfer function using polynomial algebraic methods. Our realization gives positive definite P and Q, thereby showing that J-lossless behaviours are equivalent to conservative time-reversible Hamiltonian system of [1]. Further, the P and Q matrices obtained by our realization procedure are diagonal and tridiagonal respectively. We show that this special structure of P and Q can be utilized to obtain a synthesis of a lossless linear mechanical system with a given transfer function using springs and masses.

The paper is organized as follows: We introduce important concepts and algebraic tools in section 2. In section 3, we introduce the notion and discuss properties of J-lossless behaviours. We then discuss in section 4 the main result of the paper, namely an algorithm to compute a realization (1) of a J-lossless behaviour. In section 5, we show the application of our method to synthesis of lossless mechanical systems. We conclude the paper with a discussion of the current research direction in section 6.

Notation

The space of $\mathtt{n}$ dimensional real, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and the space of $\mathtt{m}\times\mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. The space of $\mathtt{m}\times\mathtt{m}$ symmetric real matrices is denoted by $\mathbb{R}_s^{\mathtt{m}\times\mathtt{m}}$. If one of the dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times\mathtt{n}}$ denotes the set of real matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^\bullet$ whose elements are denoted with w, the notation $\mathbb{R}^{\mathtt{w}}$

(note the typewriter font type!) is used and when dealing with a vector space $\mathbb{R}^\bullet$ whose elements are denoted with ℓ, the notation $\mathbb{R}^l$ is used; similar considerations hold for matrices representing linear operators on such spaces. Given two matrices A and B with the same number of columns, we denote with $\mathrm{col}(A, B)$ the matrix obtained by stacking A over B. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of polynomials with real coefficients in the indeterminate ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The set of $\mathtt{n} \times \mathtt{m}$ polynomial matrices in ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n} \times \mathtt{m}$ polynomial matrices in ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. $\deg(p)$ denotes the degree of $p \in \mathbb{R}[\xi]$. $\mathrm{diag}(a_1, \ldots, a_n)$ denotes the diagonal matrix whose diagonal entries are $a_1, \ldots, a_n$ in the given order if $a_1, \ldots, a_n \in \mathbb{R}$ and the block diagonal matrix with entries $a_1, \ldots, a_n$ along the diagonal in the given order if $a_1, \ldots, a_n$ are real square matrices. I_N stands for identity matrix of size N. $0_{\mathtt{w}\times l}$ denotes a matrix of size $\mathtt{w} \times l$ consisting of zeroes. $\jmath$ denotes the imaginary square root of -1. $A_{i,j}$ denotes the entry corresponding to the i^{th} row, j^{th} column of a given real matrix A.

II. Background

A. Linear differential behaviors

A *linear differential behavior* $\mathfrak{B}$ is a linear subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all solutions w of a given system of linear constant-coefficient differential equations. Such a set is represented as

$$R\left(\frac{d}{dt}\right) w = 0 \tag{3}$$

where $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$; (3) is called a *kernel representation* of the behavior $\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid w \text{ satisfies (3) }\}$, and w is called the $manifest$ or *external variable* of $\mathfrak{B}$. The class of all such behaviors is denoted with $\mathfrak{L}^{\mathtt{w}}$.

When modeling physical systems from first principles, we often introduce a number of *latent* (or *auxiliary*) variables ℓ besides the manifest ones: thus *latent variable representations*

$$R\left(\frac{d}{dt}\right) w = M\left(\frac{d}{dt}\right) \ell \tag{4}$$

are obtained. Equation (4) describes the *full behavior*

$$\mathfrak{B}_{\mathfrak{f}} := \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+l}) \ |(4) \text{ holds}\}$$

and we call the projection of $\mathfrak{B}_{\mathfrak{f}}$ on the w variable, i.e.

$$\mathfrak{B} := \{w \mid \exists \ell \text{ such that (4) holds}\}$$

the *manifest behavior* associated with (4).

When the matrix R in (4) is the $\mathtt{w}$-dimensional identity, we call

$$w = M\left(\frac{d}{dt}\right) \ell \tag{5}$$

an *image representation* of $\mathfrak{B}$. A behavior can be represented by (5) if and only if it is controllable in the behavioral sense (see Chapter 5 of [4]). The latent variable ℓ in (5) is called *observable* from w if $[w = M(\frac{d}{dt})\ell = 0] \Longrightarrow [\ell = 0]$. It can be shown that this is the case if and only if the matrix $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. If $\mathfrak{B}$ is controllable, then it admits an observable image representation (5) with $M \in \mathbb{R}^{\mathtt{w}\times l}[\xi]$; an i/o partition (see [4] for the definition of input and output in the behavioural context) then corresponds to a partition of M as $M = \mathrm{col}(U, Y)$ with $U \in \mathbb{R}^{l\times l}[\xi]$ non singular. In such a case the transfer function from u to y is the matrix of rational functions $G = YU^{-1}$. Note that for a controllable system, there always exists an image representation with the number of latent variables equal to the number of inputs of the system.

In this paper we also use the concept of state and of state representation (see [7] for a thorough discussion). A latent variable ℓ is a *state variable* for $\mathfrak{B}$ if and only if $\mathfrak{B}$ admits a representation (4) of first order in ℓ and zeroth order in w: $E\frac{d\ell}{dt} + F\ell + Gw = 0$. Such a representation is called a *state representation* of $\mathfrak{B}$; in this case we denote the latent variable with x. The minimal number of state variables that can be used in order to represent $\mathfrak{B}$ in state-space form is an invariant called the *McMillan degree* of $\mathfrak{B}$ and is denoted with $\mathtt{n}(\mathfrak{B})$. By combining the notion of state with that of inputs and outputs we arrive at the *input/state/output representation* (*i/s/o*) $\frac{d}{dt}x = Ax + Bu$, $y = Cx + Du$, $w = \mathrm{col}(u, y)$.

It has been argued in [7] that state variables can be computed from the external and/ or latent variables by applying a polynomial differential operator called a *state map* to them. For the purposes of this paper, we consider only state maps acting on the latent variables of an image representation of $\mathfrak{B}$. Since we restrict our attention in this paper to observable SISO systems, a state map is of the form

$$X(\xi) = col(X_i(\xi)) \qquad i = 1, ..., N$$

where $X_i \in \mathbb{R}[\xi]$. The problem of computing a state map from an image representation (5) has been dealt with in [7]; in this paper we will propose an alternative solution based on two-variable polynomial algebra. We call a state map minimal if it induces a minimal state variable.

B. Quadratic differential forms

We briefly review the concepts of [8] necessary for the results presented here. A quadratic functional acting on an infinitely differentiable trajectory w can be written as

$$Q_\Phi(w) = \sum_{h,k=0}^{N} \left(\frac{d^h w}{dt^h}\right)^T \Phi_{h,k} \left(\frac{d^k w}{dt^k}\right) \tag{6}$$

where $\Phi_{h,k}$ are $\mathtt{w} \times \mathtt{w}$ -dimensional real matrices, and N is a non-negative integer. Such a functional is called a *quadratic differential form* (QDF). With the QDF (6), we associate the two-variable polynomial matrix

$$\Phi(\zeta,\eta)=\sum_{h,k=0}^{N}\Phi_{h,k}\zeta^h\eta^k$$

The main advantage of associating two-variable polynomial matrices with QDF's is that they allow for convenient calculus. We now illustrate this using the notion a derivative of a QDF. A QDF Q_Ψ is the derivative of QDF Q_Φ if and only if for the corresponding two-variable polynomial matrices, there holds $(\zeta+\eta)\Phi(\zeta,\eta)=\Psi(\zeta,\eta)$ (see [8]).

Defined below are the notions of nonnegativity and positivity of QDFs.

Definition 1: Let $\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Q_Φ is said to be nonnegative, denoted by $Q_\Phi \geq 0$ if $Q_\Phi(w)\geq 0$ for all $w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$; and positive, denoted by $Q_\Phi > 0$, if $Q_\Phi \geq 0$, and $[Q_\Phi(w)=0] \implies [w=0]$.

C. Autonomous, oscillatory and lossless systems

An autonomous system is a system with no inputs. For such a system, the future of every trajectory is completely determined by its past.

Definition 2: A linear differential behaviour $\mathfrak{B}$ is called autonomous *if for all $w_1, w_2 \in \mathfrak{B}$,*

$$[w_1(t)=w_2(t)\quad\forall\quad t\leq 0]\Rightarrow[w_1=w_2]$$

The *invariant polynomials* of a polynomial matrix $P \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi]$ are the diagonal elements of the Smith form (see Section 6.3-3, [3] for a definition) of P. Let $\mathfrak{B}=\ker\big(R(\frac{d}{dt})\big)$ be a minimal kernel representation of an autonomous behaviour $\mathfrak{B}$. Then the invariant polynomials of R are also called the invariant polynomials of $\mathfrak{B}$. The roots of det(R) are called the *characteristic frequencies* of $\mathfrak{B}$.

An oscillatory behaviour is defined below.

Definition 3: A linear differential behavior $\mathfrak{B}$ is oscillatory if

- $\mathfrak{B}$ is the set of solutions of a system of linear constant-coefficient differential equations
$$R\left(\frac{d}{dt}\right)w=0,\qquad R\in\mathbb{R}^{\bullet\times\mathtt{w}}[\xi];$$
equivalently, $\mathfrak{B}$ belongs to the class of linear differential behaviors with $\mathtt{w}$ external variables;
- Every solution $w:\mathbb{R}\to\mathbb{R}^{\mathtt{w}}$ is bounded on $(-\infty,\infty)$.

From the definition, it follows that an oscillatory system is necessarily autonomous: if there were any input variables in w, then those components of w could be chosen to be unbounded. It was proved in proposition 2 of [6] that any behavior $\mathfrak{B}$ is oscillatory if and only if every non-zero invariant polynomial of $\mathfrak{B}$ has distinct and purely imaginary roots. In the following, a polynomial matrix will be called oscillatory if all its invariant polynomials have distinct and purely imaginary roots.

The notion of a *conserved quantity* was first defined in [6], and it is used for defining lossless systems. This definition is given below.

Definition 4: Let $\mathfrak{B}$ be a linear differential behaviour. A QDF Q_Φ is a *conserved quantity* for $\mathfrak{B}$ if

$$\frac{d}{dt}Q_\Phi(w)=0\qquad\forall\quad w\in\mathfrak{B}$$

Thus, conserved quantity is a QDF, whose derivative is zero along the trajectories of a given behaviour. The notion of an autonomous lossless system as in [5], is defined below.

Definition 5: A linear autonomous behavior $\mathfrak{B}\in\mathcal{L}^{\mathtt{w}}$ is *lossless* if there exists a conserved quantity Q_E associated with $\mathfrak{B}$, such that $Q_E>0$. Such a Q_E is called an *energy function* for the system.

The main result of [5] which is used in this paper, is given below.

Theorem 6: A linear autonomous behaviour $\mathfrak{B}\in\mathcal{L}^{\mathtt{w}}$ is lossless if and only if it is oscillatory.

Proof: See Theorem 3, p. 1529 of [5]. ■

An open lossless system is one for which there exists an energy function which is positive for non-zero trajectories of the system and the rate of change of the energy function is zero whenever the inputs of the system are equal to zero. In order to study lossless systems to a greater level the authors suggest consulting [5].

III. J-LOSSLESS BEHAVIOURS

In this section, we provide the definition and study the properties of a J-lossless behaviour. For a certain J that is associated with conservative mechanical systems, we then obtain a realization in a particular form, which also has relevance in the synthesis of lossless mechanical systems using springs and masses. Below, we define a J-lossless behaviour.

Definition 7: Let $J\in\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ be such that $J\neq 0$. A behaviour $\mathfrak{B}\in\mathcal{L}^{\mathtt{w}}$ is said to be *J-lossless* if there exists a QDF $Q_E>0\ \forall w\in\mathfrak{B}$ with $E\in\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, such that for every trajectory $w\in\mathfrak{B}$, $Q_J(w)=\frac{d}{dt}Q_E(w)$. The QDF Q_E is called the *principal energy function* of $\mathfrak{B}$.

Following from [5], Q_E denotes an energy function, because it is strictly positive and conserved. Here Q_J is to be interpreted as the power entering the system. Hereafter, in this paper, J is defined as below:

$$J(\zeta,\eta):=\begin{bmatrix}0&\zeta\\ \eta&0\end{bmatrix}$$

The above J is associated with behaviours of conservative SISO mechanical systems as the power entering such a system is the product of the input variable and the derivative of the output variable. In the following lemma, we give the algebraic conditions on the representation of a controllable SISO system under which it is J-lossless. This lemma will be instrumental in proving the main result of this paper.

Lemma 8: Let $\mathfrak{B}=\text{Im}\begin{pmatrix}n(\frac{d}{dt})\\ d(\frac{d}{dt})\end{pmatrix}$ with $n,d\in\mathbb{R}[\xi]$ and $\deg(d)>\deg(n)$, be an observable image representation of a behaviour $\mathfrak{B}$. The following statements are equivalent.

1) $\mathfrak{B}$ is J-lossless.

2) The following hold:
   - n and d are even and oscillatory, i.e, they have distinct and purely imaginary roots $\pm \jmath\omega_i$ with $i = 1, ..., N_n$ and $\pm \jmath\omega'_i$ with $i = 1, ..., N_d$ respectively, where $N_n = (\deg(n))/2$ and $N_d = (\deg(d))/2$.
   - $\deg(d)=\deg(n)+2$.
   - The ω_i^2 interlace with $\omega_i'^2$, i.e along the real axis, exactly one root of f occurs between any two consecutive roots of r, where $f(\xi^2) := n(\xi)$ and $r(\xi^2) := d(\xi)$.
   - $n(\jmath\omega'_1) > 0$.

Proof: We state the following theorem from p. 1527 of [5], which will be used in proving the lemma.

Theorem 9: Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2 + \omega_0^2)(\xi^2+\omega_1^2)\dots(\xi^2+\omega_{n-1}^2)$, where $\omega_0 < \omega_1 \dots < \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $r(\xi) := (\xi+\omega_0^2)(\xi+\omega_1^2)\dots(\xi+\omega_{n-1}^2)$. Let $f(\xi)$ be a polynomial of degree less than or equal to $n-1$. Define

$$\phi_1(\zeta,\eta) := \frac{\eta r(\zeta^2)f(\eta^2)+\zeta r'(\eta^2)f(\zeta^2)}{\zeta+\eta}$$

Then $Q_{\phi_1} > 0$ if and only if $f(-\omega_0^2) > 0$ and the roots of f are interlaced between those of r, i.e along the real axis, exactly one root of f occurs between any two consecutive roots of r.

We now resume the proof of Lemma 8. Define

$$\begin{aligned} M: &= \mathrm{col}(n,d) \\ J'(\zeta,\eta): &= M(\zeta)^\top J(\zeta,\eta)M(\eta) \\ &= \zeta n(\zeta)d(\eta)+d(\zeta)n(\eta)\eta \end{aligned}$$

((2) $\Longrightarrow$ (1)): Consider a trajectory $w = M(\frac{d}{dt})\ell$ where $\ell \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$. It is easy to see that $Q_J(w) = Q_{J'}(\ell)$. Define

$$E'(\zeta,\eta) := \frac{J'(\zeta,\eta)}{\zeta+\eta}$$

Since n and d are both even, it follows that $E' \in \mathbb{R}[\zeta,\eta]$ (see Theorem 3.1, p. 1711, [8]). Since n and d are co-prime, from the observability of the image representation, there exists $F \in \mathbb{R}^{1\times 2}[\xi]$, such that $\ell = F(\frac{d}{dt})w$. Define

$$E(\zeta,\eta) := F(\zeta)^\top E'(\zeta,\eta)F(\eta)$$

It is easy to see that $Q_{E'}(\ell) = Q_E(w)$ and that $\frac{d}{dt}Q_E(w) = Q_J(w)$. From Theorem 9, it follows that $Q_{E'} > 0$, from which it follows that $Q_E(w) > 0 \;\forall\; w \in \mathfrak{B}$. Hence from Definition 7, it follows that $\mathfrak{B}$ is J-lossless.

((1) $\Longrightarrow$ (2)): Assume that $\mathfrak{B}$ is J-lossless. Define $\mathfrak{B}_1 := \ker(n(\frac{d}{dt}))$ and $\mathfrak{B}_2 := \ker(d(\frac{d}{dt}))$. Since $\mathfrak{B}$ is J-lossless, there exists $E \in \mathbb{R}^{2\times 2}[\zeta,\eta]$, such that $\frac{d}{dt}Q_E(w) = Q_J(w) = Q_{J'}(\ell)$. Define $E'(\zeta,\eta) := M(\zeta)^\top E(\zeta,\eta)M(\eta)$. Then it is easy to see that

$$E'(\zeta,\eta) = \frac{\zeta n(\zeta)d(\eta)+d(\zeta)n(\eta)\eta}{\zeta+\eta}$$

Since $Q_E(w) > 0$ for all $w \in \mathfrak{B}$, it follows that $Q_{E'} > 0$. It is easy to see that $Q_{E'}$ is a conserved quantity for both $\mathfrak{B}_1$ and $\mathfrak{B}_2$. Hence both $\mathfrak{B}_1$ and $\mathfrak{B}_2$ are lossless. From Theorem 6, it follows that n and d are oscillatory. Since $J'(\zeta,\eta)$ is divisible by $(\zeta+\eta)$, it follows that either n and d are both even or both odd. If n is odd, then r' defined by $r'(\xi) := \xi n(\xi)$ is not oscillatory, which implies that there cannot exist a conserved quantity for $\mathfrak{B}' := \ker\big(r'(\frac{d}{dt})\big)$ that is positive. But Q_E is both conserved and positive along $\mathfrak{B}'$ which is a contradiction. Hence it follows that n and d are both even. Since $\deg(d) > \deg(n)$, it now follows from Theorem 9 that the roots of f are interlaced between those of r, $\deg(d)=\deg(n)+2$ and $n(\jmath\omega'_1) > 0$. ■

IV. Main Result

For the case of a J-lossless behaviour $\mathfrak{B}$ given by $w = \big(\mathrm{col}(n(\frac{d}{dt}), d(\frac{d}{dt}))\ell$ with $\deg(d) > \deg(n)$, we will show in the following that if $Q_{E'}(\ell)$ is the principal energy function of $\mathfrak{B}$, then there exists a state map $X'(\xi) = \begin{bmatrix} X_1(\xi) \\ \vdots \\ X_N(\xi) \\ a_1\xi X_1(\xi) \\ \vdots \\ a_N\xi X_N(\xi) \end{bmatrix}$ with $N := \frac{deg(d)}{2}$, $a_1,\dots,a_N > 0$ and X_1 being equal to n, such that,

$$E'(\zeta,\eta) = X'(\zeta)^T \begin{bmatrix} K & 0 \\ 0 & M^{-1} \end{bmatrix} X'(\eta),$$

for some tridiagonal $K = K^\top > 0$ and diagonal $M > 0$. From such $X(\xi)$, a realization (1) will be readily obtained. In the following, 0_N denotes a square matrix of size N consisting of zeroes.

We now prove the main result of this paper.

Theorem 10: Let $w = \begin{pmatrix} n_0(\frac{d}{dt}) \\ d_0(\frac{d}{dt}) \end{pmatrix}\ell$ with $n_0, d_0 \in \mathbb{R}[\xi]$ and $\deg(d_0) > \deg(n_0)$, be an observable image representation of a J-lossless behaviour $\mathfrak{B}$ with McMillan degree $2N$. Then there exists a tridiagonal $K \in \mathbb{R}^{N\times N}$ with $K = K^T > 0$, a diagonal $M \in \mathbb{R}^{N\times N}$ with $M = M^T > 0$ and $X \in \mathbb{R}^{N\times 1}[\zeta]$ such that:

i) $\begin{bmatrix} X[\xi] \\ \xi MX[\xi] \end{bmatrix}$ is a minimal state map for $\mathfrak{B}$.

ii) $E'(\zeta,\eta) = X(\zeta)^T KX(\eta) + \zeta\eta X(\zeta)^T MX(\eta)$ is such that $Q_{E'}(\ell)$ is the principal energy function of $\mathfrak{B}$.

Proof: Since $\mathfrak{B}$ is J-lossless, it follows from Lemma 8 that d_0 and n_0 are even. Hence, we can write $d_0 = qn_0 + r_1$ according to the Euclidean division algorithm. Since n_0 and d_0 are even, so are r_1 and q. Since, $Q_{E'}(\ell)$ is the principal energy function of $\mathfrak{B}$, from Definition 7,

$$E'(\zeta,\eta) = \frac{d_0(\zeta)\eta n_0(\eta) + \zeta n_0(\zeta)d_0(\eta)}{\zeta+\eta}$$

Observe that

$$\begin{aligned} E'(\zeta,\eta) &= \frac{d_0(\zeta)\eta n_0(\eta)+\zeta n_0(\zeta)d_0(\eta)}{\zeta+\eta} \\ &= \frac{n_0(\zeta)n_0(\eta)(\zeta q(\eta)+\eta q(\zeta))}{\zeta+\eta} \\ &+ \frac{(\eta r_1(\zeta)n_0(\eta)+n_0(\zeta)\zeta r_1(\eta))}{\zeta+\eta} \\ &= n_0(\zeta)n_0(\eta)\psi_1(\zeta,\eta)+\psi_1^{'}(\zeta,\eta), \end{aligned}$$

where $\psi_1^{'}(\zeta,\eta) := \frac{r_1(\zeta)\eta n_0(\eta)+\zeta n_0(\zeta)r_1(\eta)}{\zeta+\eta}$ and $\psi_1(\zeta,\eta) := \frac{\zeta q(\zeta)+\eta q(\eta)}{\zeta+\eta}$. It is straightforward to verify that ψ_1 and ψ_1' are polynomials from the fact that their numerators vanish when $\zeta=-\xi$ and $\eta=\xi$. It is easy to see that $N=\deg(d_0)/2$. From Lemma 8, it follows that $deg(n_0)=2N-2$. This implies that q has degree equal to 2, and since q is even, it follows that

$$q(\zeta)\eta+\zeta q(\eta)=(\zeta+\eta)(b_1+a_1\zeta\eta)$$

where $a_1,b_1\in\mathbb{R}$. This implies that

$$E'(\zeta,\eta)=n_0(\zeta)n_0(\eta)(b_1+a_1\zeta\eta)+\psi_1^{'}(\zeta,\eta),$$

Now observe that

$$\begin{aligned} \psi_1^{'}(\zeta,\eta) &= (n_0(\zeta)r_1(\eta)+r_1(\zeta)n_0(\eta)) \\ &- \frac{(\zeta r_1(\zeta)n_0(\eta)+n_0(\zeta)\eta r_1(\eta))}{(\zeta+\eta)} \end{aligned}$$

Define

$$\psi_2(\zeta,\eta) := -\frac{\zeta r_1(\zeta)n_0(\eta)+n_0(\zeta)\eta r_1(\eta)}{\zeta+\eta} \qquad (7)$$

we now show that this polynomial induces a positive quantity.

Lemma 11: Let $\mathfrak{B}=\operatorname{Im}\begin{pmatrix} n_0(\frac{d}{dt}) \\ d_0(\frac{d}{dt})\end{pmatrix}$ with $n_0,d_0\in\mathbb{R}[\xi]$ and $\deg(d_0)>\deg(n_0)$, be an observable image representation of a J-lossless behaviour $\mathfrak{B}$. Let r_1 be the remainder of the division of d_0 by n_0. Then:

1) There exist $a_1,b_1\in\mathbb{R}$, $a_1,b_1>0$ such that $d_0(\xi)=(a_1\xi^2+b_1)n_0(\xi)+r_1(\xi)$.
2) $\mathfrak{B}_1=\operatorname{Im}\begin{pmatrix} -r_1(\frac{d}{dt}) \\ n_0(\frac{d}{dt})\end{pmatrix}$ is J-lossless.

Proof: From Lemma 8, it follows that n_0 and d_0 are of the form

$$\begin{aligned} n_0(\xi) &= c_0(\xi^2+\omega_2^2)(\xi^2+\omega_4^2)\dots(\xi^2+\omega_{2N-2}^2) \\ d_0(\xi) &= c_1(\xi^2+\omega_1^2)(\xi^2+\omega_3^2)\dots(\xi^2+\omega_{2N-1}^2) \end{aligned}$$

where $c_0,c_1>0$, and $0<\omega_1<\omega_2<\dots<\omega_{2N-1}$. Now consider the partial fraction expansion of the rational function $f(\xi) := \frac{d(\xi)}{n(\xi)}$. We obtain

$$\frac{d_0(\xi)}{n_0(\xi)}=a_1\xi^2+g\xi+b_1+\sum_{i=1}^{N-1}\frac{k_i\xi+p_i}{\xi^2+\omega_{2i}^2}$$

Since $f(\xi)=f(-\xi)$, we obtain $g=0$ and $k_i=0$ for $i=1,\dots,N-1$. This gives

$$\frac{d_0(\xi)}{n_0(\xi)}=a_1\xi^2+b_1+\sum_{i=1}^{N-1}\frac{p_i}{\xi^2+\omega_{2i}^2}$$

Observe that

$$a_1=\lim_{\xi\to\infty}\frac{d_0(\xi)}{\xi^2 n_0(\xi)}=\frac{c_1}{c_0}>0$$

$$\begin{aligned} b_1 &= \lim_{\xi\to\infty}\left\{\frac{d_0(\xi)}{n_0(\xi)}-a_1\xi^2\right\} \\ &= a_1\left(\sum_{i=0}^{N-1}\omega_{2i+1}-\sum_{i=1}^{N-1}\omega_{2i}\right) \\ &= a_1\left(\omega_1+\sum_{i=1}^{N-1}(\omega_{2i+1}-\omega_{2i})\right)>0 \end{aligned}$$

Observe also that

$$\begin{aligned} p_i &= \lim_{\xi\to \jmath\omega_{2i}}\frac{d_0(\xi)(\xi^2+\omega_{2i}^2)}{n_0(\xi)} \\ &= \lim_{\xi\to \jmath\omega_{2i}}\frac{d_0(\xi)}{c_0\prod_{q=1,q\neq i}^{N-1}(\xi^2+\omega_{2q}^2)}<0 \end{aligned}$$

because the numerator and denominator have opposite signs whenever $\xi=\jmath\omega_{2i}$. Now notice that

$$\begin{aligned} r_1(\xi) &= n_0(\xi)\sum_{i=1}^{N-1}\frac{p_i}{\xi^2+\omega_{2i}^2} \\ &= c_0\sum_{i=1}^{N-1}p_i\left(\prod_{q=1,q\neq i}^{N-1}(\xi^2+\omega_{2i}^2)\right) \end{aligned}$$

It is easy to see that $\deg(r_1)=\deg(n_0)-2$. Define $f(\xi^2) := -r_1(\xi)$ and $s(\xi^2) := n_0(\xi)$. It can be verified that

$$\begin{aligned} f(-\omega_2^2) &= -c_0p_1(\prod_{q=2}^{N-1}(\omega_{2q}^2-\omega_2^2))>0 \\ f(-\omega_4^2) &= -c_0p_2(\prod_{q\neq 2}(\omega_{2q}^2-\omega_4^2))<0 \\ f(-\omega_6^2) &= -c_0p_3(\prod_{q\neq 3}(\omega_{2q}^2-\omega_6^2))>0 \\ &\vdots \end{aligned}$$

Since f is a continuous function and can have a maximum of $N-2$ roots, it follows that the roots of f are real and interlaced between those of s. It now follows from Lemma 8 that $\mathfrak{B}_1$ is J-lossless. This completes the proof. ■

It follows from Lemma 11 that since $\mathfrak{B}_1$ is J-lossless, Q_{ψ_2} is positive. We can now exactly repeat the same steps as done before, this time with reference to the behaviour $\mathfrak{B}_1$ defined in statement (2) of Lemma 11. For $i=0,\dots,N-2$, let r_{i+2} denote the remainder and $(a_{i+2}\xi^2+b_{i+2})$ be the quotient when n_i is divided by n_{i+1}, where $n_{i+1} := -r_{i+1}$. From Lemma 11, it follows that $a_i,b_i>0$ for $i=1,\dots,N$. Define $X := \operatorname{col}_{i=0}^{N-1}(n_i)$. Let K be the tridiagonal matrix of size N whose i^{th} diagonal element is b_i and each of whose non-zero non-diagonal element is equal to -1. Define $M := \operatorname{diag}(a_1,a_2,\dots,a_N)$. Then it can be verified that

$$E'(\zeta,\eta)=X(\zeta)^TKX(\eta)+\zeta\eta X(\zeta)^TMX(\eta)$$

Observe that $M > 0$. Define

$$X'(\xi) := \begin{bmatrix} X(\xi) \\ \xi M X(\xi) \end{bmatrix} \qquad Q := \text{diag}(K, M^{-1})$$

and observe that $E'(\zeta,\eta) = X'(\zeta)^T Q X'(\eta)$. Notice that the number of components of X' is $2N$, equal to the McMillan degree $\mathtt{n}(\mathfrak{B})$; that the first N are linearly independent, since they have different degrees, and that the same holds for the last N components. Moreover, the odd and even components are linearly independent; consequently $X'(\xi)$ is a minimal state map for $\mathfrak{B}$. The fact that K is positive definite follows from the fact that

$$E'(\zeta,\eta) = \begin{pmatrix} 1 & \zeta & \cdots & \zeta^{2N-1} \end{pmatrix} \tilde{E} \begin{pmatrix} 1 \\ \eta \\ \vdots \\ \eta^{2N-1} \end{pmatrix}$$

$Q_{E'} > 0 \Leftrightarrow \widetilde{E} > 0$. The fact that $X'(\xi)$ is a minimal state map implies that there exists a nonsingular matrix T such that $X'(\xi) = T \begin{pmatrix} 1 \\ \xi \\ \vdots \\ \xi^{2N-1} \end{pmatrix}$. Consequently,

$$\begin{aligned} E'(\zeta,\eta) &= X(\zeta)^T K X(\eta) + \zeta\eta X(\zeta)^T M X(\eta) \\ &= \begin{bmatrix} X(\zeta)^T & \zeta X(\zeta)^T M \end{bmatrix} Q \begin{bmatrix} X(\eta) \\ \eta M X(\eta) \end{bmatrix} \end{aligned}$$

$\Rightarrow \widetilde{E} = T^T Q T$ and so $\text{diag}(K, M^{-1}) > 0$ therefore concluding the proof. ■

It is a matter of straightforward verification that the state space representation for $\mathfrak{B}$ associated with the state map $X(\xi)$ is

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 0_N & M^{-1} \\ -K & 0_N \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{N\times 1} \\ B \end{bmatrix} u \\ y &= B^\top q \end{aligned} \tag{8}$$

where $q = X(\frac{d}{dt})\ell$, $p = M\frac{dq}{dt}$, B is a column vector of dimension N whose first element is 1 and the rest are equal to zero, and $w = \text{col}(y,u)$ is a trajectory of $\mathfrak{B}$. Now define the following:

$$J := \begin{bmatrix} 0_N & -I_N \\ I_N & 0_N \end{bmatrix} \qquad Q := \text{diag}(K, M^{-1})$$

$$x := \text{col}(q,p) \qquad B_1 := \text{col}(0_{N\times 1}, B)$$

Then the state space representation (8) can be written as

$$\begin{aligned} \frac{d}{dt}x &= J^{-1}Qx + B_1 u \\ y &= B_1^\top J x \end{aligned}$$

and the energy function E_1 for $\mathfrak{B}$ is given by $E_1 = x^\top Q x$. The above representation is the same as the representation of time-reversible Hamiltonian systems as obtained in [2]. Indeed the transfer function G of a J-lossless behaviour obeys $G(s) = G(-s) = G(-s)^\top$. Incidentally the results of this section show that since we also have $K = K^\top > 0$ and $M = M^\top > 0$ in (8), a J-lossless behaviour $\mathfrak{B} = \text{Im}\begin{pmatrix} n(\frac{d}{dt}) \\ d(\frac{d}{dt}) \end{pmatrix}$ with $\deg(d) > \deg(n)$, is a conservative time-reversible Hamiltonian system as in [1].

V. Synthesis of lossless mechanical systems

Let $\mathfrak{B} = \text{Im}\big(\text{col}(n(\frac{d}{dt}), d(\frac{d}{dt}))\big)$ be a J-lossless behaviour with with $\deg(d) > \deg(n)$, $\mathtt{n}(\mathfrak{B}) = 2N$ and external variables y (output) and u (input). In this section, we associate to $\mathfrak{B}$ a mechanical system consisting of masses and springs with an external force acting on one of the masses and obeying the following property. There exists a mass whose displacement from its equilibrium position, together with the force, defines a set of trajectories equal to $\mathfrak{B}$. We call a mechanical system with this property a *mechanical realization* of $\mathfrak{B}$. In order to compute such a realization, first obtain X, M and K using the steps described in the proof of Theorem 10. Define $q := X(\frac{d}{dt})\ell$ and $p := \frac{d}{dt}\big(MX(\frac{d}{dt})\big)\ell$. Let B denote the column vector of dimension N whose first element is 1 and the rest are equal to zero. Then we have the following state space representation for $\mathfrak{B}$:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q \\ p \end{bmatrix} &= \begin{bmatrix} 0_N & M^{-1} \\ -K & 0_N \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix} + \begin{bmatrix} 0_{N\times 1} \\ B \end{bmatrix} u \\ y &= B^\top q \end{aligned}$$

For $i = 1, \ldots, N-1$, define $\delta_i :=$ principal i^{th} minor of K and $\delta_0 := 1$. Now define the diagonal matrix $D := \text{diag}(\delta_0, \delta_1, \ldots, \delta_{N-1})$. Observe that since K is positive definite, so is D. Define the following:

$$K' := DKD \qquad M' := DMD$$

$$\begin{bmatrix} q' \\ p' \end{bmatrix} := \begin{bmatrix} D^{-1} & 0 \\ 0 & D \end{bmatrix} \begin{bmatrix} q \\ p \end{bmatrix}$$

We obtain the following state space representation in terms of the new state vector $\text{col}(q', p')$:

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} q' \\ p' \end{bmatrix} &= \begin{bmatrix} 0_N & M'^{-1} \\ -K' & 0_N \end{bmatrix} \begin{bmatrix} q' \\ p' \end{bmatrix} \\ &\quad + \begin{bmatrix} 0_{N\times 1} \\ B \end{bmatrix} u \\ y &= B^\top q \end{aligned} \tag{9}$$

Define $\delta_{-1} := 0$ and $\delta_N := 0$. Observe that M' is diagonal and K' is tridiagonal with $K'_{i,i}=\delta_{i-1}^2 b_i$, $K'_{i,i+1} = -\delta_{i-1}\delta_i$ and $K'_{i,i-1} = -\delta_{i-2}\delta_{i-1}$ for $i = 1, \ldots, N$. It can be verified that

$$K'_{i,i} = -(K'_{i,i+1} + K'_{i,i-1}) > 0 \quad \text{for} \quad i = 1, \ldots, N-1. \tag{10}$$

We now use this property of K' to obtain a mechanical synthesis of $\mathfrak{B}$ as follows.

Consider a mechanical spring-mass system consisting of N springs with spring constants k_1, k_2, …, k_N and N

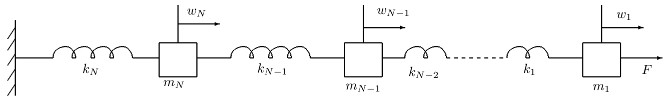

Fig. 1. A spring-mass system

masses m_1, m_2, …, m_N interconnected to each other and to the wall as shown in Figure 1.

Define $X_1 := \mathrm{col}(w_1, w_2, \ldots, w_N)$, $M_1 := \mathrm{diag}(m_1, m_2, \ldots, m_N)$, $p_1 = M_1 \frac{dX_1}{dt}$. Let B denote the column vector of dimension N whose first element is 1 and the rest are equal to zero. The equations of motion for the system can then be written as

$$\begin{aligned} \frac{d}{dt}\begin{bmatrix} X_1 \\ p_1 \end{bmatrix} &= \begin{bmatrix} 0_N & M_1^{-1} \\ -K'' & 0_N \end{bmatrix} \begin{bmatrix} X_1 \\ p_1 \end{bmatrix} \\ &+ \begin{bmatrix} 0_{N\times 1} \\ B \end{bmatrix} F \\ w_1 &= B^\top X_1 \end{aligned}$$

where K'' can be called as the stiffness matrix for the system which obeys the following property: K' is tridiagonal, $K''_{i,i+1} = -k_i$ for $i = 1, 2, \ldots, N-1$; $K''_{i,i-1} = -k_{i-1}$ for $i = 2, \ldots, N$ and

$$K''_{i,i} = -(K''_{i,i+1} + K''_{i,i-1}) > 0 \quad \text{for} \quad i = 1, \ldots, N-1. \tag{11}$$

Observe that equation (11) is similar to equation (10). Coming back to the system described by equation (9), we can now obtain a mechanical synthesis of $\mathfrak{B}$ using the matrices K' and M' as described below.

Define $m_i :=$ i^{th} diagonal entry of M' for $1 = 1, \ldots, N$, $k_i := -K'_{i,i+1}$ for $i = 1, 2, \ldots, N-1$ and $k_N := K'_{N,N} + K'_{N,N-1}$ and observe that the system described in Figure 1 with parameters $m_1, m_2, \ldots, m_N$ and $k_1, k_2, \ldots, k_N$ is a mechanical synthesis of the given behaviour $\mathfrak{B}$.

The proof of Theorem 10 and the idea for synthesis of mechanical systems with J-lossless behaviours suggests Algorithm 1 to perform the construction of a state map, of the matrices M and K and of a mechanical system corresponding to a J-lossless behaviour.

We now illustrate Algorithm 1 with an example.

Example 12: As an example, consider $\mathfrak{B} = \mathrm{Im}\big(\mathrm{col}\big(n(\frac{d}{dt}), d(\frac{d}{dt})\big)\big)$, with $n(\xi) = \xi^4 + 4\xi^2 + 3$ and $d(\xi) = 2\xi^6 + 13\xi^4 + 22\xi^2 + 8$. Application of Algorithm 1 gives the following output:

$$K = \begin{bmatrix} 5 & -5 & 0 \\ -5 & 14.0625 & -9.0625 \\ 0 & -9.0625 & 24.5292 \end{bmatrix}$$

$$M = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 6.25 & 0 \\ 0 & 0 & 14.0167 \end{bmatrix}$$

$$X(\xi) = \begin{bmatrix} \xi^4 + 4\xi^2 + 3 \\ 0.8\xi^2 + 1.4 \\ 0.5172 \\ 2\xi^5 + 8\xi^3 + 6\xi \\ 5\xi^3 + 8.75\xi \\ 7.2494\xi \end{bmatrix}$$

$k_1 = 5$, $k_2 = 9.0625$, $k_3 = 15.4667$, $m_1 = 2$, $m_2 = 6.25$, $m_3 = 14.0167$.

Algorithm 1 Input: A J-lossless behaviour $\mathfrak{B} = \mathrm{Im}(\mathrm{col}(n(\frac{d}{dt}), d(\frac{d}{dt}))$ with $\mathtt{n}(\mathfrak{B}) = 2N = \deg(d) > \deg(n)$.
Output: A tridiagonal K and a diagonal M corresponding to a state representation of the form (8) of $\mathfrak{B}$, a corresponding state map $X(\xi)$, and $\{m_i, k_i\}_{i=1,\ldots,N}$ corresponding to a mechanical synthesis of $\mathfrak{B}$ of the form shown in Figure 1.

---

1. For ($i = 1$ to N) do {
2. Find the quotient q and the remainder r of the Euclidean division of d by n.
3. Assign $a_i =$ the leading coefficient of q.
4. Assign $b_i =$ the constant term of q.
5. Assign $n_i := n$, $d = n$ and $n = -r$.}
6. Assign $M_1 := \mathrm{diag}(a_1, a_2, \ldots, a_N)$.
7. Assign $K_1 := \begin{bmatrix} b_1 & -1 & 0 & 0 & \cdots & 0 \\ -1 & b_2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & b_3 & -1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & -1 & b_{N-1} & -1 \\ 0 & \cdots & \cdots & 0 & -1 & b_N \end{bmatrix}$
8. Assign $\delta_0 := 1$ and for ($i = 1$ to $N-1$) assign $\delta_i :=$ determinant of the top left-most ($i \times i$) block of K_1.
9. Assign $D := \mathrm{diag}(\delta_0, \delta_1, \ldots, \delta_{N-1})$.
10. Assign $K := DK_1D$, $M := DM_1D$.
11. Assign $q_1(\xi) := \mathrm{col}\big(n_1(\xi), n_2(\xi), \ldots, n_N(\xi)\big)$
12. Compute $q(\xi) = D^{-1}q_1(\xi)$, $X(\xi) = \mathrm{col}(q(\xi), \xi M q(\xi))$
13. For ($i = 1$ to N) assign $m_i = M_{i,i}$.
14. Assign $k_N := K_{N,N} + K_{N,N-1}$ and for ($i = 1$ to $N-1$), assign $k_i := -K_{i,i+1}$.
15. Output $M, K, X(\xi), \{m_i, k_i\}_{i=1,\ldots,N}$.

VI. Conclusions

We have presented an algorithm for the realization of a J-lossless behaviour based on successive divisions of univariate polynomials. We have also sketched a method of synthesis of lossless mechanical systems based on our method for realization. Current research is being carried out in applying the ideas presented here in the direction of MIMO version of J-lossless behaviours.

References

[1] A. J. van der Schaft and J. E. Oeloff, "Model reduction of linear conservative mechanical systems", *IEEE Trans. Auto. Contr.*, 35:6 (1990), 729-733.

[2] A. J. van der Schaft, "Time-reversible Hamiltonian systems", *Systems and Control letters*, 1:5 (1982), 295-300.

[3] T. Kailath, *Linear Systems*, Prentice–Hall, Englewood Cliffs, NJ, 1980.

[4] J. W. Polderman and J. C. Willems, *Introduction to Mathematical System Theory: A Behavioral Approach*, Springer-Verlag, Berlin, 1997.

[5] S. Rao and P. Rapisarda, "Higher-order linear lossless systems", *International Journal of Control*, 81:10 (2008), 1519-1536

[6] P. Rapisarda and J.C. Willems, "Conserved- and zero-mean quadratic quantities in oscillatory systems", *Math. Contr. Sign. Syst.*, vol. 17 (2005), pp. 173-200.

[7] P. Rapisarda and J. C. Willems, "State maps for linear systems", *SIAM J. Control Optim., 35 (1997), pp. 1053–1091.*

[8] J. C. Willems and H. L. Trentelman, "On quadratic differential forms", *SIAM J. Control Optim., 36 (1998), pp. 1703–1749.*